A COURSE OF

HIGHER MATHEMATICS

V. I. Smirnov

Volume IV

INTEGRAL AND PARTIAL DIFFERENTIAL EQUATIONS

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A COURSE OF
Higher Mathematics

VOLUME IV

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INTRODUCTION

An account of the over-all plan of Prof. Smirnov's five-volume course on higher mathematics has been given in the Introduction to Vol. I of the present English edition.

This fourth volume of the set is devoted to subjects which lie at the very heart of classical mathematical physics and which supply the motivation of much recent work of great interest in functional analysis and in the theory of partial differential equations. The more elementary parts of the theory of the differential equations of mathematical physics have already been treated at the end of Vol. II. The present volume begins with full accounts of the theory of integral equations and of the calculus of variations which together play an important role in the discussion of the boundary value problems of mathematical physics. This is followed by a long chapter on the fundamental theory of partial differential equations and of systems of equations in which characteristics play a central role. Finally the boundary value problems of mathematical physics are treated in a complete and lucid way.

Although this volume is primarily intended for the use of mathematicians whose main interest is in the application of mathematics to the analysis and elucidation of physical problems, it contains many topics an acquaintance with which can only serve to deepen the understanding of anyone embarking on the study of functional analysis and other branches of analysis and it is to be hoped that it will find its way into their hands.

I. N. Sneddon

PREFACE TO THE SECOND EDITION

EVERY chapter except the one on the calculus of variations has been entirely revised in the present edition of Volume IV. Part of the material has been transferred to the new edition of Volume II, and a great deal of new material has been introduced.

I must sincerely thank S. M. Lozinskii for reading the chapter on integral equations in manuscript and making numerous valuable suggestions, which I utilized in the final revision. I often discussed the last two chapters with O. A. Ladyzhenskaya and Kh. L. Smolitskii, who gave me valuable assistance. Certain sections of these chapters were written by these authors at my request, as indicated in the text.

O. A. Ladyzhenskaya read Chapter III in manuscript and Kh. L. Smolitskii Chapter IV. They made a number of helpful suggestions as regards the final proofs. I thank them most sincerely.

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THE present edition of Volume IV does not differ essentially from the previous one. The most substantial change is in the treatment of integral equations with kernels depending on a difference, over a semi-infinite interval. The new treatment based on the theory of boundary value problems for analytic functions, is due to Yu. I. Cherskii. Apart from this, minor modifications have been introduced into the text in various places.

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CHAPTER I

INTEGRAL EQUATIONS

1. Examples of the formation of integral equations. Any equation containing the required function under the integral sign is an integral equation. Suppose we want to find the solution of the differential equation $y' = f(x, y)$, satisfying the initial condition $y(x_0) = y_0$. We saw previously [II, 51] that the problem amounts to solving the integral equation:

$$y(x) = \int_{x_0}^{x} f(x, y)\, \mathrm{d}x + y_0 .$$

In the same way, the problem of integrating the second order differential equation $y'' = f(x, y)$ with the initial conditions $y(x_0) = y_0$; $y'(x_0) = y_0'$ leads to the integral equation:

$$y(x) = \int_{x_0}^{x} \mathrm{d}x \int_{x_0}^{x} f[z, y(z)]\, \mathrm{d}z + y_0 + y_0'(x - x_0) .$$

We can rewrite this as follows by transforming the double integral into a single integral [II, 15]:

$$y(x) = \int_{x_0}^{x} (x - z)\, f[z, y(z)]\, \mathrm{d}z + y_0 + y_0'(x - x_0) .$$

The general solution of $y'' = f(x, y)$ is obtained from the integral equation

$$y(x) = \int_{0}^{x} (x - z)\, f[z, y(z)]\, \mathrm{d}z + c_1 + c_2 x , \tag{1}$$

where c_1 and c_2 are arbitrary constants and the lower limit of integration has been taken equal to zero. We now consider a boundary value problem for our second order differential equation; we seek the solution satisfying the boundary conditions $y(0) = a$; $y(l) = b$. On substituting first $x = 0$ then $x = l$ in equation (1), we obtain two equations for the arbitrary constants; these give us

$$c_1 = a; \quad c_2 = \frac{b - a}{l} - \frac{1}{l} \int_{0}^{l} (l - z)\, f[z, y(z)]\, \mathrm{d}z .$$

By substituting the values obtained in (1), we reduce our boundary value problem to the integral equation:

$$y(x) = F(x) + \int_0^x (x-z)\, f[z, y(z)]\, \mathrm{d}z - \frac{x}{l}\int_0^l (l-z)\, f[z, y(z)]\, \mathrm{d}z, \tag{2}$$

where

$$F(x) = a + \frac{b-a}{l}\, x.$$

We can rewrite (2) as follows:

$$y(x) = F(x) - \int_0^x \frac{z(l-x)}{l} f[z, y(z)]\, \mathrm{d}z - \int_x^l \frac{x(l-z)}{l} f[z, y(z)]\, \mathrm{d}z. \tag{3}$$

We introduce the function of two variables:

$$K(x, z) = \begin{cases} \dfrac{z(l-x)}{l} & \text{for } z \leqslant x \\[2ex] \dfrac{x(l-z)}{l} & \text{for } x \leqslant z. \end{cases} \tag{4}$$

Equation (3) can be written as follows with the aid of this function:

$$y(x) = F(x) - \int_0^l K(x, z)\, f[z, y(z)]\, \mathrm{d}z. \tag{5}$$

We apply the results obtained to the linear equation

$$y'' + p(x)\, y = \omega(x). \tag{6}$$

We can say that the problem of finding the solution of this equation with the boundary conditions

$$y(0) = a; \quad y(l) = b \tag{7}$$

is equivalent to finding the function $y(x)$ satisfying the linear integral equation

$$y(x) = F_1(x) + \int_0^l K(x, z)\, p(z)\, y(z)\, \mathrm{d}z, \tag{8}$$

where

$$F_1(x) = F(x) - \int_0^l K(x, z)\, \omega(z)\, \mathrm{d}z$$

is a known function of the independent variable x.

Notice that the upper limit of integration is variable in equation (1), whereas in (8) both limits are constant. Notice also that, both in (1) and in (8), the required function appears outside as well

as under the integral sign. As we saw previously [II, 50], this is of great importance in enabling us to apply the method of successive approximations to the solution of the equation.

We multiply the coefficient $p(x)$ in (6) by a parameter λ and consider the homogeneous equation

$$y'' + \lambda p(x)\, y = 0 \tag{9}$$

with the homogeneous boundary conditions

$$y(0) = 0; \quad y(l) = 0\,. \tag{10}$$

This homogeneous boundary value problem leads us to a homogeneous integral equation containing the parameter λ:

$$y(x) = \lambda \int_0^l K(x, z)\, p(z)\, y(z)\, \mathrm{d}z\,. \tag{11}$$

One of the main problems that will confront us later is this: for what values of the parameter λ has our problem solutions that do not vanish identically? We have already met this question when applying Fourier's method to boundary value problems of mathematical physics. We notice further some characteristic properties of the function $K(x, z)$, which is known as the *kernel* of the integral equation. The kernel is continuous in the square k_0, defined by the inequalities $0 \leqslant x \leqslant l$ and $0 \leqslant z \leqslant l$. On the diagonal of the square where $x = z$ the first derivative of the kernel possesses a discontinuity:

$$K_x(x, z)\,|_{x=z+0} - K_x(x, z)\,|_{x=z-0} = -1\,.$$

Furthermore, outside the diagonal the kernel, considered as a function of x, is the solution of the homogeneous equation $y'' = 0$ satisfying the boundary conditions (10). We observe finally the symmetrical nature of the kernel, expressed by the equation:

$$K(z, x) = K(x, z)\,. \tag{12}$$

All these properties of the kernel follow immediately from (4).

The kernel $K(x, z)$ has a simple physical meaning. We recall that, when a concentrated force acts at a point $x = z$ of a string fixed at both ends, the condition must hold at the point of application [II, 163]:

$$T_0[(u_x)_{x=z+0} - (u_x)_{x=z-0}] = -P\,,$$

where P is the magnitude of the force. The function

$$u(x) = \frac{P}{T_0} K(x, z)$$

may easily be seen to give the statical shape of the bent string due to this concentrated force. It may be mentioned here that the equation of vibration of the string in the statical case amounts simply to the equation $u_{xx} = 0$. All these ideas concerning the reduction of a boundary value problem to an integral equation, discussed here for an elementary case, will be analysed in detail in Chapter IV.

We shall further mention a characteristic method of reducing the boundary value problems of mathematical physics to integral equations. We defined earlier the potential of a spherical layer by the expression:

$$u(M) = \iint\limits_S \frac{\varrho(M')}{d}\, ds\,,$$

where $\varrho(M')$ is a function given on the surface of the sphere S, ds is an elementary area of the sphere, and d is the distance from a variable point M of space to a variable point M' of the sphere. Let n be the normal direction at a point M_0 of the sphere. Let $(\partial u(M_0)/\partial n)_i$ and $(\partial u(M_0)/\partial n)_e$ denote respectively the limits of the derivative $\partial u(M)/\partial n$ as the variable point M of space approaches the point M_0 from inside and outside the sphere. We previously [III_2, 138] deduced the following expressions:

$$\begin{aligned} \left(\frac{\partial u(M_0)}{\partial n}\right)_i &= -\iint\limits_S \varrho(M')\,\frac{\cos\omega}{d^2}\, ds + 2\pi\varrho(M_0)\,, \\ \left(\frac{\partial u(M_0)}{\partial n}\right)_e &= -\iint\limits_S \varrho(M')\,\frac{\cos\omega}{d^2}\, ds - 2\pi\varrho(M_0)\,, \end{aligned} \tag{13}$$

where d is the distance from the point M_0 to the variable point M' of the sphere and ω is the angle formed by the radius vector $M'M_0$ with the direction n.

We shall see in the next chapter that these expressions are not only valid for a sphere. We now pose the interior Neumann problem for the sphere, i.e. we take it that the function is required which is harmonic inside the sphere and whose normal derivative has assigned boundary values on the surface of the sphere:

$$\left.\frac{\partial u}{\partial n}\right|_S = f(M_0)\,. \tag{14}$$

We shall seek u as the potential of a spherical layer. This potential is a harmonic function inside the sphere and we only have to choose the density $\varrho(M')$ of the potential such that the boundary condition

(14) is satisfied. On taking into account the first of expressions (13) and condition (14), we obtain the following integral equation for the required density:

$$2\pi\varrho(M_0) = f(M_0) + \int\limits_S\int \varrho(M')\frac{\cos\omega}{d^2}\,\mathrm{d}s\,.$$

Notice that here the functions $f(M)$ and $\varrho(M)$ must be defined on the surface of the sphere, and that the integration is over the spherical surface and not over an interval of the x axis, as above.

2. The classification of integral equations. For the present, we shall only consider linear integral equations in which the required function has to be determined on the x axis. We write the integral equation:

$$y(x) = \int_a^x K(x, z)\,y(z)\,\mathrm{d}z + f(x)\,, \tag{15}$$

where $y(x)$ is the required function and $f(x)$, $K(x, z)$ are given functions. As already mentioned, the function $K(x, z)$ is known as the *kernel of the integral equation.*

The equation written is called a *Volterra equation of the second kind.* The analogous equation with constant limits:

$$y(x) = \int_a^b K(x, z)\,y(z)\,\mathrm{d}z + f(x) \tag{16}$$

is called a *Fredholm equation of the second kind.* If the required function only appears under the integral sign, we obtain a Volterra or Fredholm equation of the *first kind.* These have the form:

$$\int_a^x K(x, z)\,y(z)\,\mathrm{d}z = f_1(x); \qquad \int_a^b K(x, z)\,y(z)\,\mathrm{d}z = f_1(x). \tag{17}$$

The Abel equation that we discussed earlier in [II, 79] offers an example of a Volterra equation of the first kind:

$$\varphi(h) = \frac{1}{\sqrt{2g}}\int_0^h \frac{u(y)\,\mathrm{d}y}{\sqrt{h-y}}\,.$$

An example may be quoted of a Fredholm equation of the first kind. Let $u(x)$ be the statical bending of a string in the presence of a continuously distributed load $p(z)$ per unit length. We shall consider the continuously distributed load as the sum of concentrated

loads $p(z)\mathrm{d}z$. From the remarks of the previous section, each such concentrated load leads to statical bending of the form

$$\frac{1}{T_0} K(x, z)\, p(z)\, \mathrm{d}z\,,$$

where $K(x, z)$ is given by (4). Integration gives us the statical bending for the continuously distributed load:

$$u(x) = \frac{1}{T_0} \int_0^l K(x, z)\, p(z)\, \mathrm{d}z\,.$$

This is a Fredholm integral equation of the first kind if the bending $u(x)$ is given and the corresponding load $p(z)$ is required.

It may be observed that the Volterra equation is a particular case of the Fredholm equation. For we can perform the integration with respect to z from $z = a$ to $z = b$ in the Volterra equation, provided we first additionally define the kernel by the condition $K(x, z) = 0$ for $z > x$.

We shall be almost exclusively concerned below with equations of the second kind, and in the main, with Fredholm equations of the second kind, these being most commonly encountered in boundary value problems of mathematical physics. The theory of integral equations is much simpler for those of the second kind than for those of the first kind. As we have already remarked, the presence of the required function outside the integral sign leads naturally to the possibility of using the method of successive approximations.

The theory of integral equations has many analogies with linear algebra, which we dealt with in Vol. III. We recall that a linear transformation in n-dimensional space is of the form [III_1, 25]:

$$y_i = a_{i1} u_1 + \ldots + a_{in} u_n \qquad (i = 1, \ldots, n)\,,$$

and is characterized by the matrix formed from the transformation coefficients a_{ik}. This transformation has been written in the alternative form

$$\mathbf{y} = A\mathbf{u}\,,$$

where $\mathbf{u}(u_1, \ldots, u_n)$ is the original vector, $\mathbf{y}(y_1, \ldots, y_n)$ is the transformed vector, and A is the matrix with coefficients a_{ik}. In the case of integral equations we have functions usually defined in some interval $[a, b]$ instead of vectors in n-dimensional space. We have the kernel $K(x, z)$ instead of the matrix with coefficients a_{ik} and a

process of integration instead of summation, so that the linear transformation is now expressed by

$$y(x) = \int_a^b K(x, z)\, u(z)\, \mathrm{d}z\,, \tag{18}$$

where $u(z)$ is the original function and $y(x)$ the transformed function.

We also recall that the eigenvalues of a matrix A were defined as those values of the parameter λ for which the equation

$$Ax = \lambda x$$

has non-zero solutions x. The values of the parameter λ for which the homogeneous integral equation

$$y(x) = \lambda \int_a^b K(x, z)\, y(z)\, \mathrm{d}z \tag{19}$$

has solutions not identically zero will be referred to in future as the eigenvalues of the kernel $K(x, z)$ or of the corresponding transformation. It must be noted that, as regards the parameter λ, there is not a complete analogy with the algebraic case. For a complete analogy, we should have to write instead of (19):

$$\int_a^b K(x, z)\, y(z)\, \mathrm{d}z = \lambda y(x)\,.$$

We shall nevertheless adhere to (19) in the theory of integral equations.

Notice that the identity transformation, where $u(x)$ corresponds to $u(x)$, i.e. where $y(x)$ is the same as $u(x)$, is not expressible in the integral form (18).

We naturally have to make some assumptions regarding the kernel $K(x, z)$, as also regarding the functions $f(x)$ and $y(x)$, when discussing the theory of integral equations.

As already mentioned, we shall be concerned for the present with integral equations in the one-dimensional case. Methods of passing to the multi-dimensional case will be indicated below.

Finally, it may be mentioned that the given and required functions will occasionally be assumed complex:

$$K(x, z) = K_1(x, z) + K_2(x, z)\, i;$$
$$f(x) = f_1(x) + f_2(x)\, i;$$
$$y(x) = y_1(x) + y_2(x)\, i\,,$$

where $K_s(x, z)$, $f_s(x)$, $y_s(x)$ $(s = 1, 2)$ are real functions. The independent variable will always be assumed real.

We recall the properties of systems of orthogonal functions in the next section and add certain remarks, necessary for the discussion of integral equations.

We shall often be concerned below with closed finite intervals of the type $a \leqslant x \leqslant b$, i.e. intervals in which the ends are included; these are always denoted by the symbol $[a, b]$.

3. Orthogonal systems of functions. The real functions

$$\varphi_1(x), \varphi_2(x), \ldots, \tag{20}$$

which we shall assume continuous in the interval $[a, b]$, are said to form an *orthogonal and normalized (orthonormal)* system in the interval if

$$\int_a^b \varphi_p(x)\, \varphi_q(x)\, dx = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q. \end{cases} \tag{21}$$

Let $f(x)$ be any real function, continuous in the interval $[a, b]$. The numbers

$$c_k = \int_a^b f(x)\, \varphi_k(x)\, dx \tag{22}$$

are known as the *Fourier coefficients* of the function $f(x)$ with respect to system (20) [cf. II, 156]. We have by definition of c_k:

$$\int_a^b \left[f(x) - \sum_{k=1}^n c_k\, \varphi_k(x)\right]^2 dx = \int_a^b [f(x)]^2\, dx - \sum_{k=1}^n c_k^2, \tag{23}$$

which expresses as a difference the mean square error obtained on replacing the function $f(x)$ by the segment $s_n(x)$ of its Fourier series. The convergence of the infinite series with terms c_k^2 follows from (23), together with the *Bessel inequality:*

$$\sum_{k=1}^\infty c_k^2 \leqslant \int_a^b [f(x)]^2\, dx. \tag{24}$$

System (20) is said to be *closed* if the sign of equality holds in (24) for any continuous function $f(x)$, i.e. if the *closure equation* holds for any continuous function:

$$\int_a^b [f(x)]^2\, dx = \sum_{k=1}^\infty c_k^2. \tag{25}$$

This equation expresses the fact that, when $f(x)$ is replaced by the segment $s_n(x)$ of its Fourier series, the mean square error tends to

zero as n tends to infinity. We also recall that the value of the integral

$$\int_a^b [f(x) - \sum_{k=1}^n a_k \varphi_k(x)]^2 \, dx,$$

where the a_k are arbitrary real numbers, is a minimum if the a_k are taken equal to the Fourier coefficients c_k of $f(x)$ [II, 148].

We have so far assumed continuity of the functions $\varphi_k(x)$ and $f(x)$. All that has been said above remains valid in more general cases. We can suppose, for instance, that the functions are bounded and possess a finite number of discontinuities. It is clear, of course, that all the integrals written above have a meaning in this case.

Suppose that the functions $\varphi_k(x)$ are continuous, whilst $f(x)$ is continuous in $[a,b]$ except for a point $x = d$, in the neighbourhood of which $f(x)$ is unbounded, whilst

$$|f(x)| \leqslant \frac{C}{|x-d|^{\alpha}}, \tag{26}$$

where C and α are constants and $0 < \alpha < 1/2$. In this case $[f(x)]^2$ is integrable [II, 82], and the proof of inequality (24) is fully preserved, all the integrals having a meaning. The most natural method of extending the theory of orthogonal functions requires a different concept of integral. This is discussed in Vol. V.

We shall always assume functions to be continuous unless the contrary is stated.

We prove an elementary lemma. *If the function $\omega(x)$ is continuous and non-negative in the interval $[a, b]$ and*

$$\int_a^b \omega(x) \, dx = 0, \tag{27}$$

$\omega(x)$ is identically zero in $[a, b]$. Suppose the statement false and that $\omega(c) > 0$ at some point $x = c$ of the interval. Given a sufficiently small ε, $\omega(x)$ will be positive in the interval $[c - \varepsilon, c + \varepsilon]$; let $m > 0$ be its least value in this interval. Since $\omega(x)$ is non-negative, we have:

$$\int_a^b \omega(x) \, dx \geqslant \int_{c-\varepsilon}^{c+\varepsilon} \omega(x) \, dx \geqslant \int_{c-\varepsilon}^{c+\varepsilon} m \, dx = 2\varepsilon m,$$

which contradicts condition (27).

We saw in [III$_1$, 31] that, given m linearly independent vectors, the same number of mutually orthogonal normalized vectors can always be found such that the original vectors are linearly expressible

in terms of the new and vice versa. This entire process can be translated word for word to the case of functions. Let the functions,

$$\psi_1(x), \ldots, \psi_m(x)$$

be real continuous in $[a, b]$, and linearly independent, i.e. the identity

$$a_1 \psi_1(x) + \ldots + a_m \psi_m(x) \equiv 0$$

with constant coefficients a_k can only hold when all the coefficients are zero. We form new functions, orthogonal and normalized in $[a, b]$:

$$\varphi_1(x), \ldots, \varphi_m(x),$$

such that $\varphi_k(x)$ is expressible linearly in terms of $\psi_1(x), \ldots, \psi_k(x)$, and conversely each $\psi_k(x)$ is expressible linearly in terms of $\varphi_1(x), \ldots, \varphi_k(x)$. We introduce for brevity a notation similar to our previous algebraic notation, and write (f, F) for the integral of the product $f(x)\, F(x)$ over the interval $[a, b]$:

$$(f, F) = \int_a^b f(x)\, F(x)\, \mathrm{d}x.$$

The orthogonalization of functions $\psi_k(x)$, i.e. the construction of the $\varphi_k(x)$, proceeds as follows:

$$\varphi_1(x) = \frac{\psi_1(x)}{\sqrt{(\psi_1, \psi_1)}}$$

$$\chi_2(x) = \psi_2(x) - (\psi_2, \varphi_1)\, \varphi_1(x); \qquad \varphi_2(x) = \frac{\chi_2(x)}{\sqrt{(\chi_2, \chi_2)}}$$

$$\chi_3(x) = \psi_3(x) - (\psi_3, \varphi_2)\, \varphi_2(x) - (\psi_3, \varphi_1)\, \varphi_1(x); \qquad \varphi_3(x) = \frac{\chi_3(x)}{\sqrt{(\chi_3, \chi_3)}}$$

. . . .

$$\chi_m(x) = \psi_m(x) - (\psi_m, \varphi_{m-1})\, \varphi_{m-1}(x) - \ldots - (\psi_m, \varphi_1)\, \varphi_1(x);$$

$$\varphi_m(x) = \frac{\chi_m(x)}{\sqrt{(\chi_m, \chi_m)}}.$$

The functions $\varphi_k(x)$ differ from the functions $\chi_k(x)$ only by numerical factors, which appear in the $\chi_k(x)$for the purpose of normalizing them, i.e. so that the integral of the square of $\chi_k(x)$ over $[a, b]$ is unity. The above-mentioned linear relationship between $\psi_k(x)$ and $\varphi_k(x)$ follows at once from the equations written. We observe further that none of the $\chi_k(x)$ can vanish identically, so that $(\chi_k, \chi_k) \neq 0$, since if we had say $\chi_2(x) \equiv 0$, this would lead us to a linear relationship between

$\varphi_1(x)$ and $\psi_2(x)$:

$$\psi_2(x) - (\psi_2, \varphi_1)\, \varphi_1(x) \equiv 0\,,$$

which amounts to a linear relationship between $\psi_1(x)$ and $\psi_2(x)$, which in turn contradicts our assumption of the linear independence of the $\psi_k(x)$. It follows at once from the lemma that $(\chi_k, \chi_k) \neq 0$, since otherwise we should have to have $\chi_k = 0$. Hence all the expressions defining the functions $\varphi_k(x)$ have a meaning, and we can verify successively the orthogonality of the functions $\chi_k(x)$ to the system $\varphi_1(x), \ldots, \varphi_{k-1}(x)$ already constructed. For instance:

$$(\chi_2, \varphi_1) = (\psi_2, \varphi_1) - (\psi_2, \varphi_1)\,(\varphi_1, \varphi_1) = (\psi_2, \varphi_1) - (\psi_2, \varphi_1) = 0\,.$$

We have, since $\varphi_1(x)$ and $\varphi_2(x)$ are orthogonal and normalized:

$$(\chi_3, \varphi_1) = (\psi_3, \varphi_1) - (\psi_3, \varphi_2)\,(\varphi_2, \varphi_1) - (\psi_3, \varphi_1)\,(\varphi_1, \varphi_1) =$$
$$= (\psi_3, \varphi_1) - (\psi_3, \varphi_1) = 0\,,$$

and similarly $(\chi_3, \varphi_2) = 0$ and so on.

Some further properties of orthonormal systems may be mentioned.

Let system (20) be closed and let all the Fourier coefficients of a given continuous function $f(x)$ be zero, or in other words, the continuous function $f(x)$ is orthogonal to all the functions $\varphi_k(x)$:

$$\int_a^b f(x)\, \varphi_k(x)\, \mathrm{d}x = 0 \qquad (k = 1, 2, \ldots)\,.$$

The closure equation gives us:

$$\int_a^b [f(x)]^2\, \mathrm{d}x = 0\,,$$

and hence, by the lemma, $f(x)$ vanishes identically.

We return to the general case and let

$$\sum_{k=1}^{\infty} c_k\, \varphi_k(x) \tag{28}$$

be the Fourier series of a function $f(x)$. We cannot assert that series (28) converges, whilst if it does converge, we cannot say that its sum is equal to $f(x)$. Suppose it happens that series (28) converges uniformly in the interval $[a, b]$. We form the difference

$$f_1(x) = f(x) - \sum_{k=1}^{\infty} c_k\, \varphi_k(x)\,,$$

which is a continuous function, inasmuch as we have assumed $f(x)$ continuous. We multiply both sides of the equation by $\varphi_p(x)$ and integrate, term-by-term integration being possible in view of the uniform convergence of the series; we obtain, since functions (20) are orthogonal and normalized:

$$\int_a^b f_1(x)\,\varphi_p(x)\,\mathrm{d}x = \int_a^b f(x)\,\varphi_p(x)\,\mathrm{d}x - c_p\,.$$

The difference on the right is zero since the c_k are the Fourier coefficients of $f(x)$, and consequently, if the Fourier series (28) of $f(x)$ converges uniformly, the difference $f_1(x)$ between the function and its Fourier series has Fourier coefficients that all vanish. If, moreover, system (20) is closed, the following proposition can be stated, from what has been said above: *if system* (20) *is closed and the Fourier series of the continuous function* $f(x)$ *converges uniformly in the interval* $[a, b]$, *its sum is equal to* $f(x)$.

We also note the elementary fact that orthogonal functions are always linearly independent. For suppose we had the relationship:

$$a_1\,\varphi_1(x) + \ldots + a_m\,\varphi_m(x) \equiv 0\,.$$

Multiplication of both sides by $\varphi_k(x)$ $(k = 1, 2, \ldots, m)$ followed by integration gives us $a_k = 0$ by the orthogonality and normality of functions (20), i.e. all the coefficients a_k must in fact vanish.

All the above arguments can be generalized directly for the case of complex functions of the real variable x, i.e. functions of the form:

$$\varphi_k(x) = \varrho_k(x) + \sigma_k(x)\,i \qquad (k = 1, 2, \ldots)\,.$$

This matter has already been discussed [III_1, 49]. The fact that the functions in this case are orthogonal and normalized is expressed by the equations

$$\int_a^b \varphi_p(x)\,\overline{\varphi_q(x)}\,\mathrm{d}x = \begin{cases} 0 \text{ for } p \neq q \\ 1 \text{ for } p = q\,, \end{cases} \tag{21_1}$$

where $\bar{a}$ denotes as usual the complex conjugate of a. The Fourier coefficients of any complex continuous function are given by

$$c_k = \int_a^b f(x)\,\overline{\varphi_k(x)}\,\mathrm{d}x\,. \tag{22_1}$$

In the complex case, we always have to write the square of the modulus instead of the square of the magnitude itself. We have

instead of (23), for instance:

$$\int_a^b | f(x) - \sum_{k=1}^n c_k \varphi_k(x) |^2 \, dx = \int_a^b | f(x) |^2 \, dx - \sum_{k=1}^n | c_k |^2 , \qquad (23_1)$$

whilst Bessel's inequality becomes:

$$\sum_{k=1}^\infty | c_k |^2 \leqslant \int_a^b | f(x) |^2 \, dx . \qquad (24_1)$$

The process of orthogonalization proceeds as above except that the symbol (f, F) is defined by

$$(f, F) = \int_a^b f(x) \overline{F(x)} \, dx .$$

The definition of closure remains as above, as do all the proofs of our propositions.

The proof of (23_1) is the same as for (23). If we remove the brackets in the integral

$$\int_a^b | f(x) - \sum_{k=1}^n c_k \varphi_k(x) |^2 \, dx = \int_a^b \Big[f(x) - \sum_{k=1}^n c_k \varphi_k(x) \Big] \Big[\overline{f(x)} - \sum_{k=1}^n \overline{c_k \varphi_k(x)} \Big] \, dx$$

and use (21_1) and (22_1), we get (23_1).

Notice that, if $\omega(x)$ is a continuous complex function, not identically zero:

$$(\omega, \omega) = \int_a^b \omega(x) \overline{\omega(x)} \, dx = \int_a^b | \omega(x) |^2 \, dx > 0 .$$

The ordinary properties of integrals (taking a constant factor outside, integration of sums, etc.) obviously apply to the integrals of complex functions.

We recall that the limit $u_n(x) + v_n(x) i \to u(x) + v(x) i$, that is, $| [u(x) + v(x) i] - [u_n(x) + v_n(x) i] | \to 0$, is equivalent to the separate limits $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ [III_2, 1]. A similar remark naturally applies to uniform convergence. Furthermore, passing to the limit is possible under the integral sign for a uniformly convergent sequence [I, 145]. The other theorems of the integral calculus remain in force, as for instance, the theorem concerning parametrically dependent integrals, and that concerning integration under the integral sign. We always obtain the corresponding theorem for real functions by separating into real and imaginary parts.

We also observe that Buniakowski's inequality [III_1, 29] is equally applicable to complex functions.

For we have [III_2, 4]:

$$\left| \int_a^b f_1(x)\, f_2(x)\, \mathrm{d}x \right| \leqslant \int_a^b |f_1(x)|\, |f_2(x)|\, \mathrm{d}x\,.$$

whence we obtain, on applying Buniakowski's inequality to the real and imaginary parts:

$$\left| \int_a^b f_1(x)\, f_2(x)\, \mathrm{d}x \right|^2 \leqslant \left(\int_a^b |f_1(x)|\, |f_2(x)|\, \mathrm{d}x\right)^2 \leqslant$$

$$\leqslant \int_a^b |f_1(x)|^2\, \mathrm{d}x \int_a^b |f_2(x)|^2\, \mathrm{d}x\,.$$

In all the above arguments we have been concerned with functions of a single independent variable within the interval $[a, b]$. We can repeat all our arguments word for word for functions defined in some finite domain of a plane, in three-dimensional or n-dimensional space, or on a surface. The integrals naturally have to be over the corresponding domains.

Let P be a variable point of a finite closed domain B on a plane, in space or on a surface, i.e. a domain which includes all its boundary points. The (generally complex) functions $\varphi_k(P)$ form an orthonormal system if

$$\int_B \varphi_p(P)\, \overline{\varphi_q(P)}\, \mathrm{d}\omega_P = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q\,, \end{cases}$$

where only one integral sign has been written, though the integral must be reckoned double, triple, or over a surface. We have used $\mathrm{d}\omega_P$ to denote an element of the corresponding integral, taken with respect to the variable point P. In the case of a double integral in Cartesian coordinates, we have for instance $\mathrm{d}\omega_P = \mathrm{d}x\, \mathrm{d}y$. The Fourier coefficients of a function $f(P)$ are:

$$c_k = \int_B f(P)\, \overline{\varphi_k(P)}\, \mathrm{d}\omega_P\,,$$

and Bessel's inequality takes the form:

$$\sum_{k=1}^{\infty} |c_k|^2 \leqslant \int_B |f(P)|^2\, \mathrm{d}\omega_P\,.$$

If $f(P)$ has a discontinuity at a point Q, we have to assume, instead of condition (26),

$$|f(P)| \leqslant \frac{C}{r^a},$$

where r is the distance $\overline{PQ}$ and $a < n/2$, where $n = 2$ for a double integral or an integral over a surface and $n = 3$ for a triple integral.

4. Fredholm equations of the second kind. Our discussion of the theory of integral equations and of Fredholm equations of the second kind starts with the one-dimensional case:

$$\varphi(s) = f(s) + \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t\,. \tag{29}$$

Our basic assumptions are as follows. The kernel $K(s, t)$ is taken to be a continuous complex function of two variables (s, t) in a square k_0 defined by the inequalities $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, whilst the given function $f(s)$ is a continuous complex function in the interval $[a, b]$. We seek a solution likewise among continuous functions. The kernel $K(s, t)$ is naturally assumed not identically zero in the square k_0; if this were the case, equation (29) would reduce to $\varphi(s) = f(s)$.

Given the continuity of the kernel, the integral

$$v(s) = \int_a^b K(s, t)\, u(t)\, \mathrm{d}t \tag{30}$$

yields a continuous function $v(s)$ for any choice of continuous function $u(t)$, i.e. the above equation transforms a continuous function $u(t)$ to a likewise continuous function $v(s)$. But furthermore, if $u(t)$ is assumed bounded ($|u(t)| \leqslant C$) with a finite number of discontinuities, integral (30) still has a meaning, and we can write:

$$v(s + h) - v(s) = \int_a^b [K(s + h, t) - K(s, t)]\, u(t)\, \mathrm{d}t\,, \tag{31}$$

whence

$$|v(s + h) - v(s)| \leqslant C \int_a^b |K(s + h, t) - K(s, t)|\, \mathrm{d}t\,.$$

By virtue of the continuity of the kernel the right-hand side tends to zero as $h \to 0$, so that $|v(s + h) - v(s)| \to 0$ also, i.e. $v(s)$ is a continuous function. Integral (30) therefore transforms bounded

functions with a finite number of discontinuities, as well as continuous functions, to continuous functions.

We obtain by applying Buniakowskii's inequality [3] to (31):

$$| v(s+h) - v(s) |^2 \leqslant \int_a^b | K(s+h,t) - K(s,t) |^2 \, \mathrm{d}t \int_a^b | u(t) |^2 \, \mathrm{d}t \,, \tag{32}$$

whence it is clear that, provided that the integral

$$\int_a^b | u(t) |^2 \, \mathrm{d}t$$

has a meaning, $v(s)$ will still be a continuous function, even though $u(t)$ becomes unbounded in the neighbourhood of a point (e.g. $u(t)$ satisfies condition (26)).

Let us return to equation (29) and recall our assumption of the continuity of the kernel $K(s, t)$ and of the function $f(s)$. We can assert, by what has been said above, and by taking, for example, $\varphi(s)$ as bounded with a finite number of discontinuities, that both terms on the right-hand side will be continuous; in other words, $\varphi(s)$ in this case must in fact be continuous. It is therefore natural that we should look only for continuous solutions of equation (29).

If, for instance, we were to assume that $K(s, t)$ is continuous, but that $f(s)$ is bounded with a finite number of discontinuities, then the solutions $\varphi(s)$ would naturally also have to be sought among bounded functions with a finite number of discontinuities. We shall assume that $f(s)$ is continuous. The case frequently occurring in mathematical physics, when the kernel is discontinuous, will be discussed later, and meantime we shall assume that $K(s, t)$ and $f(s)$ are continuous as indicated above, the solutions $\varphi(s)$ being sought in the class of continuous functions.

We shall consider instead of equation (29) the equation with a numerical parameter:

$$\varphi(s) = f(s) + \lambda \int_a^b K(s,t)\, \varphi(t) \, \mathrm{d}t \,. \tag{33}$$

Equation (29) is obtained from (33) with $\lambda = 1$. We shall suppose that λ can take complex as well as real values. We put $\lambda = \lambda_1 + \lambda_2 i$. We have to seek a complex function $\varphi(s)$ as a solution: $\varphi(s) = \varphi_1(s) + \varphi_2(s)\, i$. If, for example, the kernel $K(s, t)$ and $f(s)$ are real,

on substituting in (33) and separating real and imaginary parts, we obtain the following system of equations for $\varphi_1(s)$ and $\varphi_2(s)$:

$$\varphi_1(s) = f(s) + \lambda_1 \int_a^b K(s,t)\,\varphi_1(t)\,dt - \lambda_2 \int_a^b K(s,t)\,\varphi_2(t)\,dt\,,$$

$$\varphi_2(s) = \lambda_1 \int_a^b K(s,t)\,\varphi_2(t)\,dt + \lambda_2 \int_a^b K(s,t)\,\varphi_1(t)\,dt\,.$$

We shall use equation (33) directly instead of this system in the further development of the theory. We write the corresponding homogeneous equation:

$$\varphi(s) = \lambda \int_a^b K(s,t)\,\varphi(t)\,dt\,. \tag{34}$$

It has the obvious solution $\varphi(s) \equiv 0$, which we call the zero solution. As we mentioned in [2], *the values* $\lambda = \lambda_0$ *for which equation* (34) *has a non-zero solution are called the eigenvalues of the kernel* $K(s, t)$ *or of the corresponding integral equation, whilst every non-zero solution of the equation*

$$\varphi(s) = \lambda_0 \int_a^b K(s,t)\,\varphi(t)\,dt\,, \tag{35}$$

is called an eigenfunction corresponding to the eigenvalue $\lambda = \lambda_0$. The number $\lambda_0 = 0$ obviously cannot be an eigenvalue, since it follows in this case from (35) that $\varphi(s) = 0$.

By virtue of the linearity and homogeneity of equation (35), if $\varphi_1(s), \varphi_2(s), \ldots, \varphi_m(s)$ are eigenfunctions corresponding to the same eigenvalue $\lambda = \lambda_0$, any linear combination of them with constant complex coefficients

$$\varphi(s) = c_1\,\varphi_1(s) + c_2\,\varphi_2(s) + \ldots + c_m\,\varphi_m(s) \tag{36}$$

also satisfies equation (35), and hence is also an eigenfunction, provided that formula (36) is not identically zero. If $\varphi_1(s), \varphi_2(s), \ldots, \varphi_m(s)$ are linearly independent, this can only be the case if all the coefficients c_p vanish. As we shall show later, for any eigenvalue $\lambda = \lambda_0$ there exists a finite number of linearly independent eigenfunctions $\varphi_1(s), \varphi_2(s), \ldots, \varphi_k(s)$ such that formula (36) gives all the solutions of equation (35), provided that all possible values are assigned to the coefficients c_p.

These complete sets of eigenfunctions corresponding to an eigenvalue $\lambda = \lambda_0$ can be formed in a different way. Suppose that we have

constructed two such sets, the first consisting of k functions and the second of l functions:

$$\varphi_1^{(1)}(s), \varphi_2^{(1)}(s), \ldots, \varphi_k^{(1)}(s); \; \varphi_1^{(2)}(s), \varphi_2^{(2)}(s), \ldots, \varphi_l^{(2)}(s) .$$

On observing that every function $\varphi_p^{(1)}(s)$ $(p = 1, 2, \ldots, k)$ is a solution of equation (34) and hence must be linearly expressible in terms of functions of the second set, whilst every function $\varphi_q^{(2)}(s)$ $(q = 1, 2, \ldots, 1)$ must similarly be linearly expressible in terms of functions of the first set, we can easily conclude [III_1, 10] that $k = l$, i.e. the total set of eigenfunctions always consists of the same number of functions. This number k is called the *rank of the eigenvalue* λ_0. Obviously, different eigenvalues can have different ranks.

Suppose that the kernel $K(s, t)$ and the eigenvalue λ_0 are real, and let $\varphi(s) = \omega_1(s) + \omega_2(s)\, i$ be a corresponding eigenfunction; we obtain by substituting in (35) and separating real and imaginary parts:

$$\omega_1(s) = \lambda_0 \int_a^b K(s, t)\, \omega_1(t)\, \mathrm{d}t; \quad \omega_2(s) = \lambda_0 \int_a^b K(s, t)\, \omega_2(t)\, \mathrm{d}t ,$$

i.e. $\omega_1(s)$ and $\omega_2(t)$ separately satisfy equation (35), whilst $\varphi(s) = \omega_1(s) + \omega_2(s)\, i$ is a linear combination of them. Thus, *with a real kernel and real eigenvalues, the eigenfunctions can be assumed real.*

5. Method of successive approximations and the resolvent. We shall apply the method of successive approximations to the solution of equation (33):

$$\varphi(s) = f(s) + \lambda \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t. \tag{33}$$

We shall do this by seeking the solution as a series expanded in positive integral powers of λ:

$$\varphi(s) = \varphi_0(s) + \varphi_1(s)\, \lambda + \varphi_2(s)\, \lambda^2 + \ldots \tag{37}$$

If this series is uniformly convergent with respect to s in the interval $[a, b]$, on substituting it for $\varphi(s)$ in (33), we can integrate term by term in the resulting equation; we then equate coefficients of like powers of λ on both sides of the equation and obtain formulae for successively determining the $\varphi_n(s)$:

$$\begin{gathered} \varphi_0(s) = f(s); \qquad \varphi_1(s) = \int_a^b K(s, t)\, \varphi_0(t)\, \mathrm{d}t; \\ \varphi_2(s) = \int_a^b K(s, t)\, \varphi_1(t)\, \mathrm{d}t , \end{gathered} \tag{38}$$

and in general

$$\varphi_n(s) = \int_a^b K(s, t)\, \varphi_{n-1}(t)\, \mathrm{d}t \qquad (n = 1, 2, \ldots), \tag{39}$$

all the functions defined by these formulae being continuous [4]. We now show that series (37) is absolutely and uniformly convergent with respect to s if the modulus of λ is sufficiently small. It will follow from this that the sum of the series for these values of λ is a continuous function and represents a solution of equation (33).

In the interval $[a, b]$ and in the square k_0, where functions $f(s)$ and $K(s, t)$ are continuous, we have

$$|f(s)| \leqslant m; \quad |K(s, t)| \leqslant M,$$

where m and M are positive numbers, viz. the greatest values of $|f(s)|$ and $|K(s, t)|$. We obtain successively the following upper bounds for the functions $\varphi_n(s)$:

$$|\varphi_0(s)| \leqslant m; \quad |\varphi_1(s)| \leqslant \int_a^b |K(s, t)|\, |\varphi_0(t)|\, \mathrm{d}t \leqslant mM \int_a^b \mathrm{d}t = mM(b-a),$$

$$|\varphi_2(s)| \leqslant \int_a^b |K(s, t)|\, |\varphi_1(t)|\, \mathrm{d}t \leqslant mM^2(b-a) \int_a^b \mathrm{d}t = mM^2(b-a)^2,$$

and in general

$$|\varphi_n(s)| \leqslant m\,[M(b-a)]^n,$$

so that the general term of series (37) is subject to the inequality:

$$|\varphi_n(s)\, \lambda^n| \leqslant m\,[|\lambda|\, M(b-a)]^n.$$

It is clear from this that series (37) is absolutely and uniformly convergent with respect to s under the condition

$$|\lambda| < \frac{1}{M(b-a)}, \tag{40}$$

and that its sum is a continuous solution of equation (33).

We can write the solution obtained in another form by introducing the so-called *iterated* kernels, which are given successively by the formulae:

$$K_1(s, t) = K(s, t); \quad K_n(s, t) = \int_a^b K_{n-1}(s, t_1)\, K(t_1, t)\, \mathrm{d}t_1. \tag{41}$$

By virtue of the continuity of the fundamental kernel $K(s, t)$, each of the iterated kernels is a continuous function in the square k_0 [II, 80]. The iterated kernel $K_n(s, t)$ is expressible in terms of the

fundamental kernel $K(s, t)$ with the aid of $(n-1)$ quadratures:

$$K_2(s, t) = \int_a^b K(s, t_1) K(t_1, t)\, dt_1 ,$$

$$K_3(s, t) = \int_a^b K_2(s, t_1) K(t_1, t)\, dt_1 = \int_a^b \Big[\int_a^b K(s, t_2) K(t_2, t_1)\, dt_2\Big] K(t_1, t)\, dt_1 ,$$

i.e.

$$K_3(s, t) = \int_a^b \int_a^b K(s, t_2) K(t_2, t_1) K(t_1, t)\, dt_1\, dt_2 ,$$

and in general

$$K_n(s, t) = \int_a^b \int_a^b \dots \int_a^b K(s, t_{n-1}) K(t_{n-1}, t_{n-2}) \dots K(t_2, t_1) \times \\ \times K(t_1, t)\, dt_1\, dt_2 \dots dt_{n-1} . \tag{42}$$

The order of the quadratures is a matter of indifference [II, 98].

By using these remarks, we easily obtain the formula:

$$K_{p+q}(s, t) = \int_a^b K_p(s, \tau) K_q(\tau, t)\, d\tau . \tag{43}$$

It is sufficient to carry out the $(p-1)$ quadratures for forming $K_p(s, \tau)$ and the $(q-1)$ quadratures for forming $K_q(\tau, t)$. There remains one quadrature with respect to τ.

On using formula (42) and the inequality $|K(s, t)| \leqslant M$, we get the inequality

$$|K_n(s, t)| \leqslant M^n (b-a)^{n-1} \tag{44}$$

in the square k_0, from which it follows that the series

$$R(s, t; \lambda) = K_1(s, t) + K_2(s, t)\lambda + K_3(s, t)\lambda^2 + \dots \\ = \sum_{n=0}^{\infty} K_{n+1}(s, t)\lambda^n \tag{45}$$

is absolutely and uniformly convergent in the square k_0 with condition (40). We have written its sum as $R(s, t; \lambda)$.

We now express the function $\varphi_n(s)$ directly in terms of the function $f(s)$:

$$\varphi_1(s) = \int_a^b K(s, t) f(t)\, dt;$$

$$\varphi_2(s) = \int_a^b K(s, t) \varphi_1(t)\, dt = \int_a^b \int_a^b K(s, t) K(t, t_1) f(t_1)\, dt_1\, dt \\ = \int_a^b K_2(s, t_1) f(t_1)\, dt_1$$

and in general,

$$\varphi_n(s) = \int_a^b K_n(s, t)\, f(t)\, \mathrm{d}t .$$

We obtain by substituting in (37):

$$\varphi(s) = f(s) + \lambda \sum_{n=0}^{\infty} \int_a^b K_{n+1}(s, t)\, \lambda^n f(t)\, \mathrm{d}t .$$

On taking into account the uniform convergence of series (45) in the square k_0 and all the more its uniform convergence with respect to the single variable t in the interval $[a, b]$ with any fixed s of this interval, we can interchange the summation and integration, and obtain, on using the notation of (45):

$$\varphi(s) = f(s) + \lambda \int_a^b R(s, t; \lambda)\, f(t)\, \mathrm{d}t . \tag{46}$$

This is all proved under condition (40).

Function (45), which is independent of the function $f(s)$, is called the *resolvent* of the kernel $K(s, t)$ or of equation (33). It may easily be verified that the resolvent, considered as a function of its first or second argument, satisfies the following two integral equations:

$$\left.\begin{aligned} R(s, t; \lambda) &= K(s, t) + \lambda \int_a^b K(s, t_1)\, R(t_1, t; \lambda)\, \mathrm{d}t_1 , \\ R(s, t; \lambda) &= K(s, t) + \lambda \int_a^b K(t_1, t)\, R(s, t_1; \lambda)\, \mathrm{d}t_1 . \end{aligned}\right\} \tag{47}$$

To verify say the second equation, we multiply both sides of formula (45) by $K(t, x)$ and integrate with respect to t:

$$\int_a^b R(s, t; \lambda)\, K(t, x)\, \mathrm{d}t = \sum_{n=0}^{\infty} \lambda^n \int_a^b K_{n+1}(s, t)\, K(t, x)\, \mathrm{d}t ,$$

or, by (41):

$$\int_a^b R(s, t; \lambda)\, K(t, x)\, \mathrm{d}t = \sum_{n=0}^{\infty} K_{n+2}(s, x)\, \lambda^n .$$

We multiply both sides by λ:

$$\lambda \int_a^b R(s,t;\lambda)\,K(t,x)\,\mathrm{d}t = \sum_{n=0}^{\infty} K_{n+2}(s,x)\,\lambda^{n+1},$$

or, on replacing the variable of summation n by $(n-1)$ and starting the summation at $n=1$:

$$\lambda \int_a^b R(s,t;\lambda)\,K(t,x)\,\mathrm{d}t = \sum_{n=1}^{\infty} K_{n+1}(s,x)\,\lambda^{n}.$$

By (45), we can rewrite this equation as

$$\lambda \int_a^b R(s,t;\lambda)\,K(t,x)\,\mathrm{d}t = R(s,x;\lambda) - K(s,x),$$

which in fact yields the second of equations (47), except for a different notation for the variables. The first of the integral equations (47) for the resolvent may be verified in the same way.

It may be mentioned that the convergence of the method of successive approximations can be proved for values of λ satisfying the inequality:

$$|\lambda| < \frac{1}{\sqrt{\int_a^b \int_a^b |K(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t}},$$

which is in general less restrictive than inequality (40). We shall not make use of this fact below.

6. Existence and uniqueness theorem. We have so far only defined the resolvent for values of λ satisfying condition (40). We shall see later that the resolvent exists throughout the plane of the complex variable λ, except for certain isolated values, and that it satisfies equations (47) throughout the λ plane. It is therefore important to take equations (47) as our starting-point when proving an existence and uniqueness theorem for the solution of equation (33):

THEOREM. *If there exists for a certain λ a continuous function $R(s,t;\lambda)$ in the square k_0 satisfying equations* (47), *equation* (33) *has a unique solution for this λ, and the solution is given by equation* (46).

The proof falls into two parts. We show first that, given (47), every solution of (33) must be expressible by (46). This gives us the uniqueness. We then verify that (46) actually yields the solution of (33).

Let $\varphi(s)$ be a solution of equation (33). We multiply both sides of (33) by $\lambda\, R(x, s; \lambda)$ and integrate with respect to s:

$$\lambda \int_a^b R(x, s; \lambda)\, \varphi(s)\, \mathrm{d}s = \lambda \int_a^b R(x, s; \lambda)\, f(s)\, \mathrm{d}s + \\ + \lambda \int_a^b \Big[\int_a^b \lambda R(x, s; \lambda)\, K(s, t)\, \mathrm{d}s\Big]\, \varphi(t)\, \mathrm{d}t\,.$$

On taking the second of equations (47) into account, we can write:

$$\lambda \int_a^b R(x, s; t)\, K(s, t)\, \mathrm{d}s = R(x, t; \lambda) - K(x, t)\,,$$

and the previous formula can be rewritten as

$$\lambda \int_a^b R(x, s; \lambda)\, \varphi(s)\, \mathrm{d}s = \lambda \int_a^b R(x, s; \lambda)\, f(s)\, \mathrm{d}s + \\ + \lambda \int_a^b R(x, t; \lambda)\, \varphi(t)\, \mathrm{d}t - \lambda \int_a^b K(x, t)\, \varphi(t)\, \mathrm{d}t\,.$$

On cancelling like terms on the right and left-hand sides in this formula, and substituting, by virtue of (33):

$$\lambda \int_a^b K(x, t)\, \varphi(t)\, \mathrm{d}t = \varphi(x) - f(x)\,,$$

we obtain formula (46).

We now show that the function $\varphi(s)$ defined by (46) in fact satisfies equation (33) when equations (47) hold.

On substituting expression (46) in (33) and taking all the terms to the left-hand side, we obtain:

$$f(s) + \lambda \int_a^b R(s, t; \lambda)\, f(t)\, \mathrm{d}t - \\ - f(s) - \lambda \int_a^b K(s, t)\, \Big[f(t) + \lambda \int_a^b R(t, t_1; \lambda)\, f(t_1)\, \mathrm{d}t_1\Big]\, \mathrm{d}t = 0$$

or

$$\int_a^b R(s, t; \lambda)\, f(t)\, \mathrm{d}t - \\ - \int_a^b K(s, t)\, f(t)\, \mathrm{d}t - \lambda \int_a^b \int_a^b K(s, t)\, R(t, t_1; \lambda)\, f(t_1)\, \mathrm{d}t\, \mathrm{d}t_1 = 0\,,$$

which can be rewritten as

$$\int_a^b \Big[R(s,t;\lambda) - K(s,t) - \lambda \int_a^b K(s,t_1)\, R(t_1,t;\lambda)\, \mathrm{d}t_1\Big] f(t)\, \mathrm{d}t = 0\,,$$

and this last equation in fact holds, since the square bracket is identically zero by virtue of the first of equations (47). The theorem is thus fully proved.

On observing that we have constructed the resolvent satisfying equations (47) for the values of λ satisfying condition (40), we can assert that *equation* (33) *has a unique solution for the* λ *satisfying condition* (40), *and that this solution is given by formula* (46). This could also be proved directly.

7. Fredholm's determinant. We next form an entire function $D(\lambda)$ such that, when series (45) is multiplied by it, we obtain another entire function of λ. This means that the resolvent is the quotient of two entire functions, the denominator of the quotient being equal to $D(\lambda)$, i.e. the quotient of two power series in λ, convergent for all complex values of λ. In other words, the resolvent proves to be a rational or meromorphic function of λ throughout the complex λ plane. We obtain $D(\lambda)$ by replacing the integral in equation (33) by a finite sum. Strictly speaking, this replacement is impermissible, but none of the working that follows is meant to have the force of a proof and merely serves as a guide so that we can guess the form of $D(\lambda)$.

We divide the interval $[a, b]$ into n equal parts, the length of each of which will be $\delta = (b - a)/n$. We introduce the following notation for the points of division and for the values of the functions appearing in (33) at these points:

$$s_i = a + i\,\frac{b-a}{n}\,; \quad f_i = f(s_i); \quad \varphi_i = \varphi(s_i); \quad K_{pq} = K(s_p, s_q)$$
$$(i, p, q = 1, \ldots, n)\,.$$

On replacing the integral in (33) by the corresponding Riemann sum, we obtain the approximate equation:

$$\varphi(s) = f(s) + \lambda \sum_{q=1}^{n} K(s, s_q)\, \varphi_q\, \delta\,.$$

We replace the independent variable s in this equation by s_p. We thus obtain a system of n equations of the first degree in the unknowns $\varphi_1, \ldots, \varphi_n$:

$$\varphi_p = f_p + \lambda \sum_{q=1}^{n} K_{pq}\, \varphi_q\, \delta \qquad (p = 1, \ldots, n)\,.$$

We obtain the following determinant in the denominator on solving this system by Cramer's theorem [III_1, 8]:

$$D_n(\lambda) = \begin{vmatrix} 1-\lambda K_{11}\,\delta & -\lambda K_{12}\,\delta, & \dots, & -\lambda K_{1n}\,\delta \\ -\lambda K_{21}\,\delta, & 1-\lambda K_{22}\,\delta, & \dots, & -\lambda K_{2n}\,\delta \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda K_{n1}\,\delta & -\lambda K_{n2}\,\delta, & \dots, & 1-\lambda K_{nn}\,\delta \end{vmatrix}.$$

We apply to this determinant the expansion formula for a determinant of the form (see [III_1, 5]):

$$\begin{vmatrix} a_{11}+x, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}+x, & \dots, & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \dots, & a_{nn}+x \end{vmatrix},$$

where, in this last, $x = 1$ and $a_{ij} = -\lambda K_{ij}\,\delta$. We thus obtain

$$D_n(\lambda) = 1 - \frac{\lambda}{1!}\sum_{p_1=1}^{n} K_{p_1 p_1}\,\delta + \frac{\lambda^2}{2!}\sum_{p_1,\,p_2=1}^{n} \begin{vmatrix} K_{p_1 p_1}, & K_{p_1 p_2} \\ K_{p_2 p_1}, & K_{p_2 p_2} \end{vmatrix} \delta^2 + \dots +$$

$$+ (-1)^n \frac{\lambda^n}{n!} \sum_{p_1,\,\dots,\,p_n=1}^{n} \begin{vmatrix} K_{p_1 p_1}, & K_{p_1 p_2}, & \dots, & K_{p_1 p_n} \\ K_{p_2 p_1}, & K_{p_2 p_2}, & \dots, & K_{p_2 p_n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{p_n p_1}, & K_{p_n p_2}, & \dots, & K_{p_n p_n} \end{vmatrix} \delta^n. \tag{48}$$

We introduce the following notation for convenience in future working:

$$K\begin{pmatrix} x_1, x_2, \dots, x_n \\ y_1, y_2, \dots, y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1), & K(x_1, y_2), & \dots, & K(x_1, y_n) \\ K(x_2, y_1), & K(x_2, y_2), & \dots, & K(x_2, y_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(x_n, y_1), & K(x_n, y_2), & \dots, & K(x_n, y_n) \end{vmatrix}. \tag{49}$$

$$(n = 1,\ 2,\ 3,\ \dots)$$

We consider the successive terms on the right-hand side of (48). The sum

$$\sum_{p_1=1}^{n} K_{p_1 p_1}\,\delta = \sum_{i=1}^{n} K(x_i, x_i)\,\delta$$

represents the Riemann sum for the integral

$$\int_a^b K_1(t_1, t_1)\,\mathrm{d}t_1\,,$$

and tends to this integral as $n \to \infty$. Similarly, the sum

$$\sum_{p_1,\,p_2=1}^{n} \begin{vmatrix} K_{p_1 p_1}, & K_{p_1 p_2} \\ K_{p_2 p_1}, & K_{p_2 p_2} \end{vmatrix} \delta^2$$

represents the Riemann sum for the integral

$$\int_a^b \int_a^b \begin{vmatrix} K(t_1, t_1), & K(t_1, t_2) \\ K(t_2, t_1), & K(t_2, t_2) \end{vmatrix} \mathrm{d}t_1\,\mathrm{d}t_2$$

and so on.

Formula (48) thus leads us naturally in the limit to the following power series in λ:

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} d_n , \tag{50}$$

where

$$d_n = \int_a^b \int_a^b \ldots \int_a^b K\begin{pmatrix} t_1, t_2, \ldots, t_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} dt_1\, dt_2 \ldots dt_n , \tag{51}$$

and

$$K\begin{pmatrix} t_1, t_2, \ldots, t_n \\ t_1, t_2, \ldots, t_n \end{pmatrix}$$

is given by (49).

We have arrived at series (50) by means of imprecise arguments. On returning to the strict theory, two facts need to be proved: firstly, that series (50) is convergent throughout the complex λ plane, i.e. is an entire function of λ, and secondly, that we obtain an entire function of λ on multiplying series (45) by series (50).

Let us write an inequality for the coefficient d_n. A determinant of order n stands under the integral sign in (51), each element of which $K(t_i, t_k)$ has a modulus not exceeding the positive number M. On applying Hadamard's theorem [III_1, 16] and the usual upper bound for an iterated integral, we obtain:

$$| d_n | \leqslant n^{\frac{n}{2}} [M(b-a)]^n .$$

The terms of series (50) therefore have moduli not exceeding the positive numbers

$$\frac{|\lambda|^n}{n!} n^{\frac{n}{2}} [M(b-a)]^n . \tag{52}$$

We show, by using d'Alembert's test [I, 121] that these positive numbers form a convergent series. We find on taking the ratio of two consecutive numbers:

$$\frac{|\lambda|}{n+1} \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}}} M(b-a) = \frac{|\lambda| M(b-a)}{\sqrt{n+1}} \left(1 + \frac{1}{n}\right)^{\frac{n}{2}} .$$

On indefinite increase of n the expression $(1 + 1/n)^{n/2}$ tends to $\sqrt{e}$ [I, 38], whilst all the ratios written tend to zero, whence follows the convergence for any λ of the series formed by the terms (52). Function (50) is therefore an entire function of λ.

The function $D(\lambda)$ was obtained from Cramer's denominator by passing to the limit. It is natural to suppose that $D(\lambda)$ is the denominator of the resolvent $R(s, t; \lambda)$, i.e. that, on multiplying series (45) by $D(\lambda)$, we get an entire function of λ. As a result of the multiplication we get a series, the terms of which are no longer numbers, as in $D(\lambda)$, but functions of (s, t). We introduce the special notation for this series:

$$D(s, t; \lambda) = K(s, t) + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} d_n(s, t). \tag{53}$$

Both the power series (45) and (50) are convergent in the circle (40). Hence series (53), obtained by multiplying them together, is also convergent in this circle. Power series, when absolutely convergent, can be multiplied together term by term, and we could obtain the expressions for the coefficients $d_n(s, t)$ by means of simple cross-multiplication of the series indicated; but, for the sake of convenience in future working, we shall adopt a different procedure. We obtain on multiplying both sides of the first of equations (47) by $D(\lambda)$:

$$D(s, t; \lambda) = K(s, t) D(\lambda) + \lambda \int_a^b K(s, t_1) D(t_1, t; \lambda) \, dt_1. \tag{54}$$

On substituting series (50) and (53) for $D(\lambda)$ and $D(s, t; \lambda)$ in this formula, and comparing coefficients of like powers of λ, we arrive at the formula:

$$d_n(s, t) = K(s, t) d_n - n \int_a^b K(s, t_1) d_{n-1}(t_1, t) \, dt_1 \quad (n = 1, 2, 3, \ldots), \tag{55}$$

which enables us to evaluate the coefficients $d_n(s, t)$ successively; here, we have to take $d_0(s, t) = K(s, t)$. We observe that series (53) is always absolutely and uniformly convergent with respect to (s, t) under condition (40), since the terms of the cross-multiplied series (45) and (50) are less than the positive terms of a convergent numerical series. This makes it possible for us to integrate term by term in the right-hand side of formula (54). On putting $n = 1$ in (55), we have:

$$d_1(s, t) = K(s, t) \int_a^b K(t_1, t_1) \, dt_1 - \int_a^b K(s, t_1) K(t_1, t) \, dt_1 =$$

$$= \int_a^b \begin{vmatrix} K(s, t), & K(s, t_1) \\ K(t_1, t), & K(t_1, t_1) \end{vmatrix} dt_1,$$

i.e. having regard to the notation (49):

$$d_1(s,t) = \int_a^b K\begin{pmatrix} s, & t_1 \\ t, & t_1 \end{pmatrix} dt_1 .$$

With $n = 2$, formula (55) gives:

$$d_2(s,t) = K(s,t) \int_a^b \int_a^b K\begin{pmatrix} t_1, & t_2 \\ t_1, & t_2 \end{pmatrix} dt_1\, dt_2 - 2 \int_a^b \int_a^b K(s,t_1)\, K\begin{pmatrix} t_1, & t_2 \\ t, & t_2 \end{pmatrix} dt_1\, dt_2 .$$

After elementary transformations, we obtain a formula similar to the previous one:

$$d_2(s,t) = \int_a^b \int_a^b K\begin{pmatrix} s, & t_1, & t_2 \\ t, & t_1, & t_2 \end{pmatrix} dt_1\, dt_2 .$$

Let us show that, for any positive integer n:

$$d_n(s,t) = \int_a^b \int_a^b \cdots \int_a^b K\begin{pmatrix} s, & t_1, & t_2, & \ldots, & t_n \\ t, & t_1, & t_2, & \ldots, & t_n \end{pmatrix} dt_1\, dt_2 \ldots dt_n . \tag{56}$$

We proved above that this formula holds with $n = 1$. We write $d_n^*(s,t)$ for the right-hand side of (56). We have, from what has been said: $d_1^*(s,t) = d_1(s,t)$. We now show that $d_n^*(s,t)$ satisfies the same relationship as $d_n(s,t)$

$$d_n^*(s,t) = K(s,t)\, d_n - n \int_a^b K(s,t_1)\, d_{n-1}^*(t_1,t)\, dt_1 . \tag{55$_1$}$$

By (55) and (55_1), $d_n(s,t)$ and $d_n^*(s,t)$ ($n = 2, 3, \ldots$) are successively determined uniquely, so that it will follow from $d_1^*(s,t) = d_1(s,t)$ that $d_n^*(s,t) = d_n(s,t)$ for any n. The proof of (56) therefore reduces to the proof of (55_1), where $d_n^*(s,t)$ is the right-hand side of (56).

We notice first of all that, if we transpose two of the x_i or two of the y_i in the symbol on the left-hand side of (49), the value of the determinant on the right-hand side only changes sign, because the operation amounts to interchanging two rows or columns of this determinant. On expanding the determinant of (56) by elements of

the first row and taking into account the remark just made, we can write:

$$K\begin{pmatrix} s, t_1, \ldots, t_n \\ t, t_1, \ldots, t_n \end{pmatrix} =$$

$$= K(s,t)\, K\begin{pmatrix} t_1, \ldots, t_n \\ t_1, \ldots, t_n \end{pmatrix} - K(s,t_1)\, K\begin{pmatrix} t_1, t_2, \ldots, t_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} -$$

$$- K(s,t_2)\, K\begin{pmatrix} t_1, t_2, \ldots, t_n \\ t_1, t, \ldots, t_n \end{pmatrix} - \ldots - K(s,t_n)\, K\begin{pmatrix} t_1, \ldots, t_{n-1}, t_n \\ t_1, \ldots, t_{n-1}, t \end{pmatrix}.$$

On integrating both sides of this relationship with respect to all the t_i and changing the notation for the variables of integration on the right-hand side, and at the same time using the remarks made above, we obtain:

$$d_n^*(s,t) =$$

$$= K(s,t)\, d_n - n \int_a^b \int_a^b \ldots \int_a^b K(s,t_1)\, K\begin{pmatrix} t_1, t_2, \ldots, t_n \\ t, t_2, \ldots, t_n \end{pmatrix} dt_1\, dt_2 \ldots dt_n,$$

which in fact leads us to relationship (55_1). Formula (56) is therefore proved. On applying Hadamard's theorem to the determinant in (56), we get the following inequality:

$$|\, d_n(s,t)\,| \leqslant (n+1)^{\frac{n+1}{2}} M^{n+1}(b-a)^n,$$

and we can prove from this, precisely as for (50), that series (53) gives an entire function of λ and that, for any λ, it is absolutely and uniformly convergent with respect to (s, t) in the square k_0.

On taking into account that we have, under condition (40):

$$R(s,t;\lambda)\, D(\lambda) = D(s,t;\lambda),$$

we can write for these values of λ:

$$R(s,t;\lambda) = \frac{D(s,t;\lambda)}{D(\lambda)}. \tag{57}$$

The right-hand side of this formula gives an analytic continuation of the function $R(s, t; \lambda)$ throughout the complex λ plane and shows that the resolvent is a rational function of λ. We notice that the denominator in (57), which is usually termed the *Fredholm determinant* (or *denominator*), does not depend on the variables (s, t).

We shall mention some consequences of the formulae written above. It follows at once from (51) and (56) that

$$d_{n+1} = \int_a^b d_n(s, s)\,\mathrm{d}s\,. \tag{58}$$

We observe further that a simple successive evaluation of the coefficients d_n and $d_n(s, t)$ is possible. On putting $n = 0$ in formula (58) and taking into account that $d_0(s, t) = K(s, t)$, we get d_1 from this formula. On next considering (55) with $n = 1$, we obtain $d_1(s, t)$ from it, on recalling that $d_0 = 1$. Next, (58) with $n = 1$ gives us d_2, after which (55) with $n = 2$ gives us $d_2(s, t)$ and so on. If we put $t = s$ in (53) and integrate both sides with respect to s, we get by (58):

$$\int_a^b D(s, s; \lambda)\,\mathrm{d}s = d_1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} d_{n+1}\,,$$

i.e. by (50),

$$D'(\lambda) = -\int_a^b D(s, s; \lambda)\,\mathrm{d}s\,. \tag{59}$$

We observe that it follows from (56) that $d_n(s, t)$ are continuous functions in the square k_0, and in view of the uniform convergence mentioned above of series (53) in this square, the function $D(s, t; \lambda)$ is also continuous for any λ in k_0.

The entire functions $D(\lambda)$ and $D(s, t; \lambda)$ can be expanded throughout the λ plane in non-negative integral powers of $(\lambda - \lambda_0)$, where λ_0 is any fixed complex number. For instance,

$$D(s, t; \lambda) = D(s, t; \lambda_0) + \sum_{k=1}^{\infty} \frac{\partial^k D(s, t; \lambda_0)}{\partial \lambda^k} \; \frac{(\lambda - \lambda_0)^k}{k!}\,,$$

where

$$\frac{\partial^k D(s, t; \lambda)}{\partial \lambda^k} = \sum_{n=k}^{\infty} (-1)^n \frac{\lambda^{n-k}}{(n-k)!} d_n(s, t) \quad (0! = 1)\,.$$

It follows at once from the inequalities for the $d_n(s, t)$ that the latter series is uniformly convergent in k_0 for any λ, and we can assert that the coefficients of the expansion of $D(s, t; \lambda)$ in powers of $(\lambda - \lambda_0)$ are also continuous functions in k_0.

8. Fredholm's equation for any λ. Let us consider equation (54). It was obtained from the first of equations (47) by multiplying by $D(\lambda)$. But equations (47) were obtained subject to condition (40), i. e. we can say that both sides of equation (54) coincide, given con-

dition (40). But, by the fundamental principle of analytic continuation, if two entire functions coincide in a circular domain on the plane of the complex variable λ, they coincide throughout the complex plane [III_2, 18]. On dividing both sides of (54) by $D(\lambda)$, we see that the resolvent satisfies the first of equations (47) for any values of λ which do not cause $D(\lambda)$ to vanish. In this latter case the ratio (57) becomes meaningless. Similarly, we can show by applying analytic continuation that the resolvent also satisfies the second of equations (47) for the λ indicated. Thus, if λ differs from a root of $D(\lambda)$, we have a continuous solution of both equations (47) and we obtain, on using the existence and uniqueness theorem of [6]:

THEOREM 1. *If the value of λ is not a zero of $D(\lambda)$, given any $f(s)$, equation* (33) *has a unique solution which is given by formula* (46), *while $R(s, t; \lambda)$ is given by formula* (57).

We now take a value $\lambda = \lambda_0$ which is a zero of $D(\lambda)$. It may happen that it is also a zero of the function $D(s, t; \lambda)$ for any (s, t). We now show that the multiplicity of this zero in the numerator of expression (57) must be less than its multiplicity in the denominator, whence it will follow that every zero of $D(\lambda)$ is a pole of the resolvent.

THEOREM 2. *Every zero λ_0 of the function $D(\lambda)$ is a pole of the resolvent.*

Let λ_0 be a zero of $D(\lambda)$ of multiplicity k, i.e.

$$D(\lambda) = (\lambda - \lambda_0)^k D_0(\lambda) \qquad [D_0(\lambda_0) \neq 0] .$$

Let it also be a zero of $D(s, t; \lambda)$ of multiplicity l, i.e.

$$D(s, t; \lambda) = (\lambda - \lambda_0)^l D_0(s, t; \lambda) ,$$

where $D_0(s, t; \lambda)$ is a series arranged in positive integral powers of $(\lambda - \lambda_0)$, the absolute term of which is non-zero for certain values of s, t. We recall that the derivative $D'(\lambda)$ has a zero $\lambda = \lambda_0$ of multiplicity $(k - 1)$. We obtain by applying formula (59):

$$D'(\lambda) = - (\lambda - \lambda_0)^l \int_a^b D_0(s, s; \lambda) \, ds .$$

The left-hand side has a zero $\lambda = \lambda_0$ of multiplicity $(k - 1)$, whilst the right-hand side has a factor $(\lambda - \lambda_0)^l$, in addition to which it may happen that a further positive integral power of $(\lambda - \lambda_0)$ appears after integration with respect to s. This argument leads us to the inequality $l \leqslant k - 1$, i.e. if $\lambda = \lambda_0$ is in fact a zero of the numerator of expression (57), the multiplicity of this zero is never less than k, so that the fraction as a whole has the pole $\lambda = \lambda_0$. We observe that the

absolute term in the expansion of $D_0(s, t; \lambda)$ in powers of $(\lambda - \lambda_0)$ is a function of (s, t). It may vanish for certain particular values of s and t, but is not identically equal to zero, since if this were so $\lambda = \lambda_0$ would be a zero of $D(s, t; \lambda)$ of multiplicity greater than l. We can formulate the theorem just proved more strictly as: *there exist values of s and t for which $\lambda = \lambda_0$ is a pole of the resolvent.*

We have shown that every zero λ_0 of the function $D(\lambda)$ is a pole of the resolvent. Let λ_0 be a pole of multiplicity r. In the neighbourhood of $\lambda = \lambda_0$ we have an expansion of the form:

$$R(s, t; \lambda) = \frac{a_{-r}(s, t)}{(\lambda - \lambda_0)^r} + \frac{a_{-r+1}(s, t)}{(\lambda - \lambda_0)^{r-1}} + \ldots + \frac{a_{-1}(s, t)}{\lambda - \lambda_0} + \\ + \sum_{i=0}^{\infty} a_i(s, t)\, (\lambda - \lambda_0)^i ,$$

where the coefficient $a_{-r}(s, t)$ is not identically zero in k_0.

It follows from what was said at the end of [7], that $a_k(s, t)$ are continuous functions in the square k_0.

On substituting this last expansion in the first of equations (47), multiplying both sides by $(\lambda - \lambda_0)^r$ and then putting $\lambda = \lambda_0$, we get:

$$a_{-r}(s, t) = \lambda_0 \int_a^b K(s, t_1)\, a_{-r}(t_1, t)\, \mathrm{d}t_1 .$$

It thus turns out that the coefficient $a_{-r}(s, t)$, considered as a function of s, is a solution of the homogeneous equation

$$\varphi(s) = \lambda_0 \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t \tag{60}$$

for any value of the variable t. Since the function $a_{-r}(s, t)$ is not identically zero, we arrive at the following theorem:

THEOREM 3. *If λ_0 is a zero of $D(\lambda)$, homogeneous equation* (60) *has solutions which are not identically zero.*

Thus every zero of $D(\lambda)$ is an eigenvalue of the integral equation; i.e. the homogeneous equation

$$\varphi(s) = \lambda \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t \tag{61}$$

now has non-zero solutions. Whereas if λ is not a zero of $D(\lambda)$, by Theorem 1, equation (33) has a unique solution for any $f(s)$ and, in particular, homogeneous equation (61) has now only a zero solution.

In other words, if λ is a zero of $D(\lambda)$, this is an eigenvalue, whilst if λ is not a zero of $D(\lambda)$, λ is not an eigenvalue.

We therefore obtain:

THEOREM 4. *The eigenvalues of the integral equation are the zeros of* $D(\lambda)$.

The entire function $D(\lambda)$ can only have a finite number of zeros in any bounded domain of the plane of complex variable λ, i.e.

THEOREM 5. *Only a finite number of eigenvalues can exist in any bounded domain of the* λ *plane.*

We shall mention a further formula which is useful in applications. Suppose that the term $f(s)$ of equation (33) can be written in the form

$$f(s) = \int_a^b K(s,t)\,\omega(t)\,\mathrm{d}t\,, \tag{62}$$

where $\omega(t)$ is a function of t.

Taking λ to differ from an eigenvalue, we obtain by (46) the solution of equation (33) in the form

$$\varphi(s) = \int_a^b K(s,t)\,\omega(t)\,\mathrm{d}t + \lambda \int_a^b \int_a^b R(s,t;\lambda)\,K(t,t_1)\,\omega(t_1)\,\mathrm{d}t\,\mathrm{d}t_1\,.$$

But the second of equations (47) gives us

$$\lambda \int_a^b R(s,t;\lambda)\,K(t,t_1)\,\mathrm{d}t = R(s,t_1;\lambda) - K(s,t_1);$$

on substituting this in the previous formula, we finally obtain the following simple expression for the solution of (33):

$$\varphi(s) = \int_a^b R(s,t;\lambda)\,\omega(t)\,\mathrm{d}t\,, \tag{63}$$

if the term $f(s)$ of the equation is given by (62).

9. Adjoint integral equation. For further development of the theory, we shall consider in addition to equation (33) another integral equation which differs from (33) in that the integration is performed with respect to the first variable of the kernel. The term outside the integral of this equation will be written as $g(s)$, and the required solution as $\psi(s)$:

$$\psi(s) = g(s) + \lambda \int_a^b K(t,s)\,\psi(t)\,\mathrm{d}t\,. \tag{64}$$

This equation is called the *adjoint of equation* (33).

We also write down the corresponding homogeneous equation:

$$\psi(s) = \lambda \int_a^b K(t, s)\, \psi(t)\, \mathrm{d}t\,. \tag{65}$$

Using the previous notation for the arguments of the kernel, we must define the kernel of this equation as follows:

$$K_0(s, t) = K(t, s)\,.$$

Symbol (49) for the kernel $K_0(s, t)$ is obtained from the same symbol for $K(s, t)$ by replacing x_i by y_i and vice versa, i.e.

$$K_0\begin{pmatrix} x_1, & x_2, & \ldots, & x_n \\ y_1, & y_2, & \ldots, & y_n \end{pmatrix} = K\begin{pmatrix} y_1, & y_2, & \ldots, & y_n \\ x_1, & x_2, & \ldots, & x_n \end{pmatrix}.$$

Formulae (51) then show that the coefficients d_n for kernel $K_0(s, t)$ are the same as for kernel $K(s, t)$, whilst it follows from (56) that the coefficients $d_n(s, t)$ for kernel $K_0(s, t)$ are obtained from the analogous coefficients for $K(s, t)$ by a simple interchange of arguments s and t. We thus see that the numerator and denominator in formula (57) are given, in the case of the adjoint equation (64), in terms of the analogous quantities for equation (33) by the formulae

$$D_0(s, t; \lambda) = D(t, s; \lambda)\,; \qquad D_0(\lambda) = D(\lambda)\,,$$

i.e. the numerator is obtained by interchange of the arguments s and t, whilst the Fredholm determinant for the adjoint equation (64) is the same as for equation (33). Hence it follows, *inter alia*, that the adjoint equation has the same eigenvalues as the original equation.

Obviously, all the theorems stated in [9] are valid for the adjoint equation. We can assert in addition, on the basis of what has been said above, that:

THEOREM 6. *Homogeneous equation* (60) *and its adjoint* (65) *have only zero or else have non-zero solutions simultaneously.*

10. The case of an eigenvalue. Theorem 1 gives the complete answer regarding the solution of equation (33) in the case when λ is not an eigenvalue. We are concerned in the present section with the problem when λ is an eigenvalue.

Let λ be an eigenvalue, and let non-homogeneous equation (33) have a solution $\varphi(s)$. We multiply both sides of (33) by a solution

$\psi(s)$ of the adjoint homogeneous equation (65) and integrate with respect to s:

$$\int_a^b \varphi(s)\, \psi(s)\, \mathrm{d}s = \int_a^b f(s)\, \psi(s)\, \mathrm{d}s + \int_a^b \Big[\lambda \int_a^b K(s,t)\, \psi(s)\, \mathrm{d}s\Big]\, \varphi(t)\, \mathrm{d}t\,.$$

We obtain by using (65):

$$\int_a^b \varphi(s)\, \psi(s)\, \mathrm{d}s = \int_a^b f(s)\, \psi(s)\, \mathrm{d}s + \int_a^b \psi(t)\, \varphi(t)\, \mathrm{d}t\,,$$

whence

$$\int_a^b f(s)\, \psi(s)\, \mathrm{d}s = 0\,, \tag{66}$$

i.e. the necessary condition for solubility of equation (33) is that $f(s)$ satisfy condition (66), where $\psi(s)$ is any solution of equation (65), which certainly has non-zero solutions because λ is an eigenvalue by hypothesis. If λ is not an eigenvalue, equation (33) has, by Theorem 1, solutions for any $f(s)$. This gives us

THEOREM 7. *There are two possibilities: either integral equation* (33) *is soluble for any $f(s)$ and homogeneous equation* (35) *has only a zero solution, or homogeneous equation* (35) *has solutions different from zero and equation* (33) *is not soluble for every $f(s)$.*

With the first possibility, the non-homogeneous equation has a unique solution. This follows from Theorem 1, as also from the following simple arguments: if the non-homogeneous equation were to have two different solutions, their difference would be a non-zero solution of the homogeneous equation.

Note. If non-homogeneous equation (33) is known to have one and only one solution for a certain λ and certain $f(s)$, λ is not an eigenvalue. For, if λ were an eigenvalue, by adding to the solution of the non-homogeneous equation any non-zero solution of the corresponding homogeneous equation, we should obtain a new solution of the non-homogeneous equation.

We shall see further that condition (66) is sufficient as well as necessary for the solubility of equation (33). As a preliminary we must discuss the question of the rank of the eigenvalue [4].

Let λ be an eigenvalue and let

$$\varphi_1(s),\ \varphi_2(s),\ \ldots,\ \varphi_m(s) \tag{67}$$

be any linearly independent eigenfunctions, i.e. solutions of (61) different from zero:

$$\frac{\varphi_j(s)}{\lambda} = \int_a^b K(s,t)\,\varphi_j(t)\,\mathrm{d}t \qquad (j = 1, 2, \ldots, m). \tag{68}$$

If λ or the kernel is not real, functions (67) must also be assumed complex. We recall that $\lambda = 0$ cannot be an eigenvalue [4]. Since any linear combination with constant coefficients of eigenfunctions (67) is also an eigenfunction, we can apply a process of orthogonalization to functions (67). We can thus assume that functions (67) are mutually orthogonal and normalized, i.e.

$$\int_a^b \varphi_p(s)\,\overline{\varphi_q(s)}\,\mathrm{d}s = 0 \quad (p \neq q); \qquad \int_a^b |\varphi_p(s)|^2\,\mathrm{d}s = 1. \tag{69}$$

On passing to the conjugates, we can rewrite (68) as

$$\frac{\overline{\varphi_j(s)}}{\bar{\lambda}} = \int_a^b \overline{K(s,t)}\;\overline{\varphi_j(t)}\,\mathrm{d}t.$$

Hence it follows that the left-hand side of this equation is the Fourier coefficient of $\overline{K(s,t)}$, regarded as a function of the argument t, with respect to the orthonormal system (67) consisting of a finite number of functions. We can write, by Bessel's inequality [3]:

$$\sum_{j=1}^{m} \frac{|\varphi_j(s)|^2}{|\lambda|^2} \leqslant \int_a^b |K(s,t)|^2\,\mathrm{d}t.$$

We remark that $|\alpha| = |\bar{\alpha}|$ for any complex α. On integrating both sides of this inequality with respect to s and taking (69) into account we obtain

$$\sum_{j=1}^{m} \frac{1}{|\lambda|^2} \leqslant \int_a^b \left[\int_a^b |K(s,t)|^2\,\mathrm{d}t\right]\mathrm{d}s,$$

or

$$\frac{m}{|\lambda|^2} \leqslant \int_a^b \left[\int_a^b |K(s,t)|^2\,\mathrm{d}t\right]\mathrm{d}s,$$

whence

$$m \leqslant |\lambda|^2 \int_a^b \left[\int_a^b |K(s,t)|^2\,\mathrm{d}t\right]\mathrm{d}s,\dagger$$

† Since we must have $m < 1$ with $|\lambda| < [\int_a^b [\int_a^b |K(s,t)|^2\,\mathrm{d}t]\,\mathrm{d}s]^{-1/2} = r$, there are no eigenvalues inside the circle $|\lambda| = r$, so that series (37) is convergent.

where, by virtue of the continuity of the kernel, the integral on the right can be interpreted as a double integral. It follows from the inequality written that the number of linearly independent eigenfunctions corresponding to the eigenvalue λ cannot exceed the number on the right-hand side of this inequality, i.e.

THEOREM 8. *Only a finite number of linearly independent eigenfunctions correspond to any given eigenvalue, i.e. the rank of any eigenvalue is finite.*

We remark that, for eigenvalues λ, distant from the origin $\lambda = 0$, the right-hand side of the last inequality becomes large in view of the factor $|\lambda|^2$.

Let λ be an eigenvalue. Equations (61) and (65) simultaneously have non-zero solutions. We show that the ranks of the eigenvalues of these equations are the same.

THEOREM 9. *Homogeneous equation* (61) *and the adjoint equation* (65) *have the same number of linearly independent solutions, i.e. the ranks of coincident eigenvalues are the same.*

We use *reductio ad absurdum*. Let the rank of equation (61) be m, and the rank of equation (65) be n, and let $m < n$. We prove a contradiction. Let

$$\varphi_1(s),\ \varphi_2(s),\ \ldots,\ \varphi_m(s) \tag{70}$$

be linearly independent solutions of equation (61), and

$$\psi_1(s),\ \psi_2(s),\ \ldots,\ \psi_n(s) \tag{71}$$

linearly independent solutions of (65). As above, we can assume that functions (70) and (71) both form orthonormal systems. We have:

$$\left.\begin{aligned} \varphi_j(s) &= \lambda \int_a^b K(s,t)\,\varphi_j(t)\,\mathrm{d}t \qquad (j = 1, 2, \ldots, m)\,; \\ \psi_j(s) &= \lambda \int_a^b K(t,s)\,\psi_j(t)\,\mathrm{d}t \qquad (j = 1, 2, \ldots, n)\,. \end{aligned}\right\} \tag{72}$$

We form the new kernel:

$$L(s,t) = K(s,t) - \sum_{j=1}^{m} \overline{\varphi_j(t)}\ \overline{\psi_j(s)}\,, \tag{73}$$

and write the two adjoint equations:

$$\varphi(s) = \lambda \int_a^b L(s, t)\, \varphi(t)\, dt\,, \tag{74}$$

$$\psi(s) = \lambda \int_a^b L(t, s)\, \psi(t)\, dt\,. \tag{75}$$

By (73), we can rewrite these equations as

$$\varphi(s) = \lambda \int_a^b K(s, t)\, \varphi(t)\, dt - \lambda \sum_{j=1}^{m} \overline{\psi_j(s)} \int_a^b \overline{\varphi_j(t)}\, \varphi(t)\, dt\,, \tag{74_1}$$

$$\psi(s) = \lambda \int_a^b K(t, s)\, \psi(t)\, dt - \lambda \sum_{j=1}^{m} \overline{\varphi_j(s)} \int_a^b \overline{\psi_j(t)}\, \psi(t)\, dt\,. \tag{75_1}$$

Let $\varphi(s)$ be any solution of equation (74_1). We multiply both sides of (74_1) by $\psi_k(s)$, where k is one of the numbers $1, 2, \ldots, m$, and integrate with respect to s:

$$\int_a^b \varphi(s)\, \psi_k(s)\, ds = \int_a^b \Big[\lambda \int_a^b K(s, t)\, \psi_k(s)\, ds\Big]\, \varphi(t)\, dt - \\ - \lambda \sum_{j=1}^{m} \int_a^b \overline{\varphi_j(t)}\, \varphi(t)\, dt \int_a^b \overline{\psi_j(s)}\, \psi_k(s)\, ds\,.$$

On taking (72) into account, as also the fact that functions (71) are orthogonal and normalized, we can rewrite this equation as:

$$\int_a^b \varphi(s)\, \psi_k(s)\, ds = \int_a^b \psi_k(s)\, \varphi(s)\, ds - \lambda \int_a^b \overline{\varphi_k(s)}\, \varphi(s)\, ds\,,$$

whence it follows, since $\lambda \neq 0$, that

$$\int_a^b \overline{\varphi_k(s)}\, \varphi(s)\, ds = 0 \qquad (k = 1, 2, \ldots, m)\,. \tag{76}$$

Thus every solution of equation (74_1) satisfies conditions (76). But, by virtue of these conditions, equation (74_1) can be rewritten as

$$\varphi(s) = \lambda \int_a^b K(s, t)\, \varphi(t)\, dt\,,$$

i.e. every solution of equation (74_1) (i.e. of (74)) satisfies equation (61) also. Hence $\varphi(s)$ must be expressible as a linear combination of functions (70):

$$\varphi(s) = \sum_{j=1}^{m} c_j\, \varphi_j(s)\,. \tag{77}$$

We show that all the coefficients c_j must vanish. We multiply both sides of (77) by $\overline{\varphi_k(s)}$ and integrate with respect to s:

$$\int_a^b \varphi(s)\, \overline{\varphi_k(s)}\, ds = \sum_{j=1}^{m} c_j \int_a^b \varphi_j(s)\, \overline{\varphi_k(s)}\, ds .$$

Using (76) and the fact that functions (70) are orthogonal and normalized, we get $0 = c_k$. Hence it follows from (77) that $\varphi(s) \equiv 0$, i.e. homogeneous equation (74) has only a trivial solution. We show that the adjoint equation (75) has non-zero solutions. We substitute $\psi(s) = \psi_k(s)$ in (75_1), where $k > m$. Using the fact that functions (71) form an orthonormal system, we obtain

$$\psi_k(s) = \lambda \int_a^b K(s, t)\, \psi_k(t)\, dt ,$$

whence, by (72), it is clear that $\psi(s) = \psi_k(s)$ with $k > m$ satisfies equation (75). We have thus obtained a contradiction with Theorem 7: equation (74) has only a trivial solution, whilst the adjoint equation (75) has non-zero solutions. The case $m < n$ is therefore impossible.

It can similarly be shown that the case $m > n$ is also impossible, and hence $m = n$, which proves Theorem 9.

We remark that it follows from what has been said that homogeneous equations (74) and (75) have only a trivial solution, i.e. λ is not an eigenvalue of the kernel $L(s, t)$.

We now turn to the question of solving the non-homogeneous equation

$$\varphi(s) = f(s) + \lambda \int_a^b K(s, t)\, \varphi(t)\, dt , \tag{78}$$

if λ is an eigenvalue. We have seen that the necessary condition for (78) to be soluble is that $f(s)$ satisfies

$$\int_a^b f(s)\, \psi(s)\, ds = 0 , \tag{79}$$

where $\psi(s)$ is any solution of the equation

$$\psi(s) = \lambda \int_a^b K(t, s)\, \psi(t)\, dt . \tag{80}$$

We now turn to the proof of the sufficiency of condition (79). Let (79) be fulfilled. We form the kernel $L(s, t)$ in accordance with formula (73). As we have shown, λ is not an eigenvalue of this kernel,

so that the equation

$$\varphi(s) = f(s) + \lambda \int_a^b L(s, t)\, \varphi(t)\, dt \tag{81}$$

has a solution. We rewrite (81) as

$$\varphi(s) = f(s) + \lambda \int_a^b K(s, t)\, \varphi(t)\, dt - \lambda \sum_{j=1}^{m} \overline{\psi_j(s)} \int_a^b \overline{\varphi_j(t)}\, \varphi(t)\, dt\,. \tag{81_1}$$

On multiplying by $\psi_k(s)$ as in the proof of Theorem 9, and integrating with respect to s, we obtain

$$\int_a^b \varphi(s)\, \psi_k(s)\, ds = \int_a^b f(s)\, \psi_k(s)\, ds + \int_a^b \psi_k(s)\, \varphi(s)\, ds - \lambda \int_a^b \overline{\varphi_k(t)}\, \varphi(t)\, dt\,,$$

whence we obtain, by (79),

$$\int_a^b \overline{\varphi_k(t)}\, \varphi(t)\, dt = 0 \qquad (k = 1, 2, \ldots, m)\,.$$

Equation (81_1), or what amounts to the same thing, (81), therefore reduces to equation (78), i.e. the solution $\varphi(s)$ of equation (81) is also a solution of (78). This proves the sufficiency of condition (79).

If this condition is satisfied, any solution of this linear non-homogeneous equation can be written in the usual way as the sum of a particular solution $\varphi_0(s)$ and the general solution of the corresponding homogeneous equation

$$\varphi(s) = \varphi_0(s) + \sum_{j=1}^{m} c_j\, \varphi_j(s)\,, \tag{82}$$

where the c_j are arbitrary constants. Equation (78) thus has in this case an infinite set of solutions. The solution $\varphi_0(s)$ can be formed with the aid of the resolvent of the kernel $L(s, t)$.

The foregoing discussion leads us to the following theorem.

THEOREM 10. *If λ is an eigenvalue, the necessary and sufficient condition for equation* (78) *to be soluble is that the function $f(s)$ satisfies condition* (79), *in which $\psi(s)$ is any eigenfunction of the adjoint equation, i.e. any solution of equation* (80). *If condition* (79) *is satisfied, the equation has an infinite set of solutions, all of which are given by formula* (82).

Note 1. To test condition (79), it is sufficient to replace $\psi(s)$ by a complete set of linearly independent solutions $\psi_1(s)$, $\psi_2(s)$, $\ldots$, $\psi_m(s)$, of equation (80), since every other solution is a linear com-

bination of them. Thus, if condition (79) is fulfilled with $\psi(s) = \psi_k(s)$ $(k = 1, 2, \ldots, m)$, it is also fulfilled for any solution $\psi(s)$ of equation (80).

Note 2. Instead of taking (80) as the adjoint homogeneous equation, we often take

$$\omega(s) = \bar{\lambda} \int_a^b \overline{K(t, s)}\, \omega(t)\, \mathrm{d}t\,. \tag{83}$$

Equations (80) and (83) obviously have solutions which are conjugate in pairs, i.e. if $\psi(s)$ is a solution of equation (80), then $\omega(s) = \overline{\psi}(s)$ is a solution of (83), and vice versa. If λ is an eigenvalue of the homogeneous equation

$$\varphi(s) = \lambda \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t\,,$$

$\bar{\lambda}$ is an eigenvalue of (83), and vice versa.

With this definition of the adjoint equation the condition for (79) to be soluble must be written as

$$\int_a^b \overline{\omega(s)}\, f(s)\, \mathrm{d}s = 0\,, \tag{84}$$

where $\omega(s)$ is any solution of equation (83).

The treatment of [7], on the basis of which the fundamental theorems were proved, was first given by Fredholm in 1903. The theorems proved above are entirely analogous to the theorems on the solution of systems of linear algebraic equations [III, 8, 9 and 10].

11. Fredholm minors. The above argument enables us to obtain the complete set of linearly independent eigenfunctions of the equation with kernel $K(s, t)$ corresponding to a given eigenvalue. We shall merely give the results, without dwelling on the proof†. We use the notation of (49), and introduce the quantities

$$B_n\begin{pmatrix} s_1, \ldots, s_p \\ t_1, \ldots, t_p \end{pmatrix} = \int_a^b \cdots \int_a^b K\begin{pmatrix} s_1, \ldots, s_p, r_1, \ldots, r_n \\ t_1, \ldots, t_p, r_1, \ldots, r_n \end{pmatrix} \mathrm{d}r_1 \ldots \mathrm{d}r_n\,,$$

$$B_0\begin{pmatrix} s_1, \ldots, s_p \\ t_1, \ldots, t_p \end{pmatrix} = K\begin{pmatrix} s_1, \ldots, s_p \\ t_1, \ldots, t_p \end{pmatrix}.$$

By definition, the pth Fredholm minor is given by the series

$$D_p(s, t; \lambda) = D\begin{pmatrix} s_1, \ldots, s_p \\ t_1, \ldots, t_p \end{pmatrix}; \lambda\Big) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+p-1}}{n!} B_n\begin{pmatrix} s_1, \ldots, s_p \\ t_1, \ldots, t_p \end{pmatrix}.$$

† See I. I. Privalov, *Integral Equations* (Integral'nye uravneniya), p. 61.

We observe that this series is the same as (53) with $p = 1$. Let λ_0 be a zero of $D(\lambda)$ of multiplicity r. We take the sequence

$$D(\lambda_0), D\begin{pmatrix} s_1 \\ t_1 \end{pmatrix}; \lambda_0\Big), \quad D\begin{pmatrix} s_1, s_2 \\ t_1, t_2 \end{pmatrix}; \lambda_0\Big), \ldots,$$

and find the first term in it which does not vanish identically. Let

$$D\left(\begin{matrix} s_1, \ldots, s_q \\ t_1, \ldots, t_q \end{matrix}; \lambda_0\right) \not\equiv 0.$$

The number q, which can be shown not to exceed r, where r is the multiplicity of the root $\lambda = \lambda_0$ of the equation $D(\lambda) = 0$, is the rank of the eigenvalue λ_0. If s_i' and t_i' are values of the variables s_i and t_i such that the numerical inequality

$$D\left(\begin{matrix} s_1', \ldots, s_q' \\ t_1', \ldots, t_q' \end{matrix}; \lambda_0\right) \neq 0$$

holds, the complete set of linearly independent (in general, non-orthogonal and non-normalized) eigenfunctions corresponding to the eigenvalue λ_0 is given by the expression

$$\varphi_k(s) = D\left(\begin{matrix} s_1', \ldots, s_{k-1}', s, s_{k+1}', \ldots, s_q' \\ t_1', \ldots, t_{k-1}', t_k', t_{k+1}', \ldots, t_q' \end{matrix}; \lambda_0\right) \quad (k = 1, 2, \ldots, q),$$

whilst for the adjoint equation the following set of eigenfunctions corresponds to the same eigenvalue:

$$\psi_k(s) = D\left(\begin{matrix} s_1', \ldots, s_{k-1}', s_k', s_{k+1}', \ldots, s_q' \\ t_1', \ldots, t_{k-1}', s, t_{k+1}', \ldots, t_q' \end{matrix}; \lambda_0\right) \quad (k = 1, 2, \ldots, q).$$

12. Degenerate equations. We shall now mention a class of integral equations, the solutions of which reduce to algebraic equations of the first degree. The kernel $K(s, t)$ is said to be *degenerate* if it consists of a finite sum of products of functions of s only and of t only:

$$K(s,t) = \sum_{k=1}^{n} \varrho_k(s)\, \sigma_k(t). \tag{85}$$

The functions $\varrho_k(s)$, like the functions $\sigma_k(t)$, can be assumed to be linearly independent. For, if some $\varrho_p(s)$ could be expressed linearly in terms of the remaining $\varrho_k(s)$, we could substitute this expression for $\varrho_p(s)$ in (85). The number of terms would thus be diminished.

Let us take an equation with such a kernel and the adjoint equation:

$$\left.\begin{aligned} \varphi(s) &= f(s) + \lambda \int_a^b K(s, t)\, \varphi(t)\, dt; \\ \psi(s) &= g(s) + \lambda \int_a^b K(t, s)\, \psi(t)\, dt. \end{aligned}\right\} \tag{86}$$

We obtain on taking (85) into account:

$$\left.\begin{aligned} \varphi(s) &= f(s) + \lambda \sum_{k=1}^{n} \varrho_k(s) \int_a^b \sigma_k(t)\, \varphi(t)\, \mathrm{d}t, \\ \psi(s) &= g(s) + \lambda \sum_{k=1}^{n} \sigma_k(s) \int_a^b \varrho_k(t) \psi(t)\, \mathrm{d}t, \end{aligned}\right\} \tag{87}$$

or

$$\varphi(s) = f(s) + \lambda \sum_{k=1}^{n} x_k\, \varrho_k(s), \quad \psi(s) = g(s) + \lambda \sum_{k=1}^{n} y_k\, \sigma_k(s), \tag{88}$$

where x_k and y_k are certain *numbers* given by

$$x_k = \int_a^b \sigma_k(t)\, \varphi(t)\, \mathrm{d}t; \qquad y_k = \int_a^b \varrho_k(t)\, \psi(t)\, \mathrm{d}t.$$

Thus every solution of equations (87) must have the form (88), and the entire problem reduces to finding numbers x_k and y_k instead of functions.

On substituting expressions (88) in equations (87) and equating coefficients for the linearly independent functions $\varrho_k(s)$ and $\sigma_k(s)$, we get two systems of equations for x_k and y_k:

$$x_i - \lambda \sum_{k=1}^{n} a_{ik}\, x_k = f_i, \tag{89_1}$$

$$y_i - \lambda \sum_{k=1}^{n} a_{ki}\, y_k = g_i, \tag{89_2}$$

where

$$a_{ik} = \int_a^b \sigma_i(s)\, \varrho_k(s)\, \mathrm{d}s; \quad f_i = \int_a^b f(s)\, \sigma_i(s)\, \mathrm{d}s; \quad g_i = \int_a^b g(s)\, \varrho_i(s)\, \mathrm{d}s. \tag{90}$$

The determinants of systems (89_1) and (89_2) only differ in the rows being replaced by columns.

If, for instance, the determinant of system (89_1) differs from zero, we obtain with any f_i definite values for the x_i. On substituting these in (88), we obtain $\varphi(s)$. For the homogeneous equations

$$\varphi(s) = \lambda \int_a^b K(s,t)\, \varphi(t)\, \mathrm{d}t; \quad \psi(s) = \lambda \int_a^b K(t,s)\, \psi(t)\, \mathrm{d}t$$

we have the corresponding homogeneous systems:

$$x_i - \lambda \sum_{k=1}^{n} a_{ik}\, x_k = 0, \tag{91_1}$$

$$y_i - \lambda \sum_{k=1}^{n} a_{ki}\, y_k = 0 \quad (i = 1,\ 2, \ldots,\ n). \tag{91_2}$$

On equating the determinant of one of these systems (no matter which) to zero, we get an algebraic equation for the eigenvalues. If $\lambda = \lambda_0$ is any root of this equation, system (91_1) has a solution $(x_1, x_2, \ldots, x_n)$ different from zero, and substitution of it in the formula

$$\varphi(s) = \lambda_0 \sum_{k=1}^{n} x_k \varrho_k(s), \tag{92}$$

gives us the eigenfunction.

The theorems proved above reduce in the present case to the familiar theorems of linear algebra [III_1; 8, 9, 10, 15].

It may be mentioned that a homogeneous system (91_1) can also be obtained for non-homogeneous equations (87) provided all the numbers f_i vanish, i.e.

$$\int_a^b f(s)\, \sigma_i(s)\, ds = 0 \quad (i = 1,\ 2, \ldots, n). \tag{93}$$

If λ is not an eigenvalue in this case, system (91_1) gives us only a zero solution, and, by (88), we get $\varphi(s) = f(s)$. This solution can be checked by substituting it directly in (87), if we take (93) into account. Degenerate kernels are used for the approximate solution of integral equations, the given kernel being replaced by a degenerate kernel close to it, then the resulting degenerate equation solved with the aid of the above algebraic method. This method of approximate solution of integral equations is described, with other methods, in *Approximation Methods of Advanced Analysis* (Priblizhennye metody vysshego analiza) by L. V. Kantorovich and V. I. Krylov (1950).

Methods of reduction to degenerate equations are also used in expounding the theory of integral equations. Books using such methods include S. L. Sobolev, *Partial Differential Equations of Mathematical Physics*, Pergamon Press, 1964, I. G. Petrovskii, *Lectures on the Theory of Integral Equations*, Courant and Hilbert, "Methoden der mathematischen Physik", Vol. 1.

13. Examples. 1. Let

$$K(s, t) = \cos(s+t) = \cos s \cos t - \sin s \sin t \begin{pmatrix} 0 \leqslant s \leqslant \pi \\ 0 \leqslant t \leqslant \pi \end{pmatrix}.$$

In this case

$$\varrho_1(s) = \sigma_1(s) = \cos s; \quad \varrho_2(s) = \sigma_2(s) = i \sin s,$$

where the imaginary factor i only features in the intermediate working. We obtain for the a_{ik}:

$$a_{11} = \int_0^\pi \cos^2 s\, ds = \frac{\pi}{2}; \quad a_{12} = a_{21} = 0; \quad a_{22} = -\int_0^\pi \sin^2 s\, ds = -\frac{\pi}{2}.$$

System (89_1) becomes

$$\left(1 - \lambda\frac{\pi}{2}\right) x_1 = f_1; \quad \left(1 + \lambda\frac{\pi}{2}\right) x_2 = f_2.$$

There are two eigenvalues $\lambda_{1,2} = \pm 2/\pi$, and the corresponding normalized eigenfunctions are

$$\varphi_1(s) = \sqrt{\frac{2}{\pi}} \cos s, \quad \varphi_2(s) = \sqrt{\frac{2}{\pi}} \sin s.$$

2. Let

$$K(s, t) = st + s^2 t^2 \quad \begin{pmatrix} -1 \leqslant s \leqslant 1 \\ -1 \leqslant t \leqslant 1 \end{pmatrix}.$$

In this case $\varrho_1(s) = \sigma_1(s) = s$; $\varrho_2(s) = \sigma_2(s) = s^2$ and

$$a_{11} = \frac{2}{3}; \quad a_{12} = a_{21} = 0; \quad a_{22} = \frac{2}{5}.$$

There are two eigenvalues $\lambda_1 = 3/2$ and $\lambda_2 = 5/2$, the corresponding eigenfunctions being

$$\varphi_1(s) = \sqrt{\frac{3}{2}}\, s; \quad \varphi_2(s) = \sqrt{\frac{5}{2}}\, s^2.$$

In both examples the kernel $K(s, t)$ has been real and has satisfied the condition $K(t, s) = K(s, t)$. Such kernels only have real eigenvalues.

The theory of integral equations with symmetric kernels will be given below. Such equations have wide applications in mathematical physics.

3. We now give an example of a degenerate real kernel with imaginary eigenvalues. Let

$$K(s, t) = s - t \quad \begin{pmatrix} 0 \leqslant s \leqslant 1 \\ 0 \leqslant t \leqslant 1 \end{pmatrix}.$$

Here we can take

$$\varrho_1(s) = s; \quad \varrho_2(s) = -1; \quad \sigma_1(t) = 1; \quad \sigma_2(t) = t,$$

so that

$$a_{11} = \frac{1}{2}; \quad a_{12} = -1; \quad a_{21} = \frac{1}{3}; \quad a_{22} = -\frac{1}{2}.$$

We obtain the following equation for the eigenvalues:

$$\begin{vmatrix} 1 - \frac{1}{2}\lambda & \lambda \\ -\frac{1}{3}\lambda & 1 + \frac{1}{2}\lambda \end{vmatrix} = \frac{1}{12}\lambda^2 + 1 = 0,$$

which has pure imaginary roots. In this example the real kernel satisfies the condition $K(t, s) = -K(s, t)$.

Such skew-symmetric kernels only have pure imaginary eigenvalues.

4. A further example may be mentioned, of a degenerate kernel having no eigenvalues. Let

$$K(s, t) = \sin s \sin 2t \quad \begin{pmatrix} 0 \leqslant s \leqslant \pi \\ 0 \leqslant t \leqslant \pi \end{pmatrix}.$$

Here, $n = 1$ and the only element a_{ik} will be

$$a_{11} = \int_0^\pi \sin s \sin 2s \, \mathrm{d}s = 0.$$

Homogeneous systems (91_1) and (92_2) give us $x_1 = y_1 = 0$, and the homogeneous equation only has a zero solution with any λ. The equation for the eigenvalues reduces here to the absurdity $1 = 0$.

14. Generalization of the results obtained. We have assumed in our description of the theory of integral equations that the required function $\varphi(s)$ and the function $f(s)$ are functions of a single independent variable which can vary in an interval $[a, b]$. This interval was also the interval of variation for both the arguments of the kernel $K(s, t)$. The entire theory remains quite unchanged if we assume that $\varphi(M)$ and $f(M)$ are functions of a point in a bounded domain B of any number of dimensions or on a surface or a curve. The kernel $K(M, N)$ now becomes a function of a pair of points M and N, each of which can vary in the domain or on the surface or curve, whilst the integral sign in the equation has to be understood as referring to integration over the domain or surface or curve, so that the equation becomes

$$\varphi(M) = f(M) + \int_B K(M, N)\, \varphi(N) \, \mathrm{d}\omega_N. \tag{94}$$

We have only written one integral sign though it has to be remembered that the integral may be iterated over the domain, and that $\mathrm{d}\omega_N$ denotes an element of area or volume of the domain or an elementary length of arc of the curve. For instance, if the domain of variation is a bounded domain B on the (x, y) plane, equation (94) can be written in coordinate form as follows:

$$\varphi(x, y) = f(x, y) + \iint_B K(x, y;\ \xi, \eta)\, \varphi(\xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta.$$

We assume the function $f(M)$ to be continuous in the closed domain B and seek solutions $\varphi(M)$ continuous in this domain. The kernel

$K(M, N)$ is taken to be a continuous function of the pair of points. (M, N), with each varying in the closed domain B.

We now consider a system of m integral equations in the same number of required functions:

$$\varphi_i(s) = f_i(s) + \int_a^b \sum_{j=1}^{m} K_{ij}(s,t)\,\varphi_k(t)\,dt \quad (i = 1, 2, \ldots, m).$$

We have here, instead of a kernel, a matrix of functions $K_{ik}(s, t)$

The above system is readily reduced to a single integral equation with a single required function. To avoid unnecessary complexity in the notation we shall put $m = 2$:

$$\begin{aligned} \varphi_1(s) &= f_1(s) + \int_a^b [K_{11}(s,t)\,\varphi_1(t) + K_{12}(s,t)\,\varphi_2(t)]\,dt, \\ \varphi_2(s) &= f_2(s) + \int_a^b [K_{21}(s,t)\,\varphi_1(t) + K_{22}(s,t)\,\varphi_2(t)]\,dt. \end{aligned} \tag{95}$$

We remarked above that the entire theory of integral equations remains unchanged if, instead of taking an interval as the basic domain, we take any bounded domain on a plane, on a surface or in space. It can also be snpposed that the variable point runs over several separate segments or domains instead of over a single segment or domain. This again leaves the theory quite unchanged. To reduce system (95) to a single equation, we take as the basic domain two specimens of the interval $[a, b]$. These specimens are not connected in any way. We take $f(M) = f_1(M)$ if the point M is on the first specimen, and $f(M) = f_2(M)$ if on the second specimen. We define $\varphi(M)$ similarly in terms of $\varphi_1(M)$ and $\varphi_2(M)$. The kernel $K(M, N)$ is defined as follows:

$$\begin{array}{cc} K(M,N) = K_{11}(M,N) & K(M,N) = K_{12}(M,N) \\ (M \text{ and } N \text{ on 1st specimen}) & (M \text{ on 1st spec., } N \text{ on 2nd}) \\ K(M,N) = K_{21}(M,N) & K(M,N) = K_{22}(M,N) \\ (M \text{ on 2nd spec., } N \text{ on 1st}) & (M \text{ and } N \text{ on 2nd spec.}) \end{array} \tag{96}$$

System (94) now reduces to a single integral equation with continuous kernel in the basic domain J, consisting of the two specimens

of the segment $[a, b]$:

$$\varphi(M) = f(M) + \int_J K(M, N)\, \varphi(N)\, \mathrm{d}\omega_N . \tag{97}$$

The integration is carried out over both specimens of $[a, b]$, and we can take $\mathrm{d}\omega_N = \mathrm{d}x$.

The theory described remains valid with more general assumptions regarding the kernel than continuity. Suppose, for instance, that the kernel $K(s, t)$ has a finite number of points and curves of discontinuity, but remains bounded in the square k_0, and that a number N exists such that, with any fixed value $s = s_0$, there are at most N points on the segment $s = s_0$ $(a \leqslant t \leqslant b)$, at which $K(s, t)$ is not a continuous function in both its variables.

It may be easily shown that in this case the integral

$$\omega(s) = \int_a^b K(s, t)\, h(t)\, \mathrm{d}t \tag{98}$$

defines a continuous function of s, provided $h(t)$ is a bounded function with a finite number of discontinuities. We shall assume for simplicity that all the discontinuities of the kernel lie on a diagonal of the square $t = s$. Taking into account that $|\, h(t)\, | \leqslant m$ by hypothesis, where m is a positive number, we can write

$$|\omega(s) - \omega(s')\,| \leqslant m \int_a^b |K(s, t) - K(s', t)|\, \mathrm{d}t. \tag{99}$$

Given any positive ε, there exists by virtue of the boundedness of the integrand, a positive δ such that the integral written is less than ε in the interval $[s - \delta, s + \delta]$. Let s' be situated inside this interval. The integrand will be a continuous function of the two variables s' and t during the integration over the remaining intervals $[a, s - \delta]$ and $[s + \delta, b]$, so that, for all s' sufficiently close to s, the integrals over these two intervals will also be less than ε. Hence it follows that the left-hand side of inequality (99) will be less than $3m\varepsilon$ for all s' sufficiently close to s, which shows, in view of the arbitrariness of ε, that $\omega(s)$ is a continuous function. It can similarly be shown that, if $K'(s, t)$ and $K''(s, t)$ are two kernels satisfying the conditions stated above, the function

$$K'''(s, t) = \int_a^b K''(s, t_1)\, K'(t_1, t)\, \mathrm{d}t_1$$

will be a continuous function of both its arguments. Hence, if the kernel $K(s, t)$ satisfies the above conditions, the second iterated kernel will in fact be continuous. It may easily be seen that, under the assumptions made regarding the kernel, the entire theory of the equations is preserved without any modifications to the proofs.

It may further be remarked that, if we seek bounded solutions with a finite number of discontinuities, on condition that $f(s)$ is continuous, it now follows from the continuity of integral (98) proved above that, by virtue of the equation itself, the solution must in fact be continuous [cf. 4].

When proving the fundamental theorems, we have had occasion to change the order of integration in the iterated integrals. This is justifiable with the assumptions made above regarding the kernel. Everything that has been said still holds in the theory of integral equations with multiple integrals. Corresponding to discontinuities on the diagonal $s = t$, we now have discontinuities of the kernel $K(M, N)$ when points M and N coincide.

The problem becomes far more difficult if the kernel is unbounded. Such kernels are, however, frequently encountered in the applications of integral equations to mathematical physics. It is of importance to distinguish those unbounded kernels for which the theorems proved above for continuous kernels remain valid. We shall now go into this matter.

15. The selection principle. We shall discuss in this section and the next the so-called selection principle, which we need for the further development of the theory of integral equations.

Let $\mathfrak{E}$ be an infinite set of real numbers, the absolute values of which do not exceed some definite positive number. We know that a convergent subsequence a_{n_k} can be chosen from any infinite sequence a_n of numbers of $\mathfrak{E}$ [II, 89]. The same can evidently be said for any infinite part of $\mathfrak{E}$. This assertion is known as the *selection principle* for sets of real numbers bounded in absolute value by the same number. The same principle holds for sets of complex numbers, whose moduli are bounded by the same positive number. This may be seen simply by first applying the selection principle for the real parts of the numbers of $\mathfrak{E}$, then next, for the imaginary parts of the numbers of the sequence obtained. Our task is to answer the question as to the conditions in which the selection principle holds for sets of functions, the discussion being confined here to uniform convergence

of the sequence of functions to the limit function. To answer this question we require a number of new concepts and auxiliary propositions.

Suppose we have an infinite set of any objects (elements). This set is said to be *denumerable* if all the objects can be enumerated by positive integers, i.e. for every positive integer there is a corresponding object of the set, and conversely, for every object of the set there is a corresponding definite positive integer.

In other words, a set is denumerable if its elements can be represented as a sequence: $u_1, u_2, u_3, \ldots$ An important example for what follows is provided by the set of all real rational numbers. Let us prove that this set is denumerable. We arrange all the positive rational numbers in such an order that the sum of numerator and denominator is non-decreasing, and so that the denominator is increasing in any group where the sum in question has a constant value. We shall include reducible fractions. We thus obtain the sequence of numbers:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{4}{2}, \ldots$$

On discarding the numbers which have already made an appearance, we obtain a sequence containing all the positive rational numbers:

$$1,\ 2,\ \frac{1}{2},\ 3,\ \frac{1}{3},\ 4,\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4},\ 5, \ldots$$

Every positive rational number thus receives an index, indicating the position which it occupies in this sequence. All the real rational numbers also form a denumerable set. For, suppose we take zero as the first number, then write down the above sequence with the number of opposite sign interposed after each number of the sequence, we obtain:

$$0,\ 1,\ -1,\ 2,\ -2, \frac{1}{2},\ -\frac{1}{2},\ 3,\ -3, \frac{1}{3},\ -\frac{1}{3}, \ldots$$

If we strike out certain terms from any sequence $u_1, u_2, \ldots$ in such a way that an infinity of terms remains, these remaining terms again form an infinite sequence $u_{n_1}, u_{n_2}, \ldots$ and they can be re-enumerated. It follows from this that any part of a denumerable set containing an infinite set of elements is itself a denumerable set.

For instance, the set of rational numbers belonging to any interval $[a, b]$ is a denumerable set.

It may be remarked that there is an infinite set of rational numbers in any fixed interval of the x axis, no matter how small, or, as it is

generally expressed, *the rational numbers are everywhere dense on the x axis.*

No rational number has one following it in magnitude, and the rational numbers are not arranged in increasing or decreasing order in the above sequence.

It can be shown that all the real numbers of the interval $[a, b]$ form a set which is not denumerable.

We now consider a domain B of the plane referred to coordinates XY. We show that the set of points of B at which both coordinates (x, y) are rational numbers is a denumerable set.

We can first enumerate, say, all the points $(p/q, r/s)$ with rational coordinates. Since the rational numbers are denumerable, the points with rational coordinates can be written as (u_m, v_n) $(m, n = 1, 2, \ldots)$. These pairs of numbers can be enumerated according to the sum of the subscripts, and with the first subscript increasing in a given sum:

$$(u_1, v_1,), (u_1, v_2), (u_2 v_1), (u_1, v_3), (u_2, v_2), (u_3, v_1), \ldots$$

The set $\mathfrak{E}$ of points with rational coordinates belonging to the domain B is an infinite part of a denumerable set, i.e. is also a denumerable set. This set $\mathfrak{E}$ is everywhere dense in B, i.e. an infinite set of points of $\mathfrak{E}$ lies in any circle with centre at a point belonging to B. It may be shown in precisely the same way that the set of points $(x_1, x_2, \ldots, x_n)$ with rational coordinates belonging to a domain B of n-dimensional space is a denumerable set everywhere dense in B.

To show that there is a denumerable everywhere dense set of points on a surface S, all we need to do is say divide the surface into a finite number of pieces, each of which has an explicit equation $z = f(x, y)$ when the tangent plane at some point of the piece is taken as the XY plane. A denumerable everywhere dense set is now obtained on each piece if, for example, we choose points (x, y) of the tangent plane with rational coordinates. Let the number of pieces be p. We have an everywhere dense sequence of points on each piece:

$$a_1^{(s)}, a_2^{(s)}, a_3^{(s)}, \ldots \qquad (s = 1, 2, \ldots, p).$$

We can arrange them in a single sequence

$$a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(p)}, a_2^{(1)}, a_2^{(2)}, \ldots, a_2^{(p)}, \ldots$$

and discard any points encountered more than once. There will be an infinite set of the above enumerated points of the surface in any sphere with centre lying on S. We now prove an auxiliary proposition.

LEMMA. *If $f_n(x)$ is a sequence of functions given in an interval $[a, b]$ and bounded in modulus by the same number L, a subsequence can always be extracted from the sequence such that it is convergent on any denumerable set whatever of points x_k $(k = 1, 2, \ldots)$ of $[a, b]$.*

By hypothesis, $|f_n(x)| \leqslant L$ $(n = 1, 2, \ldots)$, and we can extract a convergent sequence from the sequence of numbers $f_n(x_1)$, i.e. we can extract from the sequence of functions $f_n(x)$ the subsequence

$$f_1^{(1)}(x),\ f_2^{(1)}(x),\ f_3^{(1)}(x), \ldots, \tag{I}$$

which is convergent at the point $x = x_1$. If we put $x = x_2$ for functions (I), we obtain numbers $f_k^{(1)}(x_2)$ whose moduli also do not exceed L. Hence we can extract from the function sequence (I) the subsequence

$$f_1^{(2)}(x),\ f_2^{(2)}(x),\ f_3^{(2)}(x), \ldots, \tag{II}$$

which is convergent at $x = x_2$ as well as at $x = x_1$, inasmuch as it is extracted from sequence (I) which is convergent at $x = x_1$. On setting $x = x_3$, all the numbers $f_k^{(2)}(x_3)$ are seen to have moduli less than or equal to L, and we can extract from sequence (II) the new subsequence

$$f_1^{(3)}(x),\ f_2^{(3)}(x),\ f_3^{(3)}(x), \ldots, \tag{III}$$

which will be convergent at the points $x = x_1$, $x = x_2$, $x = x_3$. On continuing this construction, we arrive in general at the sequences

$$f_1^{(m)}(x),\ f_2^{(m)}(x),\ f_3^{(m)}(x), \ldots \quad (m = 1,\ 2,\ 3, \ldots), \tag{m}$$

which are convergent at the points $x = x_1$, $x = x_2$, $\ldots$, $x = x_m$. We now form a new sequence by taking the first function from sequence (I), the second function from sequence (II), the third function from sequence (III), and so on:

$$f^{(1)}(x) = f_1^{(1)}(x),\ f^{(2)}(x) = f_2^{(2)}(x),\ f^{(3)}(x) = f_3^{(3)}(x), \ldots,$$
$$f^{(n)}(x) = f_n^{(n)}(x), \ldots \tag{*}$$

We show that this subsequence is now convergent at any point $x = x_k$. In fact, let us take the point $x = x_k$. All the functions of sequence (*) as from the index $m = k$, i.e. all the functions

$$f^{(k)}(x) = f_k^{(k)}(x),\ f^{(k+1)}(x) = f_{k+1}^{(k+1)}(x), \ldots, \tag{**}$$

form by virtue of the above construction a part of sequence (m) with $m = k$, so that we obtain on substituting the value $x = x_k$ in the sequence (*) a convergent sequence of numbers, i.e. the sequence of functions (**) is convergent at the point $x = x_k$. The same can be said

as regards sequence (*), which in fact proves the lemma. The process used in proving the lemma, of constructing sequences of functions convergent at all the points $x = x_k$, is generally described as a *diagonal process*. It is clearly not a concrete constructional process and is of purely theoretical value.

The above proof is suitable for complex as well as real functions $f_n(x)$. The proof of the lemma for functions $f_n(P)$, given in a domain B of n-dimensional space or on a surface, is a word-for-word repetition of the above.

16. The selection principle (continued). Let $f(x)$ be a continuous function in a finite interval $[a, b]$. We know that it is uniformly continuous, i.e. given any positive ε, there exists a positive η such that $|f(x') - f(x'')| \leqslant \varepsilon$ for any points x', x'' of $[a, b]$ such that $|x' - x''| \leqslant \eta$. Given the same ε, the numbers η will in general be different for different functions continuous in $[a, b]$. If we have a finite number of continuous functions $f_1(x), f_2(x), \ldots, f_m(x)$, there will be a minimum among the numbers $\eta_1, \eta_2, \ldots, \eta_m$ corresponding to a given ε. Let us call this η'. We can now assert that $|f_k(x') - f_k(x'')| \leqslant \varepsilon$ with $k = 1, 2, \ldots, m$, provided only that $|x' - x''| \leqslant \eta'$. But if we have an infinite set $\mathfrak{E}$ of continuous functions $f(x)$, there may not be a minimum among the positive numbers η corresponding to them. It may happen, in addition, that these positive numbers indefinitely approach zero with a given ε. It now becomes impossible to choose an η' which is the same for all the $f(x)$ of $\mathfrak{E}$. For instance, in the case of the functions $f_n(x) = \sin nx$ $(n = 1, 2, \ldots)$, given ε, the number η evidently tends to zero on indefinite increase of n. This follows at once from the fact that, when the independent variable x varies by an amount δ, the argument of the sine changes by $n\delta$.

DEFINITION. *A set $\mathfrak{E}$ of functions $f(x)$, continuous in a closed interval $[a, b]$, is said to be a set of equicontinuous functions if, given any positive ε, there exists the same positive η for all the functions of $\mathfrak{E}$ such that $|f(x') - f(x'')| \leqslant \varepsilon$ provided that x' and x'' belong to $[a, b]$ and $|x' - x''| \leqslant \eta$.*

If the functions are equicontinuous and bounded in modulus by the same number, we can prove a selection principle, convergence being understood here as uniform convergence in $[a, b]$, i.e. the following theorem holds:

THEOREM 1. *If $\mathfrak{E}$ is a set of functions $f(x)$ equicontinuous in a finite interval $[a, b]$, the moduli of all the functions being bounded by the same*

number L, i.e. $|f(x)| \leqslant L$, *a subsequence uniformly convergent in* $[a, b]$ *can be extracted from any sequence of functions of* $\mathfrak{E}$.

Suppose we have a sequence of functions of $\mathfrak{E}$. On applying Lemma 1, we can say that a subsequence can be extracted from the sequence such that it tends to a limit at all the points $x_k$of a denumerable set of points everywhere dense in $[a, b]$. These points may be, for example, all the points of $[a, b]$ with rational abscissae. Let

$$f_1(x),\ f_2(x),\ f_3(x), \ldots \qquad (*)$$

be the extracted subsequence of the given sequence of functions of the set $\mathfrak{E}$ which is convergent at all the above points x_k $(k = 1, 2, 3, \ldots)$. We show that this sequence is uniformly convergent throughout the interval $[a, b]$. We form the difference $f_p(x) - f_q(x)$ and write it as

$$f_p(x) - f_q(x) = [f_p(x) - f_p(x')] + [f_p(x') - f_q(x')] + \\ + [f_q(x') - f_q(x)], \qquad (\alpha)$$

where x' is one of the points of the above-mentioned set, everywhere dense in $[a, b]$. Let ε be any given positive number and η the number corresponding to it in the definition of equicontinuity. We take a finite set τ'consisting of points x_k and such that the points of the finite set divide the interval $[a,b]$ into subintervals, the lengths of which are $\leqslant \eta$. This is obviously possible, since the set of all the points x_k is everywhere dense in $[a, b]$. At each point of this finite set τ' the sequence (*) has a limit. Hence a number N exists such that

$$|f_p(x') - f_q(x')| < \varepsilon \text{ with } p \text{ and } q > N, \qquad (\beta)$$

if x' is a point of the finite set τ'. We shall assume that the point x' appearing in formula (α) is a point of finite set τ', and we write down the inequality

$$|f_p(x) - f_q(x)| \leqslant |f_p(x) - f_p(x')| + |f_p(x') - f_q(x')| + \\ + |f_q(x') - f_q(x)|, \qquad (\gamma)$$

which follows directly from (α). For any position of x on $[a, b]$ we can indicate an x' belonging to τ' such that $|f_n(x) - f_n(x')| < \varepsilon$ for any n. This x' will be one end of the subinterval to which x belongs. In addition, with p and $q > N$, we have inequality (β) for any x' belonging to τ'. Hence, by (γ), we can assert the following: given any positive ε, an N exists, independent of x, such that $|f_p(x) - f_q(x)| < 3\varepsilon$

for p and $q > N$ and any x of $[a, b]$, and this in fact shows that sequence (*) is uniformly convergent throughout $[a, b]$, which proves the theorem.

The proof is evidently suitable for complex as well as real functions. If the sequence $f_n(x)$ of equicontinuous functions is known to be convergent at every point of the interval $[a, b]$ or at points x_k of a set everywhere dense in $[a, b]$, the need disappears for the extraction of a subsequence convergent at all points x_k, and the following can be asserted:

THEOREM 2. *If a sequence $f_1(x), f_2(x), \ldots$ of functions equicontinuous in an interval $[a, b]$ is convergent at all points of the interval (or even only at points x_k of a set everywhere dense in $[a, b]$), the sequence is uniformly convergent in $[a, b]$.* The proof of the theorems can be carried over word for word to the case of a set $\mathfrak{E}$ of functions $f(P)$ defined in a closed domain B in n-dimensional space or on a surface. Equicontinuity is evidently defined here as follows: given any positive ε, there exists a positive number η which is the same for all the functions of $\mathfrak{E}$, such that $| f(P) - f(Q) | \leqslant \varepsilon$ if P and Q belong to B and the distance $| PQ | \leqslant \eta$. The fact that domain B is closed implies that the domain includes its boundary [II, 88]. The proof is also retained without change when B consists of several separate closed domains.

17. Unbounded kernels. The theorems on integral equations proved above may cease to be valid if the kernel is unbounded. The theorems remain valid, however, given certain supplementary conditions, even when the kernel is unbounded. Our present purpose is to distinguish the relevant class of unbounded kernels. The general theory of integral equations, including both the case of unbounded kernels and the case of an infinite domain of integration, will be given in Vol. V on the basis of a more general concept of the integral (the Lebesgue integral). For definiteness, we shall carry out the discussion for the case of a plane. The whole of it is readily extended to the case of any n-dimensional space or of integration over a surface.

We shall assume that the kernel $K(M; N)$ tends to infinity only when the points M and N coincide. This is the type of kernel most commonly encountered in mathematical physics.

Thus we take a kernel of the form

$$K(M;\ N) = \frac{L(M;\ N)}{r^{\alpha}}, \tag{100}$$

where $L(M, N)$ is a continuous function of the pair of points (M, N) in the bounded closed domain B, r is the distance between points M

and N, and the number a satisfies the condition $0 < a < 2$. We shall describe this type of kernel as *polar*. It follows from (100) that

$$|K(M; N)| \leqslant \frac{C}{r^a}, \tag{101}$$

where C is a constant. We shall establish as a preliminary certain properties of polar kernels. We shall in future write B for a closed domain, which is naturally assumed to be measurable [II, 91].

Let d be the diameter of B, i.e. the greatest distance between points of B [II, 89]. The domain B is contained in a circle of radius d and centre at any point M of B, so that we have

$$\int_B |K(M; N)|\, d\omega_N \leqslant \int_{r \leqslant d} \frac{C}{r^a}\, d\omega_N = \frac{2\pi C}{2-a} d^{2-a},$$

i.e.

$$\int_B |K(M; N)|\, d\omega_N \leqslant D, \tag{102}$$

where $D = [2\pi C/(2-a)]\, d^{2-a}$.

Let $u(N)$ be a function continuous in B and

$$v(M) = \int_B K(M; N)\, u(N)\, d\omega_N. \tag{103}$$

The integral written is obviously meaningful in view of (100). We prove that function $v(M)$ is continuous:

$$v(M') - v(M) = \int_B [K(M'; N) - K(M; N)]\, u(N)\, d\omega_N.$$

For a continuous function $u(N)$, we have the inequality in B:

$$|u(N)| \leqslant C_1, \tag{104}$$

where C_1 is constant. Further

$$|v(M') - v(M)| \leqslant C_1 \int_B |K(M';\ N) - K(M; N)|\, d\omega_N. \tag{105}$$

We shall write ω_ϱ and ω'_ϱ in future for the circles with centres M and M' and radius ϱ. Let δ be a small positive number, which will be chosen later. We draw the circle $\omega_{2\delta}$, and let β_δ be the part of domain B belonging to the circle $\omega_{2\delta}$ and B_δ the part of B lying outside $\omega_{2\delta}$. We can write:

$$\int_B |K(M'; N) - K(M; N)| d\omega_N = \int_{\beta_\delta} |K(M'; N) - K(M; N)|\, d\omega_N + \\ + \int_{B_\delta} |K(M; N) - K(M'; N)|\, d\omega_N. \tag{106}$$

We remark that the sets β_δ and B_δ are measurable, and the formula written follows at once from what was said in [II, 96].

We have by (101):

$$\int_{\beta_\delta} |K(M'; N) - K(M; N)|\, d\omega_N \leqslant C \int_{\omega_{2\delta}} \frac{1}{r'^a}\, d\omega_N + C \int_{\omega_{2\delta}} \frac{1}{r^a}\, d\omega_N,$$

where r' is the distance between M' and N. We shall assume that the distance between M' and M is less than δ. The circle $\omega_{2\delta}$ must obviously lie in this case inside the circle $\omega'_{3\delta}$, so that

$$\int_{\beta_\delta} |K(M'; N) - K(M; N)|\, d\omega_N \leqslant C \int_{\omega_{2\delta}} \frac{1}{r^a}\, d\omega_N + C \int_{\omega'_{3\delta}} \frac{1}{r'^a}\, d\omega_N.$$

On passing to polar coordinates in each circle and evaluating the integrals, we obtain:

$$\int_{\beta_\delta} |K(M'; N) - K(M; N)|\, d\omega_N \leqslant \frac{2\pi C}{2-a} [(2\delta)^{2-a} + (3\delta)^{2-a}]. \quad (107)$$

Let ε be a given positive number. We fix δ so small that the right-hand side of (107) is $\leqslant \varepsilon/2C_1$.

We turn to the second term in (106). If M' lies in the closed circle ω_δ and N in the closed domain B_δ, then r and $r' \geqslant \delta$, and

$$|K(M'; N) - K(M; N)| = \frac{|r^a L(M'; N) - r'^a L(M; N)|}{r^a r'^a} \leqslant$$
$$\leqslant \frac{|L_1(M', M; N)|}{\delta^{2a}},$$

where $L_1(M', M; N)$ is a continuous function of M', M and N in B, vanishing when M' coincides with M. Hence it follows that a positive number η exists, which does not depend on the position of the point M and which can be assumed not greater than δ, such that

$$\int_{B_\delta} |K(M'; N) - K(M; N)|\, d\omega_N \leqslant \frac{\varepsilon}{2C_1}, \quad (108)$$

if the distance $|MM'|$ is not greater than η. On taking (107) into account, we get

$$|v(M') - v(M)| \leqslant \varepsilon, \quad \text{if} \quad |MM'| \leqslant \eta,$$

which proves the continuity of $v(M)$ in B. We remark further that the number η depends only on ε and C_1, and does not depend on the

actual choice of $u(N)$, i.e. with fixed C_1 we get a family of equicontinuous functions $v(M)$ for all the functions $u(N)$ satisfying condition (104).

It follows at once from (102), (103) and (104) that

$$|v(M)| \leqslant C_1 D.$$

We thus obtain the following result:

LEMMA 1. *The right-hand side of* (103) *transforms continuous functions* $u(N)$ *into continuous functions* $v(M)$. *If the functions* $u(N)$ *are bounded in modulus by the same number* C_1, *the class of equicontinuous functions* $v(M)$ *obtained consists of functions which are bounded in modulus by the same number.*

We introduce a continuous kernel $K_\gamma(M; N)$ which is close in the familiar sense to kernel $K(M; N)$, i.e. we put

$$K_\gamma(M; N) = \begin{cases} K(M; N) & \text{with } r \geqslant \gamma \\ \dfrac{L(M; N)}{\gamma^\alpha} & \text{with } r \leqslant \gamma, \end{cases} \tag{109}$$

where γ is any positive number. The kernel $K_\gamma(M; N)$ differs from kernel $K(M; N)$ only for $r < \gamma$ and $|K_\gamma(M; N)| \leqslant |K(M; N)|$, so that by (101),

$$|K_\gamma(M; N)| \leqslant \frac{C}{r^\alpha}; \tag{110_1}$$

$$\int_B |K_\gamma(M; N)| \, d\omega_N \leqslant D. \tag{110_2}$$

Let us consider, along with transformation (103), the transformation

$$v_\gamma(M) = \int_B K_\gamma(M; N) \, u(N) \, d\omega. \tag{111}$$

The continuity of $v_\gamma(M)$ is evident from the continuity of the kernel $K_\gamma(M; N)$. We write down inequalities similar to the above. By (110_1), we have the previous inequality for the integral over β_δ. It remains to write an inequality, with fixed positive δ, for the integral

$$\int_{B_\delta} |K_\gamma(M'; N) - K_\gamma(M; N)| \, d\omega_N. \tag{112}$$

This can be done in precisely the same way as above for the integral, except that, when $r < \gamma$ (or $r' < \gamma$), we must replace r (or r') by γ in the expression

$$L_1(M', M; N) = r^\alpha L(M'; N) - r'^\alpha L(M; N).$$

On taking $K_0(M; N) = K(M; N)$, we see that $L_1(M', M; N)$ is a continuous function of the points M', M and N of B and of the parameter γ belonging to the interval $0 \leqslant \gamma \leqslant \varepsilon_1$, where ε_1 is any positive number. Thus the number η mentioned in regard to the inequality for the integral (108) can be taken here independently of γ, and the previous proof of the equicontinuity of $v_\gamma(M)$ and the boundedness of $|v_\gamma(M)|$ is fully preserved.

LEMMA 2. *If the continuous functions $u(N)$ have moduli bounded by the same number, and γ takes any positive values, a class of equicontinuous functions $v_\gamma(M)$ with moduli bounded by the same number is defined by formula* (111) *with $0 \leqslant \gamma \leqslant \varepsilon_1$, where ε_1 is any positive number, and by $K_0(M; N) = K(M; N)$.*

We now prove the formula needed in future for changing the order of integrations:

$$\int_B \Big[\int_B K(M, N)\, u_1(N)\, d\omega_N \Big] u_2(M)\, d\omega_M = \int_B \Big[\int_B K(M, N)\, u_2(M)\, d\omega_M \Big] u_1(N)\, d\omega_N, \tag{113}$$

where $u_1(N)$ and $u_2(M)$ are arbitrary functions continuous in B. This formula is familiar for a continuous kernel $K_\gamma(M, N)$ [II, 97]:

$$\int_B \Big[\int_B K_\gamma(M; N)\, u_1(N)\, d\omega_N \Big] u_2(M)\, d\omega_M = \int_B \Big[\int_B K_\gamma(M; N)\, u_2(M)\, d\omega_M \Big] u_1(N)\, d\omega_N. \tag{114}$$

It may readily be shown that, as $\gamma \to 0$,

$$\int_B K_\gamma(M; N)\, u_1(N)\, d\omega_N \to \int_B K(M; N)\, u_1(N)\, d\omega_N \tag{115}$$

uniformly with respect to M. For,

$$\Big| \int_B [K(M; N) - K_\gamma(M; N)]\, u_1(N)\, d\omega_N \Big| \leqslant \max_{\text{in } B} |u_1(N)| \int_B |K(M; N) - K_\gamma(M; N)|\, d\omega_N.$$

But the difference written vanishes outside the circle ω_γ, so that

$$\Big| \int_B [K(M; N) - K_\gamma(M; N)]\, u_1(N)\, d\omega_N \Big| \leqslant \max_{\text{in } B} |u_1(N)| \int_{\omega'_\gamma} [|K(M; N)| + |K_\gamma(M; N)|]\, d\omega_N,$$

where ω'_γ is the part of ω_γ belonging to B.

We obtain on taking (101) and (110) into account:

$$\left|\int_B [K(M;N) - K_\gamma(M;N)]\, u_1(N)\, d\omega_N\right| \leqslant \max_{\text{in } B} |u_1(N)| \frac{4\pi C\gamma^{2-\alpha}}{2-\alpha},$$

whence (115) follows (uniformly with respect to M).

Similarly, as $\gamma \to 0$:

$$\int_B K_\gamma(M;N)\, u_2(M)\, d\omega_M \to \int_B K(M;N)\, u_2(M)\, d\omega_M$$

uniformly with respect to N, and formula (113) is obtained by passing to the limit in (114).

Let us justify a further convergence which will be needed later. Let the functions $\varphi_\gamma(N)$, continuous in B and dependent on the positive parameter γ, tend uniformly to the function $\varphi_0(N)$ as $\gamma \to 0$, the latter function being obviously continuous. In this case,

$$\int_B K_\gamma(M;N)\, \varphi_\gamma(N)\, d\omega_N \to \int_B K(M;N)\, \varphi_0(N)\, d\omega_N. \tag{116}$$

We have:

$$\begin{aligned}
\Big|\int_B [K(M;N)\,\varphi_0(N) &- K_\gamma(M;N)\,\varphi_\gamma(N)]\, d\omega_N\Big| \\
&\leqslant \Big|\int_B K(M;N)\,[\varphi_0(N) - \varphi_\gamma(N)]\, d\omega_N\Big| + \\
&+ \Big|\int_B [K(M;N) - K_\gamma(M;N)]\,\varphi_\gamma(N)\, d\omega_N\Big|.
\end{aligned}$$

The uniform convergence of $\varphi_\gamma(N)$ to $\varphi_0(N)$ implies that, for all γ sufficiently close to zero, $|\varphi_\gamma(N)| \leqslant D_1$, where D_1 is a constant, and we obtain by taking (102) into account:

$$\begin{aligned}
\Big|\int_B [K(M;N)\,\varphi_0(N) &- K_\gamma(M;N)\,\varphi_\gamma(N)]\, d\omega_N\Big| \\
&\leqslant D \max_{\text{in } B} |\varphi_0(N) - \varphi_\gamma(N)| + D_1 \int_B |K(M;N) - K_\gamma(M;N)|\, d\omega_N.
\end{aligned}$$

The integral on the right-hand side tends to zero along with γ, as we have just seen, so that the whole of the right-hand side tends to zero, whence (116) follows.

18. Integral equations with unbounded kernels. Let us take an integral equation with an unbounded kernel of the type considered above:

$$\varphi(M) = f(M) + \lambda \int_B K(M;N)\, \varphi(N)\, d\omega_N, \tag{117}$$

where $f(M)$ is a given and $\varphi(M)$ the required function, both being continuous in B.

We suppose firstly that λ is not an eigenvalue. We show that, in this case, the homogeneous equation

$$\varphi_\gamma(M) = \lambda \int_B K_\gamma(M; N)\, \varphi_\gamma(N)\, d\omega_N \tag{118}$$

has no non-zero solutions for sufficiently small positive γ. This will be proved by *reductio ad absurdum*. Let a sequence of positive numbers $\gamma = \gamma_1, \gamma_2, \ldots$ exist, tending to zero, such that the equations

$$\varphi_{\gamma_n}(M) = \lambda \int_B K_{\gamma_n}(M; N)\, \varphi_{\gamma_n}(N)\, d\omega_N \tag{119}$$

have non-zero solutions. In view of the fact that these solutions are determined up to a constant factor, we can take

$$|\varphi_{\gamma_n}(M)| \leqslant 1 \tag{120}$$

and can assume that the sign of equality is attained in this formula for a certain position of the point M.

By Lemma 2, the functions $\varphi_{\gamma_n}(M)$ are equicontinuous in B. On further taking (120) into account, we can say that it is possible to extract from the sequence $\varphi_{\gamma_n}(M)$ a subsequence which tends uniformly in B to some limit function $\varphi_0(M)$. On passing to the limit via this subsequence in (119), we obtain [17]:

$$\varphi_0(M) = \lambda \int_B K(M; N)\, \varphi_0(N)\, d\omega_N. \tag{121}$$

By virtue of the uniformity of the convergence and the fact that the sign of equality is attained in (120) with any n, we can say that $\varphi_0(M)$ is not indentically zero. It follows from (121) that λ is an eigenvalue of equation (117), which contradicts the assumption made at the start of this section. Hence equations (118) have only zero solutions for all γ sufficiently close to zero, and we can say that the equations

$$\varphi_\gamma(M) = f(M) + \lambda \int_B K_\gamma(M; N)\, \varphi_\gamma(N)\, d\omega_N \tag{122}$$

with continuous kernels have solutions which are in fact unique for any function $f(M)$. Let us show that these solutions have moduli bounded by the same number for all γ sufficiently close to zero. Let $m_\gamma = \max\limits_{\text{in } B} |\varphi_\gamma(M)|$. We have to show that no sequence m_{γ_n} exists

which tends to $(+\infty)$. This is proved by *reductio ad absurdum.* Let such a sequence exist, i.e. $m_{\gamma_n} \to +\infty$. We have:

$$\frac{|\varphi_{\gamma_n}(M)|}{m_{\gamma_n}} \leqslant 1, \tag{123}$$

the sign of equality being obtained for a certain choice of M. We put $\gamma = \gamma_n$ in equation (122) and divide both sides by m_{γ_n}:

$$\frac{\varphi_{\gamma_n}(M)}{m_{\gamma_n}} = \lambda \int_B K_{\gamma_n}(M; N) \frac{\varphi_{\gamma_n}(N)}{m_{\gamma_n}} \mathrm{d}\omega_N + \frac{f(M)}{m_{\gamma_n}} . \tag{124}$$

The second term on the right-hand side tends to zero uniformly in B, whilst the first term, by (123) and Lemma 2, gives a sequence of uniformly bounded and equicontinuous functions. By Arzela's theorem, we can assume that the first term tends uniformly in B to a limit function as $\gamma_n \to 0$. Hence the left-hand side must also tend uniformly in B to some limit function $\varphi_0(M)$, which is not identically zero, since the sign of equality is attained in (123). On passing to the limit in (124), we get (121) [17], i.e. λ turns out to be an eigenvalue of equation (117), which contradicts the hypothesis. Thus all the functions $\varphi_\gamma(M)$ have moduli bounded by the same number when γ is sufficiently close to zero. After this, it follows from (122) and Lemma 2 that the functions $\varphi_\gamma(M)$ are equicontinuous. On again using Arzela's theorem and passing to the limit, we obtain

$$\omega(M) = f(M) + \lambda \int_B K(M; N)\, \omega(N)\, \mathrm{d}\omega_N, \tag{125}$$

where $\omega(M)$ is a continuous function.

We have thus shown that, if λ is not an eigenvalue of equation (117), this equation has a solution for any function $f(M)$. The uniqueness of the solution follows at once from the fact that homogeneous equation (121) has only a zero solution by hypothesis.

We now consider the homogeneous equation adjoint to (121):

$$\psi(M) = \lambda \int_B K(N; M)\, \psi(N)\, \mathrm{d}\omega_N, \tag{126}$$

and show that it also has only a zero solution. We assume the converse and let $\psi(M)$ be a non-zero solution of the equation. We multiply both sides of (117) by $\psi(M)$, integrate with respect to M and change the

order of integration in the iterated integral, in accordance with [17]:

$$\int_B \varphi(M)\,\psi(M)\,\mathrm{d}\omega_M$$
$$= \int_B \Big[\lambda \int_B K(M; N)\,\psi(M)\,\mathrm{d}\omega_M\Big]\,\varphi(N)\,\mathrm{d}\omega_N + \int_B f(M)\,\psi(M)\,\mathrm{d}\omega_M,$$

whence, by (126), we obtain the condition for solubility of equation (117) (cf. the deduction of formula (79) in [10]):

$$\int_B f(M)\,\psi(M)\,\mathrm{d}\omega_M = 0. \tag{127}$$

But we saw above that (117) is soluble for any $f(M)$. This contradiction shows that homogeneous equation (126) has only a zero solution.

We have thus proved the following for kernels of the form (100): *there are two possibilities: either the equations*

$$\varphi(M) = f(M) + \lambda \int_B K(M; N)\,\varphi(N)\,\mathrm{d}\omega_N;$$

$$\psi(M) = g(M) + \lambda \int_B K(N; M)\,\psi(N)\,\mathrm{d}\omega_N$$

simultaneously have solutions which are unique for any functions $f(M)$ and $g(M)$, or the corresponding homogeneous equations have non-zero solutions.

Note. If λ is not an eigenvalue, we have shown that equation (122) has a unique solution for any positive γ sufficiently close to zero, and for all γ these solutions have moduli bounded by the same number. Further, we have obtained the solution $\varphi_1(M)$ of the initial equation (117) by extraction of a subsequence and passing to the limit.

It is easily shown, by using the uniqueness of the solution, that it is unnecessary to extract a subsequence. For, if $\varphi_\gamma(M)$ were not to have a definite limit at some point M as $\gamma \to 0$, we should be able to extract two subsequences which would tend uniformly to two continuous limit functions, having different values at the point M. We should therefore obtain two different solutions of equation (117), which is impossible if λ is not an eigenvalue. Consequently, $\varphi_\gamma(M)$ tends to the unique limit function $\omega(M)$ without any choice. The fact that it tends uniformly to the limit follows from the fact that functions $\varphi_\gamma(M)$ have bounded moduli by (122) and are equicontinuous.

19. The case of an eigenvalue. Now suppose that λ is an eigenvalue. If one of the homogeneous equations

$$\varphi(M) = \lambda \int_B K(M; N)\, \varphi(N)\, \mathrm{d}\omega_N; \tag{128_1}$$

$$\psi(M) = \lambda \int_B K(N; M)\, \psi(N)\, \mathrm{d}\omega_N \tag{128_2}$$

has a finite number of linearly independent solutions, we can show by repeating the proof of Theorem 9 [10] that the second equation also has the same number of linearly independent solutions. After this, we can show, precisely as in [10], that condition (127), where $\psi(M)$ is any solution of equation (128_2), is sufficient as well as necessary for the solubility of equation (117).

It remains to show that the number of linearly independent solutions say of equation (128_1) is finite.

We shall prove this by a contradiction. Suppose that (128_1) has an infinite set of linearly independent solutions:

$$\varphi_n(M) = \lambda \int_B K(M; N)\, \varphi_n(N)\, \mathrm{d}\omega_N \qquad (n = 1, 2, \ldots). \tag{129}$$

It can be assumed that these solutions are pairwise orthogonal:

$$\int_B \varphi_p(N)\, \overline{\varphi_q(N)}\, \mathrm{d}\omega_N = 0 \text{ with } p \neq q \tag{130}$$

and satisfy the inequality

$$|\varphi_n(M)| \leqslant 1 \qquad (n = 1,\ 2, \ldots), \tag{131}$$

where the sign of equality is obtained at certain points. It follows from (129) and (131) that the $\varphi_n(M)$ are equicontinuous in B and that there exists in the sequence $\varphi_n(N)$ a subsequence which tends uniformly in B to some limit function $\varphi_0(N)$. On passing to the limit via this subsequence $\varphi_p(x)$ in (130), we obtain

$$\int_B \varphi_0(N)\, \overline{\varphi_q(N)}\, \mathrm{d}\omega_N = 0$$

and on again passing to the limit via the subscript q:

$$\int_B |\varphi_0(N)|^2\, \mathrm{d}\omega_N = 0. \tag{132}$$

But the function $\varphi_0(N)$ cannot be identically equal to zero, since the sign of equality can hold in (131) for any n, and formula (132) leads us to a contradiction. We have thus shown that equation (128_1) has only a finite number of linearly independent solutions.

We have now also proved the following for kernels of the form (100): *if* λ *is an eigenvalue of equation* (117), *equations* (128_1) *and* (128_2) *have the same number of linearly independent solutions, and the necessary and sufficient condition for* (117) *to be soluble is that the term* $f(M)$ *satisfies condition* (127), *where* $\psi(M)$ *is any solution of equation* (128_2). Everything that has been said above can be extended without modification to the case of one-dimensional, three-dimensional, and in general n-dimensional space.

We must obviously take $0 < a < n$ instead of the condition $0 < a < 2$ in formula (100).

Now let the domain of integration be a bounded closed surface Σ subject to the following conditions:

1. At any point of the surface Σ there is a tangent plane that varies continuously on displacement along Σ.

2. A positive number d exists such that the part of the surface Σ, belonging to a sphere with centre at an arbitrary point M of Σ and radius d, is cut by straight lines parallel to the normal to Σ at M in a single point. Thus, if we take the tangent plane to Σ at the point M as the XY plane, the above-mentioned part of Σ has the explicit equation $z = f(x, y)$.

3. $f(x, y)$ has continuous first order partial derivatives. Integration over this part of the surface can now be reduced to integration over a plane domain lying in the tangent plane, and all the above discussion and the various inequalities retain their force.

The arguments follow exactly the same lines for the case of integration over a contour.

It can further be shown that for the type of kernel considered there exists only a finite number of eigenvalues in any bounded part of the λ plane. We shall prove this later for the case of symmetric kernels.

We describe without proof in the next section another method of attacking kernels of the type indicated above and kernels of more general unbounded types.

20. Equations with continuous iterated kernels. We confine the discussion to the case of a bounded plane domain B.

It can be shown that, if we form the iterated kernels $K_p(M; N)$ by starting from kernel (100), they will all be continuous when the points M and N do not coincide. But if the index p is sufficiently large, they are continuous at all positions of M and N in B. These large indices are determined by the inequality $p > 2/(2 - a)$, and for n-dimensional spaces by $p > n/(n - a)$.

It can further be shown that, given familiar provisos, the fundamental theorems proved above remain valid for unbounded kernels in which all the iterated

kernels, as from a certain index, are continuous (see S. L. Sobolev, *Partial Differential Equations of Mathematical Physics*, Pergamon Press, 1964, Lecture 18).

Let the kernels $K_p(M; N)$ be continuous from $p = m$. It is easily shown that, if λ is an eigenvalue of equation (128_1), $\mu = \lambda^m$ is an eigenvalue of the equation

$$\varphi(M) = \mu \int_B K_m(M; N)\, \varphi(N)\, d\omega_N. \tag{133}$$

Conversely, if μ is an eigenvalue of equation (133), at least one of the values of the root $\lambda = \sqrt[m]{\mu}$ is an eigenvalue of equation (128_1). Since the kernel $K_m(M; N)$ is continuous, only a finite number of eigenvalues μ can lie in any bounded domain of the plane of the complex variable μ. From what was said above, the same can be asserted as regards the eigenvalues λ of (128_1).

The question arises as to the possibility of retaining Fredholm's treatment, described in [7] and [8]. This treatment loses its meaning in the unmodified form for kernels of type (100), since the diagonal terms of determinant (49) now become infinite.

It can be shown that, if the number α appearing in formula (100) satisfies the condition $0 < \alpha < n/2$ (we only had $0 < \alpha < n$ in [19]) Fredholm's treatment can be retained with a single modification: we must take $K(N_s, N_s) = 0$ in determinants (100). The resolvent can now be written as before in the form (57) (see I. I. Privalov, *Integral Equations* (Integralnye uravneniya), 1935, p. 83).

Fredholm's treatment can be modified in a different way. As above, let the kernels $K_p(M; N)$ be continuous for $p \geqslant m$. We construct the resolvent $R_m(M, N; \lambda)$ of the continuous kernel $K_m(M; N)$ in the usual way. With λ close to zero it is given by the series

$$R_m(M, N; \lambda) = K_m(M; N) + K_{2m}(M; N)\,\lambda + K_{3m}(M; N)\,\lambda^2 + \dots,$$

and, in the case of any λ, by the fraction

$$R_m(M, N; \lambda) = \frac{D_m(M, N; \lambda)}{D_m(\lambda)}.$$

It can be shown that the basic kernel has the resolvent

$$R(M, N; \lambda) = H(M, N; \lambda) + \frac{D_m(M, N; \lambda^m)}{D_m(\lambda^m)} + $$
$$+ \lambda^m \int_B H(M, P; \lambda)\, \frac{D_m(P, N; \lambda^m)}{D_m(\lambda^m)}\, d\omega_P,$$

where

$$H(M, N; \lambda) = K(M; N) + K_2(M; N)\lambda + \dots + K_{m-1}(M; N)\,\lambda^{m-2}.$$

If λ is not a root of the equation $D_m(\lambda^m) = 0$, the resolvent satisfies relationships (47), and equation (94) has a unique solution, which is given by formula (46) (see Goursat, *Cours d'Analyse Mathematique,* Vol. III, part II, 1934). As is clear from the above formulae, the resolvent is also the quotient of two functions of λ in this case.

It can be shown that all the theorems proved above now hold when, in the case of an unbounded kernel, the integral

$$\int\limits_B \int\limits_B |K(M; N)|^2 \, d\omega_M \, d\omega_N \tag{134}$$

has a finite value. A strict statement of this condition and a proof of the above assertion will be given in Vol. V, where the Lebesgue integral will be employed.

We remark that, for kernels of type (100), integral (134) is finite when $a < n/2$. The extension of Fredholm's treatment to the case when integral (134) has a finite value was due to Carleman *(Math. Zeitschr.* Bd. 9, Heft 3/4, 1921) and S. G. Mikhlin *(Dokl. Akad. Nauk SSSR.*, vol. XLII, No. 9, 1944).

21. Symmetric kernels. Integral equations with so-called symmetric kernels find wide employment in mathematical physics.

DEFINITION. *A real kernel whose value remains unchanged when the arguments are interchanged is said to be symmetric.*

In the one-dimensional case such a kernel is real and satisfies the condition

$$K(t, s) = K(s, t) \qquad (s \text{ and } t \text{ belong to the square } k_0). \tag{135}$$

We shall now examine the theory of integral equations with symmetric kernels. All the proofs will be carried out for the one-dimensional case. They remain exactly the same in the n-dimensional case. For the present the kernel will be assumed continuous. We shall extend the theory later to unbounded kernels of the type considered in [17].

An example above showed us that kernels can exist which have no eigenvalues. This cannot happen in the case of symmetric kernels, i.e. the following fundamental theorem holds.

THEOREM I. *Any symmetric continuous kernel which is not identically zero, or any kernel of the type indicated in* [17], *when* $0 < a < n/2$, *where* n *is the multiplicity of the integral, has one or more eigenvalues.*

We shall use this theorem and give the proof later. For the moment we want to establish some simple properties of integral equations with symmetric kernels.

Let λ_1 and λ_2 be two different eigenvalues, and $\varphi_1(s)$, $\varphi_2(s)$ corresponding eigenfunctions, so that

$$\frac{1}{\lambda_1} \varphi_1(s) = \int_a^b K(s, t) \, \varphi_1(t) \, dt; \qquad \frac{1}{\lambda_2} \varphi_2(s) = \int_a^b K(s, t) \, \varphi_2(t) \, dt.$$

On multiplying the first equation by $\varphi_2(s)$ and the second by $\varphi_1(s)$, integrating with respect to s and subtracting term by term, we get

$$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\int_a^b \varphi_1(s)\,\varphi_2(s)\,\mathrm{d}s = \int_a^b\left[\int_a^b K(s,t)\,\varphi_1(t)\,\mathrm{d}t\right]\varphi_2(s)\,\mathrm{d}s -$$

$$-\int_a^b\left[\int_a^b K(s,t)\,\varphi_2(t)\,\mathrm{d}t\right]\varphi_1(s)\,\mathrm{d}s. \tag{136}$$

On changing the order of integration in one of the integrals on the right-hand side and using (135), the right-hand side is seen to vanish, i.e.

$$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\int_a^b \varphi_1(s)\,\varphi_2(s)\,\mathrm{d}s = 0,$$

whence, since $\lambda_1 \neq \lambda_2$,

$$\int_a^b \varphi_1(s)\,\varphi_2(s)\,\mathrm{d}s = 0, \tag{137}$$

i.e. *the integral over the basic interval* $[a, b]$ *of the product of any two eigenfunctions corresponding to different eigenvalues vanishes.*

We now show that all the eigenvalues are real. We use *reductio ad absurdum.* Let λ_0 be some non-real (complex) eigenvalue and $\varphi_0(s)$ a corresponding eigenfunction, which, by definition of eigenfunctions, cannot vanish identically. We have:

$$\varphi_0(s) = \lambda_0 \int_a^b K(s,t)\,\varphi_0(t)\,\mathrm{d}t.$$

We obtain on passing to the conjugates in this equation:

$$\overline{\varphi_0(s)} = \overline{\lambda}_0 \int_a^b K(s,t)\,\overline{\varphi_0(t)}\,\mathrm{d}t.$$

Hence it is clear that $\overline{\lambda}_0$ is also an eigenvalue and $\overline{\varphi_0(s)}$ is a corresponding eigenfunction, where $\overline{\lambda}_0 \neq \lambda_0$ since λ_0 is not real. The eigenfunctions $\varphi_0(s)$ and $\overline{\varphi_0(s)}$, which correspond to different eigenvalues, must satisfy condition (137), if we put $\varphi_1(s) = \varphi_0(s)$ and $\varphi_2(s) = \overline{\varphi_0(s)}$, i.e.

$$\int_a^b |\varphi_0(s)|^2\,\mathrm{d}s = 0,$$

whence it follows [3] that $\varphi_0(s)$ vanishes identically, which contradicts the fact that it is an eigenfunction.

We know that every linear combination with constant coefficients of eigenfunctions corresponding to the same eigenvalue, is also an eigenfunction corresponding to this eigenvalue. In other words, we can say that *the eigenfunctions corresponding to a given eigenvalue form a linear manifold* [4].

Since the eigenvalues have been shown to be real, we can assume that *all the eigenfunctions are also real* [4]. Equation (137) that we obtained now shows that *any two eigenfunctions corresponding to different eigenvalues are mutually orthogonal.*

To every eigenvalue λ there corresponds a finite number of linearly independent eigenfunctions [10]. We can apply a process of orthogonalization to these eigenfunctions and thus assume that they are in fact mutually orthogonal and normalized. The eigenfunctions corresponding to different eigenvalues must be mutually orthogonal, as we saw above. We can therefore assume that all the eigenfunctions are mutually orthogonal and normalized.

Further, we know that only a finite number of eigenvalues can lie in any finite interval of variation of λ. On taking into account what has just been said, we can enumerate all the eigenvalues in order of non-decreasing absolute value:

$$|\lambda_1| \leqslant |\lambda_2| \leqslant |\lambda_3| \leqslant \ldots, \tag{138}$$

where $|\lambda_n| \to +\infty$ as $n \to \infty$, if the number of eigenvalues is infinite whilst every eigenvalue is encountered in the series written a number of times equal to its rank [4] (the number of linearly independent eigenfunctions corresponding to it). All the eigenfunctions are now automatically enumerated:

$$\varphi_1(s),\ \varphi_2(s),\ \varphi_3(s), \ldots, \tag{139}$$

and it can be assumed, as we saw above, that the system consists of real, orthogonal and normalized functions.

System (139) is called the *system of eigenfunctions of the kernel $K(s, t)$ or of the corresponding integral equation.*

We have for the eigenfunctions:

$$\frac{\varphi_k(s)}{\lambda_k} = \int_a^b K(s, t)\, \varphi_k(t)\, dt, \tag{139_1}$$

whence it is clear that the left-hand side can be regarded as the Fourier coefficient of kernel $K(s, t)$ with respect to the orthonormal system (139). Bessel's inequality gives

$$\sum_{k=1}^{n}\left[\frac{\varphi_k(s)}{\lambda_k}\right]^2 \leqslant \int_a^b [K(s,t)]^2\,dt. \tag{140}$$

We obtain by integrating with respect to s:

$$\sum_{k=1}^{n}\frac{1}{\lambda_k^2} \leqslant \int_a^b\int_a^b [K(s,t)]^2\,dt\,ds, \tag{141}$$

and, on passing to the limit, if the number of eigenvalues is infinite:

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2} \leqslant \int_a^b\int_a^b [K(s,t)]^2 dt\,ds. \tag{142}$$

Everything said above up to the assertion of the finiteness of the number of eigenvalues in any finite interval of variation also holds for unbounded kernels of the type indicated in [17].

For, the essential step in the proofs was the change of the order of the integrations in one of the integrals on the right-hand side of formula (136), and such a change has been justified by us for unbounded kernels of the type of [17].

We shall assume in our future treatment of the theory of integral equations with symmetric kernels that the number α appearing in formula (100) satisfies the condition $0 < \alpha < n/2$, i.e. the condition $0 < \alpha < 1/2$ in the one-dimensional case. We shall describe such kernels as *weakly polar*. The theory still holds in essence with the more general assumption $0 < \alpha < n$, though a more natural treatment of the theory in this more general case is in terms of the Lebesgue integral, which we shall use in Vol. V. The condition $0 < \alpha < n/2$ is satisfied in applications to mathematical physics. It may be remarked that we in fact confined ourselves to the case $0 < \alpha < n/2$ in the statement of the fundamental Theorem I. We have for a weakly polar kernel:

$$[K(s,t)]^2 \leqslant \frac{C^2}{|s-t|^{2\alpha}} \qquad (2\alpha < 1).$$

On repeating for the one-dimensional case the proof from the beginning of [17], we see that the integral

$$\int_a^b [K(s,t)]^2\,\mathrm{d}t$$

has a meaning and does not exceed some definite positive number M:

$$\int_a^b [K(s,t)]^2\,\mathrm{d}t < M. \tag{143}$$

It may be shown, precisely as in [9], that only a finite number of different eigenvalues is contained in any finite interval $[-L, L]$.

We thus also have (138) and (139) for weakly polar kernels. In what follows all the proofs will first be given for continuous kernels, then for weakly polar kernels.

22. Expansion in eigenfunctions. The set of all eigenfunctions (139) may not form a closed system. For example, this will be the case for a degenerate symmetric kernel, when the number of eigenvalues is finite. We can form a Fourier series in functions (139) for a continuous function $F(s)$, or even for a discontinuous function with the type of discontinuity indicated in [14], but there is no sort of foundation for asserting that the series is convergent. Even if it is uniformly convergent in the interval $[a, b]$, it is still impossible to assert that its sum is equal to $F(s)$, since functions (139) may form a non-closed system, and the proposition proved in [3] cannot be applied. We shall start with the formation of the Fourier series for the kernel $K(s, t)$ considered as a function of t.

We have seen that the Fourier coefficients of the kernel are equal to the ratios $\varphi_k(s)\colon \lambda_k$, so that the Fourier series has the form

$$\sum_k \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k}, \tag{144}$$

the summation over k being carried out up to infinity if the number of eigenfunctions is infinite, or up to the finite number representing the total of eigenfunctions (139).

We observe that series (144) can be regarded as the Fourier series of the function $K(s, t)$ defined in k_0, in functions $\varphi_k(s)\,\psi_1(t)$ $(k, l = 1, 2, 3, \ldots)$, which are easily shown to form an orthonormal system in k_0. In this case

$$\iint_{k_0} K(s,t)\,\varphi_k(s)\,\varphi_l(t)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{\lambda_k}\int_a^b \varphi_k(s)\,\varphi_l(s)\,\mathrm{d}s = \begin{cases} 0 & \text{for } k \neq l \\ \dfrac{1}{\lambda_k} & \text{for } k = l. \end{cases}$$

Series (144) has the remarkable property that, if it is uniformly convergent in the square k_0, its sum is equal to the kernel, i.e. the fact that the system is not closed is of no detriment here. Since the sum of series (144) is a continuous function in k_0 when the series is uniformly convergent, it is natural to assume the continuity of the kernel in the proof of this property.

THEOREM 1. *If the kernel is continuous and series* (144) *is uniformly convergent in* k_0, *its sum is equal to the kernel in* k_0, i.e.

$$K(s,t) = \sum_k \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k}. \tag{145}$$

Assuming for the present that the number of eigenvalues is infinite, we consider the difference

$$\omega(s,t) = K(s,t) - \sum_{k=1}^{\infty} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k},$$

which is a continuous symmetric function in the square k_0. If we fix s and regard $\omega(s, t)$ as a function of t in the interval $[a, b]$, its Fourier coefficients with respect to the system of functions $\varphi_k(t)$ are equal to zero [3]:

$$\int_a^b \omega(s,t)\,\varphi_k(t)\,\mathrm{d}t = 0 \qquad (k = 1,\ 2, \ldots). \tag{146}$$

We have to show that $\omega(s, t)$ vanishes identically in the square k_0. We do this by *reductio ad absurdum.*

Suppose that the function $\omega(s, t)$ does not vanish identically in the square k_0; let us take it as the kernel of the integral equation

$$\psi(s) = \lambda \int_a^b \omega(s,t)\,\psi(t)\,\mathrm{d}t.$$

By the fundamental theorem stated in the previous section, this integral equation must have at least one eigenvalue λ_0, to which there corresponds some eigenfunction $\psi_0(s)$ which is not identically zero:

$$\psi_0(s) = \lambda_0 \int_a^b \omega(s,t)\,\psi_0(t)\,\mathrm{d}t. \tag{147}$$

We show that this function $\psi_0(s)$ must be orthogonal to all the eigenfunctions $\varphi_k(s)$ of kernel $K(s, t)$. We have, in fact, on multiplying both sides of (146) by $\lambda_0\psi_0(s)$ and integrating with respect to s:

$$\lambda_0 \int_a^b \int_a^b \omega(s,t)\,\psi_0(s)\,\varphi_k(t)\,\mathrm{d}s\,\mathrm{d}t = 0.$$

Hence we obtain, by (147) and the symmetry of $\omega(s, t)$:

$$\int_a^b \psi_0(t)\,\varphi_k(t)\,\mathrm{d}t = 0 \qquad (k = 1,\ 2, \ldots). \tag{148}$$

We can rewrite equation (147) as

$$\psi_0(s) = \lambda_0 \int_a^b \Big[K(s,t) - \sum_{k=1}^{\infty} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k} \Big] \psi_0(t)\,\mathrm{d}t.$$

On taking into account the uniform convergence of series (144) and formula (148), we obtain

$$\psi_0(s) = \lambda_0 \int_a^b K(s,t)\,\psi_0(t)\,\mathrm{d}t,$$

i.e. the function $\psi_0(s)$ must be an eigenfunction of the original kernel $K(s, t)$. It must therefore be a linear combination of eigenfunctions $\varphi_k(s)$ corresponding to the eigenvalue λ_0.

But this is impossible, because $\psi_0(s)$ and all the $\varphi_k(s)$ form an orthogonal system, and orthogonal functions cannot be linearly dependent [3]. This contradiction shows that our assumption that $\omega(s, t)$ is not identically zero is false, i.e. $\omega(s, t) = 0$ in k_0, and formula (145) holds.

The proof could be repeated on the assumption that the kernel $K(s, t)$ is weakly polar, but it would now follow from (145) that $K(s, t)$ is a continuous function in k_0, i.e. series (144) cannot be uniformly convergent in k_0 for weakly polar non-bounded kernels.

If the kernel has a finite number of eigenvalues, series (144) consists of a finite number of terms, and the above proof is fully preserved, i.e. we have:

$$K(s,t) = \sum_{k=1}^{m} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k}, \tag{149}$$

where m is the number of eigenvalues in sequence (138).

Formula (149) shows that $K(s, t)$ is a degenerate kernel. Thus, on the one hand, as we saw above, a degenerate kernel (symmetric in the present case) has a finite number of eigenvalues and, on the other hand, as we have just shown, if a symmetric kernel has a finite number of eigenvalues, it is degenerate.

Therefore, *in the case of a continuous symmetric kernel, the necessary and sufficient condition for it to be degenerate is that it has a finite number of eigenvalues.*

We now turn to the formation of the Fourier series of any function $F(s)$ in the eigenfunctions (139). We first introduce a new concept.

We shall say that the series

$$\sum_{k=1}^{\infty} f_k(x) \tag{150}$$

is *regularly* convergent in some domain of variation of x if the series

$$\sum_{k=1}^{\infty} |f_k(x)|$$

is uniformly convergent in this domain. Regular convergence obviously implies absolute convergence of the series. We have further:

$$\left|\sum_{k=n}^{n+p} f_k(x)\right| \leqslant \sum_{k=n}^{n+p} |f_k(x)|.$$

The regular convergence implies that, given any positive ε, an N exists such that the right-hand side of this last inequality is $< \varepsilon$ with $n > N$, with any $p > 0$ and any x of the domain in question. Now, all the more, the left-hand side is also $< \varepsilon$, i.e. *regular convergence implies uniform as well as absolute convergence.*

If the absolute values of the terms of the series do not exceed certain positive numbers: $|f_k(x)| \leqslant a_k$, and the series of these numbers is convergent, series (150) must obviously be regularly convergent. The existence of these a_k does not follow, however, from the regular convergence. If the a_k exist, the series is sometimes said to be *properly* convergent. Hence, proper convergence implies regular convergence, and regular convergence implies that the series is absolutely and uniformly convergent.

A class of continuous functions exists, such that their Fourier series in functions (139) are regularly convergent in the interval $[a, b]$. These functions are described as expressible in terms of a kernel.

DEFINITION. *A continuous real function $F(s)$ is called a function expressible in terms of a kernel if there exists a real function $h(t)$ continuous in $[a, b]$ (or bounded with a finite number of discontinuities), such that*

$$F(s) = \int_a^b K(s, t)\, h(t)\, dt. \tag{151}$$

We can assume here that the kernel $K(s, t)$ is either continuous or weakly polar.

THEOREM 2. *The Fourier series of any function expressible in terms of the kernel in functions* (139) *is regularly convergent in the interval* $[a, b]$.

Let h_k denote the Fourier coefficients of the function $h(t)$:

$$h_k = \int_a^b h(t)\, \varphi_k(t)\, \mathrm{d}t,$$

and let us find the Fourier coefficients F_k of function (151):

$$F_k = \int_a^b F(s)\, \varphi_k(s)\, \mathrm{d}s = \int_a^b \Big[\int_a^b K(s, t)\, h(t)\, \mathrm{d}t \Big]\, \varphi_k(s)\, \mathrm{d}s,$$

or, on changing the order of integration and using the symmetry of the kernel:

$$F_k = \int_a^b \Big[\int_a^b K(t, s)\, \varphi_k(s)\, \mathrm{d}s \Big]\, h(t)\, \mathrm{d}t,$$

whence, by (139_1),

$$F_k = \frac{h_k}{\lambda_k}\,. \tag{152}$$

The Fourier series of function (151) thus has the form

$$\sum_{k=1}^{\infty} \frac{h_k}{\lambda_k}\, \varphi_k(s), \tag{153}$$

on the assumption that the number of eigenvalues is infinite. We have by Cauchy's inequality:

$$\sum_{k=n}^{n+p} \left| \frac{h_k}{\lambda_k}\, \varphi_k(s) \right| \leqslant \sqrt{\sum_{k=n}^{n+p} h_k^2}\, \sqrt{\sum_{k=n}^{n+p} \left[\frac{\varphi_k(s)}{\lambda_k} \right]^2}\,. \tag{154}$$

We return to inequality (140), which holds for any value of the number n.

Using the fact that the terms on the left-hand side of (140) are positive, we can say that, for any n and p:

$$\sum_{k=n}^{n+p} \left[\frac{\varphi_k(s)}{\lambda_k} \right]^2 \leqslant \int_a^b [K(s, t)]^2\, \mathrm{d}t.$$

The integral on the right does not exceed some positive number M, either for a continuous or for a weakly polar kernel, i.e.

$$\sum_{k=n}^{n+p} \left[\frac{\varphi_k(s)}{\lambda_k} \right]^2 \leqslant M,$$

and it follows from (154) that

$$\sum_{k=n}^{n+p} \left| \frac{h_k}{\lambda_k}\, \varphi_k(s) \right| \leqslant \sqrt{M}\, \sqrt{\sum_{k=n}^{n+p} h_k^2}\,.$$

The sum on the right tends to zero on indefinite increase of n with any $p > 0$, since the squares of the Fourier coefficients h_k^2 form a convergent series [3]. On noticing further that the right-hand side is independent of s, we can say that the series

$$\sum_{k=1}^{\infty} \left| \frac{h_k}{\lambda_k} \varphi_k(s) \right|$$

is uniformly convergent in $[a, b]$, and the theorem is proved.

In view of the fact that the system of functions (139) may not be closed, we cannot assert without additional proof that the sum of series (153) is equal to $F(s)$. It can be shown, however, that this is in fact the case, i.e. the following fundamental theorem holds:

THEOREM II. *The sum of the Fourier series in functions* (139) *of any function $F(s)$ expressible in terms of the kernel is equal to $F(s)$, or, in other words, every function expressible in terms of the kernel can be expanded in a Fourier series in functions* (139), *which is regularly convergent in the interval* $[a, b]$.

This is generally known as the Hilbert–Schmidt theorem, and is valid for weakly polar as well as continuous kernels. We shall prove it later, along with Theorem I on the existence of the eigenvalues.

If the Fourier series for the continuous kernel (144) is uniformly convergent in k_0, the proof of Theorem II is very simple.

For, on multiplying both sides of (145) by $h(t)$ and integrating with respect to t, we obtain

$$F(s) = \int_a^b K(s, t)\, h(t)\, \mathrm{d}t = \sum_{k=1}^{\infty} \frac{\varphi_k(s)}{\lambda_k} \int_a^b \varphi_k(t)\, h(t)\, \mathrm{d}t,$$

i.e.

$$F(s) = \sum_{k=1}^{\infty} \frac{\varphi_k(s)}{\lambda_k} h_k,$$

which in fact gives Theorem II.

Note. There is some arbitrariness in the choice of the orthonormal system of real functions (139). If all the eigenvalues are simple, i.e. have unit rank, the arbitrariness merely amounts to the possibility of changing the sign of each eigenfunction $\varphi_k(s)$. Let us take the case of multiple eigenvalues. If say the eigenvalue λ_1 has rank three, and $\lambda_1 = \lambda_2 = \lambda_3$, instead of $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$ we can take three different functions, which are obtained from the above with the aid of a linear orthogonal transformation.

If c_1, c_2, c_3 are the Fourier coefficients of any function $F(s)$ with respect to $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$, it is easily shown that the sum

$$c_1\,\varphi_1(s) + c_2\,\varphi_2(s) + c_3\,\varphi_3(s)$$

retains the same value with any choice of functions $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$.

The theorems given above naturally hold for any choice of system (139).

23. Dini's theorem. We shall prove in the present section an auxiliary theorem which will be useful later. It is due to the Italian mathematician Dini.

THEOREM. *If the terms of the series*

$$\sum_{k=1}^{\infty} f_k(x) \tag{155}$$

are continuous non-negative functions in the interval $[a, b]$, *the series is convergent at every point of this interval and its sum is a continuous function in the interval, then series* (155) *is uniformly convergent in* $[a, b]$.

Let $R_n(x)$ denote the remainder term of series (155):

$$R_n(x) = \sum_{k=n+1}^{\infty} f_k(x).$$

Since the terms of the series and its sum are continuous functions in $[a, b]$ by hypothesis, the function $R_n(x)$ will also be a continuous function in $[a, b]$. Given any fixed x, it cannot increase as n increases, since the terms of the series are non-negative, i.e. we have $R_{n+1}(x) \leqslant R_n(x)$. Let m_n denote the greatest value attained by the non-negative continuous function $R_n(x)$ in the interval $[a, b]$, and let ξ_n be the point of the interval at which the greatest value is attained, i.e. $m_n = R_n(\xi_n)$. We show that, as n increases, the number m_n cannot increase i.e. $m_{n+1} \leqslant m_n$. In fact, $m_{n+1} = R_{n+1}(\xi_{n+1}) \leqslant R_n(\xi_{n+1})$. But the value $R_n(\xi_{n+1})$ cannot be greater than the greatest value m_n of the function $R_n(x)$ in the interval $[a, b]$, whence it follows that $m_{n+1} \leqslant m_n$. The non-increasing sequence of positive numbers m_n must have a limit, which can be zero or positive: if the limit is zero, the uniform convergence of series (155) is guaranteed, since the greatest value of its remainder term tends to zero as $n \to \infty$. It remains to show that the limit of the numbers m_n cannot be positive. We shall prove this by a contradiction. All the numbers ξ_n which we introduced above lie in the finite interval $[a, b]$, so that the interval will contain at least one

point $x = c$ of condensation of the numbers [II, 89], i.e. a point such that an infinity of the ξ_n lie in any small neighbourhood of it. The series is convergent by hypothesis at the point $x = c$, so that we can fix a subscript N large enough for $R_N(c) > l/2$, where l denotes the assumed positive limit of the sequence m_n. Since the function $R_N(x)$ is continuous, we can find a point ξ_n with $n > N$ so close to c that we have $R_N(\xi_n) < l/2$ at the point ξ_n also. Since $n > N$ by hypothesis we have $m_n = R_n(\xi_n) \leqslant R_N(\xi_n)$, i.e. it turns out that $m_n < l/2$, which contradicts the fact that the non-increasing sequence m_n tends to a limit l. This contradiction leads us to Dini's theorem.

We know that, if the terms of the series are continuous functions and the series is uniformly convergent, its sum is also a continuous function. The converse is not generally true, i.e. it is impossible to conclude from the continuity of the sum that the series is uniformly convergent. Dini's theorem says that if the terms are non-negative as well as continuous functions, the converse is true, i.e. the continuity of the sum implies the uniform convergence of the series.

24. Expansion of iterated kernels. We shall assume in the next four sections that the kernel is continuous. Hence all the iterated kernels are continuous. We see from the formula

$$K_2(s,t) = \int_a^b K(s,t_1)\,K(t_1,t)\,\mathrm{d}t_1 \tag{156}$$

that $K_2(s, t)$ as a function of s is expressible in terms of a kernel, the role of $h(t_1)$ being played by the function $K(t_1, t) = K(t, t_1)$, where t is a parameter. As we saw above, the Fourier coefficients of $K(t, t_1)$ with respect to the system of functions (139) are equal to $\varphi_k(t)$: λ_k, so that Theorem II gives

$$K_2(s,t) = \sum_{k=1}^{\infty} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k^2}\,. \tag{157}$$

This may be proved (on the basis of Theorem II) for any s of $[a, b]$ and any t of the same interval, i.e. the formula holds throughout the square k_0.

We recall the formula [5]:

$$K_n(s,t) = \int_a^b K(s,t_1)\,K_{n-1}(t_1,t)\,\mathrm{d}t, \tag{158}$$

It follows from (152) that the Fourier coefficients of $K_n(s, t)$ as a function of s are equal to the Fourier coefficients of $K_{n-1}(t_1, t)$ as a

function of t_1 divided by λ_k, i.e. $\varphi_k(t)/\lambda_k$ for $K(s, t)$, $\varphi_k^{(t)}/\lambda_k^2$ for $K_2(s, t)$ and so on, and in general, $\varphi_k^{(t)}/\lambda_k^n$ for $K_n(s, t)$, and Theorem II gives

$$K_n(s,t) = \sum_{k=1}^{\infty} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k^n} \quad (n = 2, 3, \ldots), \tag{159}$$

the series being convergent in k_0 as above. Let us investigate the nature of the convergence of these series.

By Theorem II, we can assert the regular convergence of the series written with respect to the variable s in the interval $[a, b]$ for any fixed value of t in this interval. In view of the symmetry, we also have regular convergence with respect to the variable t for fixed s. We show that *the series are regularly convergent with respect to both variables in the square* k_0. It is sufficient to carry out the proof for series (157). The proof retains its force all the more for the remaining series (with $n > 2$), because $|\lambda_n| \to +\infty$. On applying the obvious inequality

$$\left|\frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k^2}\right| \leqslant \frac{1}{2}\left[\frac{\varphi_k^2(s)}{\lambda_k^2} + \frac{\varphi_k^2(t)}{\lambda_k^2}\right],$$

we see that it is sufficient to prove that the series $\sum_{k=1}^{\infty} \varphi_k^2(s)/\lambda_k^2$ is uniformly convergent in the interval $[a, b]$. This latter series is obtained from series (157) with $t = s$, so that its sum is equal to

$$\sum_{k=1}^{\infty} \frac{\varphi_k^2(s)}{\lambda_k^2} = K_2(s, s).$$

The terms of the series written are non-negative continuous functions and its sum is a continuous function in $[a, b]$, so that the uniform convergence of the series follows directly from Dini's theorem.

Let us deduce some properties of the formulae obtained. On setting $t = s$ in (159) and integrating with respect to s, we find on taking the normalization of functions $\varphi_k(s)$ into account an expression for the so-called *traces of iterated kernels* in terms of the eigenvalues of the basic kernel:

$$\int_a^b K_n(s,s)\,ds = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^n}. \tag{160}$$

We can write, on taking (156) into account:

$$\int_a^b K_2(s,s)\,ds = \int_a^b \int_a^b [K(s,t)]^2\,ds\,dt,$$

and (160) with $n = 2$ leads us to the equation

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = \int_a^b \int_a^b [K(s, t)]^2 \, ds \, dt. \tag{161}$$

We recall that we merely proved an inequality (142) above.

Formula (159) can prove to be invalid for $n = 1$. But we now show that, with any fixed s of $[a, b]$:

$$\lim_{n \to \infty} \int_a^b \left[K(s, t) - \sum_{k=1}^{n} \frac{\varphi_k(s)\varphi_k(t)}{\lambda_k} \right]^2 dt = 0, \tag{162}$$

the convergence to zero being uniform with respect to s. Expression (162) shows that *the mean square error obtained by replacing $K(s, t)$ by a segment of its Fourier series tends to zero as $n \to \infty$*. We turn to the proof of our assertion. Regarding $K(s, t)$ as a function of t, we have $\varphi_k(s) : \lambda_k$ for its Fourier coefficients, and (23) gives us

$$\int_a^b \left[K(s, t) - \sum_{k=1}^{n} \frac{\varphi_k(s)\varphi_k(t)}{\lambda_k} \right]^2 dt = \int_a^b [K(s, t)]^2 \, dt - \sum_{k=1}^{n} \frac{\varphi_k^2(s)}{\lambda_k^2} .$$

But we have seen that

$$\int_a^b [K(s, t)]^2 \, dt = K_2(s, s),$$

and, in accordance with (157):

$$\sum_{k=1}^{n} \frac{\varphi_k^2(s)}{\lambda_k^2} \to K_2(s, s),$$

this convergence being uniform with respect to s, as we saw above. Hence it follows that expression (162) tends to zero uniformly with respect to s. All the more:

$$\int_a^b \int_a^b \left[K(s, t) - \sum_{k=1}^{n} \frac{\varphi_k(s)\varphi_k(t)}{\lambda_k} \right]^2 ds \, dt \to 0 .$$

Suppose that series (144) is uniformly convergent with respect to t in $[a, b]$ for any fixed s, and let $K^*(s, t)$ denote the sum of this series. On passing to the limit under the integral sign in (162), we obtain

$$\int_a^b [K(s, t) - K^*(s, t)]^2 \, dt = 0,$$

whence it follows at once that $K^*(s,t) = K(s,t)$, i.e. *when proving formula* (145) *it is unnecessary to assume the uniform convergence of the series with respect to both the variables in the square* k_0, *it being sufficient merely to assume that the series is uniformly convergent with respect to one of the variables with any fixed value of the other.*

We consider the difference

$$\omega_n(s,t) = K(s,t) - \sum_{k=1}^{n} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k} \tag{163}$$

as the kernel of an integral equation

$$\varphi(s) = \lambda \int_a^b \omega_n(s,t)\,\varphi(t)\,\mathrm{d}t, \tag{164}$$

and show that *the numbers* $\lambda_{n+1}, \lambda_{n+2}, \ldots$ *and functions* $\varphi_{n+1}(s)$, $\varphi_{n+2}(s), \ldots$ *represent a complete set of eigenvalues and eigenfunctions of equation* (164). We multiply both sides of (163) by $\lambda_m \varphi_m(t)$, where $m > n$, and integrate with respect to t. On taking the orthogonality of functions $\varphi_p(t)$ into account, we obtain

$$\lambda_m \int_a^b \omega_n(s,t)\,\varphi_m(t)\,\mathrm{d}t = \lambda_m \int_a^b K(s,t)\,\varphi_m(t)\,\mathrm{d}t,$$

or, on observing that $\varphi_m(t)$ is the eigenfunction of kernel $K(s,t)$ corresponding to the eigenvalue λ_m:

$$\lambda_m \int_a^b \omega_m(s,t)\,\varphi_m(t)\,\mathrm{d}t = \varphi_m(s).$$

Equation (164) is thus seen to have the same eigenvalues λ_m and corresponding eigenfunctions $\varphi_m(s)$ with $m > n$ as the basic equation. It remains to show that the system of eigenvalues and eigenfunctions of equation (164) is complete. We multiply both sides of (163) by $\varphi_m(s)$, where $m \leqslant n$. On taking the orthogonality and normalization of functions $\varphi_p(s)$ into account, we obtain

$$\int_a^b \omega_n(s,t)\,\varphi_m(s)\,\mathrm{d}t = \int_a^b K(s,t)\,\varphi_m(s)\,\mathrm{d}s - \frac{\varphi_m(t)}{\lambda_m}.$$

The difference on the right-hand side vanishes, because $\varphi_m(t)$ is an eigenfunction of kernel $K(s,t)$ corresponding to the eigenvalue λ_m, i.e.

$$\int_a^b \omega_n(s,t)\,\varphi_m(s)\,\mathrm{d}s = 0 \qquad (m \leqslant n). \tag{165}$$

Let λ be an eigenvalue of equation (164) and $\varphi(s)$ a corresponding eigenfunction. On multiplying both sides of (164) by $\varphi_m(s)$ and taking (165) into account, we obtain

$$\int_a^b \varphi(s)\,\varphi_m(s)\,\mathrm{d}s = 0 \quad (m \leqslant n). \tag{166}$$

On replacing $\omega_n(s, t)$ in (164) by its expression (163) and using (165), we can write (164) as

$$\varphi(s) = \lambda \int_a^b K(s, t)\,\varphi(t)\,\mathrm{d}t,$$

i.e. $\varphi(s)$ is an eigenfunction of the basic kernel also, and by (166), is at the same time a function orthogonal to $\varphi_m(s)$ with $m \leqslant n$, whence it follows that the corresponding eigenvalue λ is the same as one of the λ_k with $k > n$, whilst $\varphi(s)$ is one of the functions $\varphi_k(s)$ with $k > n$ or a linear combination of them in the case of an eigenvalue of rank greater than unity. Our statement about the eigenfunctions of kernel $\omega_n(s, t)$ is thus proved.

It follows from (159) that the kernels $K_n(s, t)$ are symmetric. This could be shown directly from the definition of them. We form the homogeneous integral equation

$$\varphi(s) = \lambda \int_a^b K_n(s, t)\,\varphi(t)\,\mathrm{d}t. \tag{167}$$

It is easily shown by using the uniform convergence of the series that the λ_k^n are eigenvalues of equation (167), whilst the $\varphi_k(s)$ are corresponding eigenfunctions. Let us prove that there are no other eigenvalues and eigenfunctions. If an eigenvalue λ exists, different from all the λ_k, a corresponding eigenfunction will have to be orthogonal to all the $\varphi_k(s)$, i.e.

$$\int_a^b \varphi_k(t)\,\varphi(t)\,\mathrm{d}t = 0 \quad (k = 1, 2, \ldots).$$

But now, by (159), the right-hand side of (167) vanishes for any s, i.e. $\varphi(s)$ is identically zero, which is absurd. We now suppose that the eigenvalue λ coincides with one or several of the λ_k. We have to show that $\varphi(s)$ is a linear combination of corresponding $\varphi_k(s)$. If this were not so, i.e. if $\varphi(s)$ were linearly independent of the $\varphi_k(s)$ in question, by applying a process of orthogonalization we should be able to construct a $\varphi(s)$, not identically zero and orthogonal to all these $\varphi_k(s)$.

This function is orthogonal to the remaining $\varphi_k(s)$, since these latter correspond to different eigenvalues.

Thus $\varphi(s)$ is orthogonal to all the $\varphi_k(s)$, and we should be able to prove as above that $\varphi(s)$ is identically zero, which is absurd.

Hence λ_k^n and $\varphi_k(s)$ $(k = 1, 2, \dots)$ form a complete set of eigenvalues and eigenfunctions of the kernel $K_n(s, t)$.

If we divide both sides of the basic equation

$$\varphi(s) = \lambda \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t,$$

for the eigenvalues and eigenfunctions by λ and then put $\lambda = \infty$, we obtain the equation

$$\int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t = 0 \quad (s \text{ is any point of } [a, b]), \tag{168}$$

i.e. formally speaking, the latter equation defines the eigenfunctions (if there are any) corresponding to the eigenvalue $\lambda = \infty$.

Definition. *The continuous function $\varphi(t)$ is said to be orthogonal to the kernel if it satisfies equation* (168) *for any s of $[a, b]$.*

We prove the following:

Theorem 3. *The necessary and sufficient condition for the continuous function $\varphi(t)$ to be orthogonal to the kernel is that it be orthogonal to all the eigenfunctions of the kernel.*

We have to show that (168) is equivalent to

$$\int_a^b \varphi_k(t)\, \varphi(t)\, \mathrm{d}t = 0 \quad (k = 1, 2, \dots). \tag{169}$$

We show first that (169) follows from (168). This is done by multiplying both sides of (168) by $\varphi_k(s)$ and integrating with respect to s. On changing the order of integration and using the symmetry of the kernel, we obtain:

$$\int_a^b \Big[\int_a^b K(t, s)\, \varphi_k(s)\, \mathrm{d}s \Big] \varphi(t)\, \mathrm{d}t = 0 \quad \text{or} \quad \lambda_k \int_a^b \varphi_k(t)\, \varphi(t)\, \mathrm{d}t = 0.$$

whence (169) follows.

We now show that (168) follows from (169). On taking (157) into account and the uniform convergence of the series, we obtain

$$\int_a^b K_2(t, s)\, \varphi(s)\, \mathrm{d}s = 0.$$

We multiply both sides by $\varphi(t)$ and integrate with respect to t:

$$\int_a^b \int_a^b K_2(t, s)\, \varphi(s)\, \varphi(t)\, ds\, dt = 0,$$

the order of integration being of no significance.

By using (156), we can write this as

$$\int_a^b \int_a^b \int_a^b K(t, t_1)\, K(t_1, s)\, \varphi(s)\, \varphi(t)\, ds\, dt\, dt_1 = 0, \tag{170}$$

or, using the symmetry of the kernel:

$$\int_a^b \Big[\int_a^b K(t_1\, t)\, \varphi(t)\, dt \Big] \Big[\int_a^b K(t_1, s)\, \varphi(s)\, ds \Big] dt_1 = 0.$$

The integrals in the square brackets are the same, so that

$$\int_a^b \Big[\int_a^b K(t_1, s)\, \varphi(s)\, ds \Big]^2 dt_1 = 0,$$

whence it follows [3] that

$$\int_a^b K(t_1, s)\, \varphi(s)\, ds = 0 \quad (t_1 \text{ is any point of } [a, b]),$$

i.e. we obtain (168).

25. Classification of symmetric kernels. Let $p(s)$ and $q(s)$ be two continuous functions in the interval $[a, b]$. We form the double integral

$$\int_a^b \int_a^b K(s, t)\, p(s)\, q(t)\, ds\, dt,$$

similar to the bilinear form

$$\sum_{i,k=1}^{n} a_{ik}\, x_i\, y_k \qquad (a_{ik} \text{ real};\ a_{ki} = a_{ik}),$$

which we considered in [III$_1$, 40]. We obtain by using Theorem II:

$$\int_a^b K(s, t)\, q(t)\, dt = \sum_{k=1}^{\infty} \frac{q_k}{\lambda_k}\, \varphi_k(s),$$

where the q_k are the Fourier coefficients of the function $q(t)$, and the series on the right is regularly convergent. On multiplying both sides by $p(s)$, integrating with respect to s and writing p_k for the Fourier coefficients of the functions $p(s)$, we get the following expression for

the integral:

$$\int_a^b \int_a^b K(s,t)\, p(s)\, q(t)\, \mathrm{d}s\, \mathrm{d}t = \sum_{k=1}^{\infty} \frac{p_k q_k}{\lambda_k}, \tag{171}$$

the series on the right being absolutely convergent. With $q(s) \equiv p(s)$, we obtain the analogue of the quadratic form:

$$J = \int_a^b \int_a^b K(s,t)\, p(s)\, p(t)\, \mathrm{d}s\, \mathrm{d}t = \sum_{k=1}^{\infty} \frac{p_k^2}{\lambda_k}. \tag{172}$$

This formula provides the basis of the classification of symmetric kernels [cf. III_1, 35]. The kernel $K(s, t)$ is said to be *positive* if the integral

$$\int_a^b \int_a^b K(s,t)\, p(s)\, p(t)\, \mathrm{d}s\, \mathrm{d}t \tag{173}$$

is non-negative for any choice of continuous function $p(s)$. If all the eigenvalues λ_k are positive, it follows at once from (172) that the kernel is also positive in the sense indicated. We now suppose that there is at least one negative eigenvalue and show that the kernel cannot now be positive. In fact, we take say $\lambda_1 < 0$ and replace $p(s)$ by $\varphi_1(s)$ in (172). By virtue of the functions $\varphi_k(s)$ being orthogonal and normalized, we now have $p_1 = 1$, and the remaining p_k vanish, so that the right-hand side of (172) becomes $1/\lambda_1$ and will be negative. We thus see that *the fact of the kernel being positive is equivalent to all the eigenvalues of the kernel being positive.*

Similarly, the kernel $K(s, t)$ is said to be *negative* if $J \leqslant 0$ for any choice of continuous function $p(s)$. Precisely as above, it can be shown that *the fact of the kernel being negative is equivalent to all its eigenvalues being negative.* We introduce a further new concept. The kernel $K(s, t)$ is said to be *complete in the class of continuous functions*, or simply *complete*, if there exists no continuous function, not identically zero, which is orthogonal to all the eigenfunctions, i.e. by Theorem 3 of [24], the kernel is complete if there is no continuous function not identically zero which is orthogonal to the kernel. In other words, the completeness of the kernel amounts to the requirement that integral equations (168) have no continuous solutions other than identically zero solutions. We can say that the completeness of the kernel amounts to the requirement that there are no continuous eigenfunctions corresponding to the value $\lambda = \infty$.

Let $K(s, t)$ be a positive kernel, i.e. all the λ_k are positive. The right-hand side of (172) can now vanish only when all the Fourier coefficients p_k of the function $p(s)$ vanish, i.e. when $p(s)$ is orthogonal to all the eigenfunctions of the kernel. There can be no such continuous function $p(s)$ not identically zero for a complete kernel, so that we can say, for a positive complete kernel, that integral (172) will be strictly positive for any continuous function not identically zero. Conversely, if we are given that the integral $J > 0$ for any continuous function, not identically zero, we can say that the kernel is complete. The kernel is said to be *positive definite* if $J > 0$ for any continuous function not identically zero. It follows from the previous discussion that *a positive kernel will be positive definite when and only when it is complete*. Similarly, a kernel is said to be *negative definite* if $J < 0$ for any continuous function $p(s)$ not identically zero. As above, it can be shown that *a negative kernel will be negative definite when and only when it is complete*.

Suppose that the eigenfunctions $\varphi_k(s)$ of kernel $K(s, t)$ form a closed system of functions [3]. In this case, as we know, there is no continuous function, not identically zero, which is orthogonal to all the $\varphi_k(s)$, i.e. the completeness of the kernel follows from the fact that the system of eigenfunctions is closed. The converse is not true, i.e. if the kernel is complete, it does not follow from this that its system of eigenfunctions is closed in the sense of our definition of [3].

A complete kernel can be constructed for which the system of eigenfunctions is not closed. For such a kernel, equation (168) will have no continuous solutions, but must have certain solutions consisting of discontinuous functions. By using a new concept of the integral which will be discussed later, and considering a wider class of functions than the class of continuous functions, we can define a complete kernel in such a way that its system of eigenfunctions must be closed.

Every iterated kernel $K_n(s, t)$ with even subscript has positive eigenvalues, so that it is necessarily positive. It follows from the definition of the second iterated kernel that integral (172) for the second iterated kernel can be written as

$$\int_a^b \int_a^b K_2(s, t)\, p(s)\, p(t)\, \mathrm{d}s\, \mathrm{d}t = \int_a^b \Big[\int_a^b K(s, t)\, p(s)\, \mathrm{d}s\Big]^2 \mathrm{d}t,$$

whence it follows at once that this iterated kernel is complete if the basic kernel is complete.

Let the system of eigenfunctions of the kernel be finite. It is easily seen that the kernel cannot be complete in this case. For, a polynomial

of sufficiently high degree is readily constructed which is orthogonal to all the eigenfunctions. Suppose we have say two eigenfunctions $\varphi_1(s)$ and $\varphi_2(s)$ altogether. We construct a second degree polynomial orthogonal to both these functions. We thus arrive at the two homogeneous equations

$$\alpha \int_a^b s^2 \varphi_i(s)\,\mathrm{d}s + \beta \int_a^b s\varphi_i(s)\,\mathrm{d}s + \gamma \int_a^b \varphi_i(s)\,\mathrm{d}s = 0 \quad (i = 1, 2)$$

with three unknowns α, β, γ. Such a system certainly has a non-zero solution [III_1, 10].

26. Extremal properties of the eigenfunctions. The eigenfunctions and eigenvalues of a symmetric kernel have extremal properties analogous to those in algebra for the eigenvalues of a quadratic form, the role of quadratic form being played by integral (173).

We first suppose for simplicity that the kernel is positive, i.e. that all its eigenvalues λ_k are positive. We consider the class of continuous functions normalized to unity, i.e. satisfying the condition

$$\int_a^b p^2(s)\,\mathrm{d}s = 1, \tag{174}$$

and we seek in this class the function for which integral (173) has a maximum value. We assume that the positive eigenvalues are arranged in non-decreasing order, i.e.

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots \tag{175}$$

By Bessel's inequality:

$$\sum_{k=1}^{\infty} p_k^2 \leqslant 1,$$

and, on taking (175) into account, we can write

$$\int_a^b \int_a^b K(s, t)\, p(s)\, p(t)\,\mathrm{d}s\,\mathrm{d}t \leqslant \frac{1}{\lambda_1} \sum_{k=1}^{\infty} p_k^2,$$

i.e. with condition (174) we can write for integral (173) the inequality

$$J \leqslant \frac{1}{\lambda_1}\,.$$

We shall have the sign of equality in this expression if we put $p(s) = \varphi_1(s)$, because in this case $p_1 = 1$ and $p_k = 0$ for $k > 1$, i.e. *in the*

case of a positive kernel the maximum value of integral (173) *under condition* (174) *will be equal to* $1/\lambda_1$ *and will be attained with* $p(s) = \varphi_1(s)$.

We now pose the following extremal problem. We want to find the maximum value of integral (173) on condition that the continuous function $p(s)$ is normalized to unity and orthogonal to the eigenfunction $\varphi_1(s)$, i.e.

$$\int_a^b p^2(s)\,ds = 1; \quad \int_a^b p(s)\varphi_1(s)\,ds = 0. \tag{176_1}$$

By virtue of the second of these conditions, we must assume $p_1 = 0$ in (172), and by arguing precisely as above, we then show that, in the case of a positive kernel, the maximum value of integral (173) under conditions (176_1) is equal to $1/\lambda_2$ and is attained when $p(s) = \varphi_2(s)$. Similarly, we can show that, in the case of a positive kernel, the maximum value of integral (173) with the conditions:

$$\int_a^b p^2(s)\,ds = 1; \tag{176_2}$$

$$\int_a^b p(s)\varphi_1(s)\,ds = \int_a^b p(s)\varphi_2(s)\,ds = \ldots = \int_a^b p(s)\varphi_{n-1}(s)\,ds = 0$$

will be equal to $1/\lambda_n$ and will be attained when $p(s) = \varphi_n(s)$. We can therefore characterize the reciprocals of the eigenvalues of a positive kernel as the successive maxima of integral (173) under the conditions stated above for the function $p(s)$. The eigenfunctions are simultaneously determined for which integral (173) attains its maxima.

If the kernel is negative, we have to talk about the successive minima of integral (173) under conditions (177) instead of the successive maxima. If the kernel has eigenvalues of both signs, the problem of the successive maxima of the integral leads us to the reciprocals of the positive eigenvalues, and the problem of the minima of (173) to the reciprocals of the negative eigenvalues. This extremal problem is precisely analogous to the one that we had in [III_1, 39] for a quadratic form. We have obtained the reciprocals here instead of the eigenvalues themselves, due to the fact that the parameter λ plays a different role in integral equations to that in linear algebra [2].

A different statement of the extremal problems will be of importance to us later. We take the class of continuous functions $p(s)$, satisfying the following condition:

$$\int_a^b \Big[\int_a^b K(s,t)\,p(t)\,dt\Big]^2 ds = 1, \tag{177}$$

i.e. we require that the transformations of the continuous functions with the aid of the kernel $K(s, t)$, and not the functions themselves, be normalized to unity. By Theorem II, an expansion exists of the transformed function in an absolutely and uniformly convergent Fourier series. By virtue of the uniform convergence of this series, the greatest absolute value of the difference between the expanded function and a segment of its Fourier series tends to zero on indefinite increase of the number of terms in the segment. All the more, the mean square error i.e. the integral of the square of this difference, tends to zero, and we have in this case the closure formula:

$$\int_a^b \left[\int_a^b K(s, t)\, p(t)\, \mathrm{d}t \right]^2 \mathrm{d}s = \sum_{k=1}^{\infty} \frac{p_k^2}{\lambda_k^2},$$

so that condition (177) can be written as

$$\sum_{k=1}^{\infty} \frac{p_k^2}{\lambda_k^2} = 1. \tag{178}$$

We shall assume a positive kernel and rewrite the right-hand side of (172) as follows:

$$\sum_{k=1}^{\infty} \frac{p_k^2}{\lambda_k} = \sum_{k=1}^{\infty} \frac{p_k^2}{\lambda_k^2} \lambda_k.$$

On replacing the factors λ_k by their least λ_1 and using condition (178), we obtain an inequality for the right-hand side of (172):

$$J \geqslant \lambda_1. \tag{179}$$

If we put $p(s) = \lambda_1 \varphi_1(s)$, then $p_1 = \lambda_1$ and $p_k = 0$ for $k > 1$, so that condition (178) is satisfied, and we have the sign of equality in (179). Thus, *the first eigenvalue λ_1 is the least value of integral* (173) *under condition* (177). This least value is attained if we put $p(s) = \lambda_1 \varphi_1(s)$. We can show, precisely as above, that *the eigenvalue λ_2 is the least value of integral* (173) *if the function $p(s)$ satisfies the following conditions:*

$$\int_a^b \left[\int_a^b K(s, t)\, p(t)\, \mathrm{d}t \right]^2 \mathrm{d}s = 1; \quad \int_a^b p(s)\, \varphi_1(s)\, \mathrm{d}s = 0,$$

and this least value is attained if we put $p(s) = \lambda_2 \varphi_2(s)$.

It is easily seen that the extremal principle deduced by us for obtaining the eigenvalues and eigenfunctions is applicable not merely to a positive kernel, but to one that has a finite number of

negative eigenvalues, i.e. for which the eigenvalues can be arranged in non-decreasing order, starting with the first. We remark that, if say $\lambda_1 = \lambda_2 = \lambda_3 < \lambda_4$, integral (173) will attain its least value under condition (176_2) with $p(s) = \lambda_1 \varphi_1(s)$, $p(s) = \lambda_1 \varphi_2(s)$ and $p(s) = \lambda_1 \varphi_3(s)$, as also with any linear combination $p(s) = c_1 \varphi_1(s) + c_2 \varphi_2(s) + c_3 \varphi_3(s)$ in which the coefficients satisfy the condition $(c_1^2 + c_2^2 + c_3^2)/\lambda_1^2 = 1$. This will exhaust all the functions $p(s)$ that give the integral a minimum value. A similar remark also holds for the first extremal problem mentioned above.

Suppose that the kernel has a finite number of positive eigenvalues, and let this number be $(n-1)$. By determining successively the maxima of integral (173) under condition (174) and auxiliary orthogonality conditions, we finally arrive at conditions (176_2) and discover that, under these conditions, the integral can no longer take positive values. For, under conditions (176_2), only negative terms remain in the right-hand side of (172). We assume here that $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are positive eigenvalues.

27. Mercer's Theorem. As we have already remarked above, the Fourier series (144) for the kernel may prove not to be convergent. Mercer's theorem states that the series is absolutely and uniformly convergent if the kernel is positive or negative, i.e. if all its eigenvalues have the same sign. Thus, *if* $K(s, t)$ *is a positive or negative continuous kernel, expansion* (145) *holds, and the series is regularly convergent in the square* k_0. We shall assume for clarity that the kernel is positive. We show first that the inequality $K(s, s) \geqslant 0$ holds for any positive kernel.

In fact, if a point $s = t = c$ existed on the diagonal of the square k_0 at which $K(c, c) < 0$, there would be a neighbourhood of the point, $|s - c| < \varepsilon$ and $|t - c| < \varepsilon$, such that $K(s, t) < 0$ throughout this neighbourhood. We can define a continuous function $p(s)$ having positive values in the interval $c - \varepsilon < s < c + \varepsilon$ and vanishing everywhere outside the interval. We shall have for this function:

$$J = \int_a^b \int_a^b K(s,t)\, p(s)\, p(t)\, \mathrm{d}s\, \mathrm{d}t = \int_{c-\varepsilon}^{c+\varepsilon} \int_{c-\varepsilon}^{c+\varepsilon} K(s,t)\, p(s)\, p(t)\, \mathrm{d}s\, \mathrm{d}t < 0\,,$$

which contradicts the kernel's being positive. We form the kernel

$$K(s,t) - \sum_{k=1}^{n} \frac{\varphi_k(s)\, \varphi_k(t)}{\lambda_k}. \tag{180}$$

Its eigenvalues $\lambda_{n+1}, \lambda_{n+2}, \ldots$ are positive. On applying to this kernel the fact just proved, we obtain:

$$K(s,s) - \sum_{k=1}^{n} \frac{\varphi_k^2(s)}{\lambda_k} \geqslant 0, \quad \text{i.e.} \quad \sum_{k=1}^{n} \frac{\varphi_k^2(s)}{\lambda_k} \leqslant K(s,s).$$

It follows at once from this that the series $\sum_{k=1}^{\infty} \varphi_2^k(s)/\lambda_k$ with positive terms is convergent for every value of s and that its partial sums remain less than a positive number M for any s of $[a, b]$. On using Cauchy's inequality, we can write:

$$\sum_{k=n}^{n+p} \left| \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k} \right| = \sum_{k=n}^{n+p} \left| \frac{\varphi_k(s)}{\sqrt{\lambda_k}} \right| \cdot \left| \frac{\varphi_k(t)}{\sqrt{\lambda_k}} \right|$$

$$\leqslant \sqrt{\sum_{k=n}^{n+p} \frac{\varphi_k^2(s)}{\lambda_k}} \sqrt{\sum_{k=n}^{n+p} \frac{\varphi_k^2(t)}{\lambda_k}},$$

or

$$\sum_{k=n}^{n+p} \left| \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k} \right| \leqslant \sqrt{\sum_{k=n}^{n+p} \frac{\varphi_k^2(s)}{\lambda_k}} \sqrt{M},$$

and it now follows immediately, by virtue of the convergence of $\sum_{k=1}^{\infty} \varphi_k^2(s)/\lambda_k$, that series (144) is convergent uniformly with respect to t in $[a, b]$ for a fixed s.Formula (145) follows from this, as we know from [23].

The absolute and uniform convergence of the series in k_0 can be proved by using Dini's theorem, precisely as was done in [24] for the case of the second iterated kernel. It is worth pointing out that the only essential fact for our proof of the theorem was that the eigenvalues λ_k are all positive, as from a certain index. It was this fact that showed us that kernel (180) is positive. Consequently, the proof retains its force in the case when the kernel $K(s, t)$ has a finite number of negative eigenvalues, and in general, *Mercer's theorem remains valid in the case when the kernel has only a finite number of positive or negative eigenvalues.* It may be remarked that the continuity of the kernel is an absolutely essential condition for the theorem to hold.

28. The case of a weakly polar kernel. Let us take the case of a one-dimensional weakly polar kernel:

$$K(s, t) = \frac{L(s, t)}{|s-t|^a} \qquad \left(0 < a < \frac{1}{2}\right). \tag{181}$$

We introduce, as in [17], the continuous kernels

$$K_\gamma(s, t) = \begin{cases} K(s, t) & \text{when } |s-t| \geqslant \gamma \\ \dfrac{L(s, t)}{\gamma^a} & \text{when } |s-t| \leqslant \gamma. \end{cases} \tag{182}$$

The inequalities hold:

$$|K_\gamma(s, t)| \leqslant |K(s, t)| \leqslant \frac{C}{|s-t|^a}. \tag{183}$$

We form the second iterated kernel:

$$K_2(s, t) = \int_a^b K(s, t_1) K(t_1, t)\, \mathrm{d}t_1. \tag{184$_1$}$$

This last integral has a meaning, by (181), for any positive s and t in $[a, b]$, since we have the following inequality for the integrand in the most unfavourable case when s and t coincide:

$$|K(s, t_1) K(t, s)| \leqslant \frac{C^2}{|s-t_1|^{2a}}.$$

We show that $K_2(s, t)$ is a continuous function in the square k_0.

We construct the function

$$K_2^{(\gamma)}(s, t) = \int_a^b K_\gamma(s, t_1) K_\gamma(t_1, t)\, \mathrm{d}t_1, \tag{184$_2$}$$

which is continuous in k_0. It is sufficient to show that $K_2^{(\gamma)}(s, t) \to K_2(s, t)$ uniformly in k_0 as $\gamma \to 0$. We have:

$$K_2(s,t) - K_2^{(\gamma)}(s,t) = \int_a^b \left[K(s, t_1) K(t_1, t) - K_\gamma(s,t_1) K_\gamma(t_1,t)\right] \mathrm{d}t_1.$$

The difference on the right-hand side vanishes if $|s - t_1| \geqslant \gamma$ and $|t - t_1| \geqslant \gamma$. On taking (183) into account, we can write:

$$|K_2(s, t) - K_2^{(\gamma)}(s, t))|$$
$$\leqslant 2C^2 \left[\int_{s-\gamma}^{s+\gamma} \frac{\mathrm{d}t_1}{|t_1 - s|^a |t_1 - t|^a} + \int_{t-\gamma}^{t+\gamma} \frac{\mathrm{d}t_1}{|t_1 - s|^a |t_1 - t|^a}\right]. \tag{185}$$

If $|s - t| \geqslant 2\gamma$, the intervals $[s - \gamma, s + \gamma]$ and $[t - \gamma, t + \gamma]$ do not overlap, and we obtain

$$\int_{s-\gamma}^{s+\gamma} \frac{dt_1}{|t_1 - s|^a |t_1 - t|^a} \leqslant \int_{s-\gamma}^{s+\gamma} \frac{dt_1}{|t_1 - s|^a \gamma^a} \leqslant \frac{1}{\gamma^a}\left[\int_{s}^{s+\gamma} \frac{dt_1}{(t_1 - s)^a} + \int_{s-\gamma}^{s} \frac{dt_1}{(s - t_1)^a}\right],$$

i.e.

$$\int_{s-\gamma}^{s+\gamma} \frac{dt_1}{|t_1 - s|^a |t_1 - t|^a} \leqslant \frac{2\gamma^{1-2a}}{1-a},$$

so that

$$|K_2(s,t) - K_2^{(\gamma)}(s,t)| \leqslant \frac{4C^2\gamma^{1-2a}}{1-a} \qquad (|s-t| \geqslant 2\gamma). \tag{186}$$

We now suppose $|s - t| < 2\gamma$. The intervals $[s - \gamma, s + \gamma]$ and $[t - \gamma, t + \gamma]$ now overlap, and both are contained in an interval with centre at s or t and length 6γ. By using the inequality $ab \leqslant (1/2)(a^2 + b^2)$, we get:

$$\frac{1}{|t_1 - s|^a |t_1 - t|^a} \leqslant \frac{1}{2}\left[\frac{1}{|t_1 - s|^{2a}} + \frac{1}{|t_1 - t|^{2a}}\right],$$

so that

$$\int_{s-\gamma}^{s+\gamma} \frac{dt_1}{|t_1 - s|^a |t_1 - t|^a} \leqslant \frac{1}{2}\int_{s-3\gamma}^{s+3\gamma} \frac{dt_1}{|t_1 - s|^{2a}} + \frac{1}{2}\int_{t-3\gamma}^{t+3\gamma} \frac{dt_1}{|t_1 - s|^{2a}} \leqslant \frac{1}{1-2a}(3\gamma)^{1-2a},$$

and similarly,

$$\int_{t-\gamma}^{t+\gamma} \frac{dt_1}{|t_1 - s|^a |t_1 - t|^a} \leqslant \frac{1}{1-2a}(3\gamma)^{1-2a},$$

whence, by (185),

$$|K_2(s, t) - K_2^{(\gamma)}(s, t)| \leqslant \frac{4C^2}{1-2a}(3\gamma)^{1-2a} \qquad (|s-t| < 2\gamma). \tag{187}$$

On comparing this with (186), we see that $K_2^{(\gamma)}(s, t) \to K_2(s, t)$ uniformly in k_0, so that $K_2(s, t)$ is a continuous function in k_0.

The continuity of $K_3(s, t)$ and of the remaining kernels defined by (158) is proved even more simply.

It is easy also to prove the possibility of changing the order of the integrals:

$$\int_a^b \left[\int_a^b K(t_1, t)\, u(t)\, \mathrm{d}t\right] K(s, t_1)\, \mathrm{d}t_1 = \int_a^b \left[\int_a^b K(s, t_1)\, K(t_1, t)\, \mathrm{d}t_1\right] u(t)\, \mathrm{d}t, \tag{188}$$

where $u(t)$ is a continuous function.

For, a similar formula holds for continuous kernels with positive γ_1 and γ_2:

$$\int_a^b \left[\int_a^b K_{\gamma_1}(t_1, t)\, u(t)\, \mathrm{d}t\right] K_{\gamma_2}(s, t_1)\, \mathrm{d}t_1 = \int_a^b \left[\int_a^b K_{\gamma_2}(s, t_1)\, K_{\gamma_1}(t_1, t)\, \mathrm{d}t_1\right] u(t)\, \mathrm{d}t. \tag{189}$$

When γ_1 tends to zero, the inner integral on the left-hand side tends uniformly with respect to t_1 to the integral [17]

$$\int_a^b K(t_1, t)\, u(t)\, \mathrm{d}t,$$

and the inner integral on the right to

$$\int_a^b K_{\gamma_2}(s, t_1)\, K(t_1, t)\, \mathrm{d}t_1.$$

This follows from the fact that $v_\gamma(M)$, given by (111) of [17], tends uniformly to $v(M)$. On passing to the limit in (189) as $\gamma_1 \to 0$, we get:

$$\int_a^b \left[\int_a^b K(t_1, t)\, u(t)\, \mathrm{d}t\right] K_{\gamma_2}(s, t_1)\, \mathrm{d}t_1 = \int_a^b \left[\int_a^b K_{\gamma_2}(s, t_1)\, K(t_1, t)\, \mathrm{d}t_1\right] u(t)\, \mathrm{d}t. \tag{190}$$

As $\gamma_2 \to 0$ the iterated integral on the left tends to the left-hand side of (188) in accordance with the above arguments. The inner integral on the right of (190) tends uniformly with respect to t to the inner integral on the right of (188). This can be seen, precisely as in the above case of integral (184_2). On passing to the limit in (190), we in fact obtain (188).

We return to [22]. In (156), the role of $h(t)$ is played by the function $K(t_1, t)$ with weak polarity ($a < 1/2$) for $t_1 = t$. This function depends on the parameter t. In view of the weak polarity of the kernel, the sum of the squares of the coefficients h_k^2 forms a convergent series, which is just what we used in the proof of Theorem 2 of [22]. We can thus say that series (157) is regularly convergent with respect to s for any t of $[a, b]$. In particular,

$$\sum_k \frac{[\varphi_k(s)]^2}{\lambda_k^2} = K_2(s, s) \qquad (a \leqslant s \leqslant b),$$

the right-hand side being a continuous function, as we saw above. By Dini's theorem, the series written is uniformly convergent, whilst it follows from the inequality

$$|\varphi(s)\varphi(t)| \leqslant \frac{1}{2}\{[\varphi(s)]^2 + [\varphi(t)]^2\}$$

that series (157) is regularly convergent if the point (s, t) lies in the square k_0.

As already mentioned above, Theorems I and II will also be proved for weakly polar kernels.

In (158), with $n \geqslant 3$ the role of $h(t_1)$ is played by the continuous function $K_{n-1}(t_1, t)$. Thus, *expression* (159) *and the regular convergence of the corresponding series remain true for weakly polar kernels.*

Precisely as above, we can justify changing the order of integrations in the iterated integral (170) and hence all the results of [22], [24] and [25] still hold for weakly polar kernels.

All the results can be generalized at once to the case of n-dimensional space.

29. Non-homogeneous equations. We now take a non-homogeneous equation with a continuous or weakly polar symmetric kernel:

$$\varphi(s) = f(s) + \lambda \int_a^b K(s.t)\varphi(t)\,dt \tag{191}$$

and suppose first that λ is not an eigenvalue, i.e. it differs from all the λ_k. Equation (191) now has a unique solution. Let us express it in terms of the eigenfunctions $\varphi_k(s)$. We can write

$$\varphi(s) = f(s) + g(s), \tag{192}$$

where

$$g(s) = \lambda \int_a^b K(s, t)\varphi(t)\,dt. \tag{193}$$

By Theorem II, the function $g(s)$ can be expanded as an absolutely and uniformly convergent series in eigenfunctions of the kernel:

$$g(s) = \sum_{k=1}^{\infty} g_k \varphi_k(s).$$

Let us find the coefficients of this expansion. We cannot obtain them directly from (193), since the required function $\varphi(t)$ is under the integral sign in this expression. On replacing $\varphi(t)$ by the sum $f(t) + g(t)$ in accordance with (192), we can write

$$g(s) = \lambda \int_a^b K(s,t) \left[f(t) + g(t)\right] \mathrm{d}t. \tag{194}$$

Let f_k be the Fourier coefficients of the given function $f(s)$. The sum $f(t) + g(t)$ has Fourier coefficients $(f_k + g_k)$, so that the function expressible in terms of a kernel by the integral on the right-hand side of (194) has, by (152), Fourier coefficients $(f_k + g_k) : \lambda_k$, and we have by (194):

$$g_k = \frac{\lambda (f_k + g_k)}{\lambda_k}. \tag{195}$$

We can find the coefficients g_k from this expression:

$$g_k = \frac{\lambda f_k}{\lambda_k - \lambda}, \tag{196}$$

and consequently, by (192), the solution of equation (191) must have the form

$$\varphi(s) = f(s) + \lambda \sum_{k=1}^{\infty} \frac{f_k \varphi_k(s)}{\lambda_k - \lambda}. \tag{197}$$

We now suppose that λ is an eigenvalue. We shall assume for clarity that its rank is equal to three and that $\lambda = \lambda_1 = \lambda_2 = \lambda_3$.

By virtue of the symmetry of the kernel, the adjoint equation is the same as equation (191), and the necessary and sufficient condition for this equation to be soluble is that $f(s)$ be orthogonal to $\varphi_1(s)$, $\varphi_2(s)$ and $\varphi_3(s)$, i.e. that $f_1 = f_2 = f_3 = 0$. Suppose that this condition is fulfilled. Our further arguments are as above. Expression (195) makes it possible for us to find all the g_k, starting from g_4, in accordance with (196).

Expression (195) becomes an identity with $k = 1, 2, 3$, since $\lambda = \lambda_k$ and $f_k = 0$ with $k = 1, 2, 3$. This corresponds to the fact that we can add to the solution of equation (191) any linear combination of eigenfunctions $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$.

The general solution of equation (191) in the present case therefore has the form

$$\varphi(s) = f(s) + \sum_{k=4}^{\infty} \frac{f_k \varphi_k(s)}{\lambda_k - \lambda} + c_1\varphi_1(s) + c_2\varphi_2(s) + c_3\varphi_3(s), \tag{198}$$

where c_1, c_2, c_3 are arbitrary constants.

30. Fredholm's treatment in the case of a symmetric kernel. Let us apply Fredholm's treatment, as described above, to the case of a continuous symmetric kernel.

In this case, the Fredholm determinant (53) and resolvent will also be symmetric functions. The expansions of the iterated kernels have already been given [24]. Let us substitute these expansions in expression (45), it being assumed that λ satisfies condition (40), so that $|\lambda| < |\lambda_1|$:

$$R(s, t; \lambda) = K(s, t) + \lambda \sum_{n=1}^{\infty} \frac{\varphi_n(s)\varphi_n(t)}{\lambda_n^2} + \lambda^2 \sum_{n=1}^{\infty} \frac{\varphi_n(s)\varphi_n(t)}{\lambda_n^3} + \dots \tag{199}$$

It is easy to see that, if we replace all the magnitudes by their absolute values in this series, a double series with positive terms is obtained which is convergent. For, on collecting the terms in it containing $|\varphi_n(s)||\varphi_n(t)|$ into a single group, we obtain the series:

$$|K(s,t)| + |\varphi_1(s)||\varphi_1(t)| \sum_{k=1}^{\infty} \frac{|\lambda|^k}{|\lambda_1|^{k+1}} + |\varphi_2(s)||\varphi_2(t)| \sum_{k=1}^{\infty} \frac{|\lambda|^k}{|\lambda_2|^{k+1}} + \dots =$$

$$= |K(s,t)| + \sum_{n=1}^{\infty} |\varphi_n(s)||\varphi_n(t)| \frac{|\lambda|}{|\lambda_n|(|\lambda_n| - |\lambda|)}.$$

But we see by comparing this series with the uniformly convergent series

$$\sum_{n=1}^{\infty} \frac{|\varphi_n(s)||\varphi_n(t)|}{|\lambda_n|^2}, \tag{200}$$

that the ratio of their common terms $|\lambda||\lambda_n|^2/[|\lambda_n|(|\lambda_n| - |\lambda|)]$ is independent of the variables (s, t) and tends to $|\lambda|$, whence it follows that double series (199) is absolutely convergent. We can thus collect into a single group the terms in this series which contain $\varphi_n(s)\varphi_n(t)$. We thus obtain the following expansion for the resolvent in eigenfunctions:

$$R(s, t; \lambda) = K(s, t) + \lambda \sum_{n=1}^{\infty} \frac{\varphi_n(s)\varphi_n(t)}{\lambda_n(\lambda_n - \lambda)}. \tag{201}$$

Strictly speaking, we have deduced this expansion on the assumption that λ satisfies condition (40). However, by replacing all the terms in series (201) by their absolute values, and comparing as above the series obtained with series (200), it can be verified that series (201) is absolutely and uniformly convergent with respect to (s, t) for any λ differing from λ_n. Moreover, it is uniformly convergent with respect to λ in any bounded domain of the λ plane, provided we omit the first few terms that have poles in this domain. The right-hand side of (201) is therefore the decomposition of a fraction into partial fractions, and just as in the case of (57), it provides an analytic continuation of the resolvent $R(s, t; \lambda)$ throughout the plane. In particular, it follows from (201) that, *in the case of a symmetric kernel, every eigenvalue is a simple pole of the resolvent.* Notice that, if expansion (201) is substituted in expression (46), (197) is obtained, giving the expansion of the solution in eigenfunctions.

Let us put $t = s$ in (201) and integrate with respect to s:

$$\int_a^b R(s, s; \lambda)\,\mathrm{d}s = \int_a^b K(s, s)\,\mathrm{d}s + \lambda \sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n - \lambda)}.$$

But we have, on dividing both sides of (59) by $D(\lambda)$:

$$\int_a^b R(s, s; \lambda)\,\mathrm{d}s = -\frac{D'(\lambda)}{D(\lambda)},$$

so that the previous expression can be rewritten as

$$\frac{D'(\lambda)}{D(\lambda)} = -\int_a^b K(s, s)\,\mathrm{d}s + \lambda \sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda - \lambda_n)}.$$

Let λ_0 be a zero of $D(\lambda)$ of multiplicity r. We know from [III_2, 21] that $\lambda = \lambda_0$ is a simple pole with residue r of the left-hand side of the last expression. On the right-hand side, certain of the λ_n will coincide with λ_0. Each of the corresponding fractions can be rewritten as

$$\frac{\lambda}{\lambda_n(\lambda - \lambda_n)} = \frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n},$$

i.e. each of these fractions gives a residue equal to unity at the pole $\lambda = \lambda_0$, so that r of the λ_n must be equal to λ_0. We thus have the following theorem: if, in the case of a symmetric kernel, λ_0 is a zero

of multiplicity r of $D(\lambda)$, there correspond to this eigenvalue precisely r linearly independent eigenfunctions, i.e. *in the case of a symmetric kernel the multiplicity of a zero of* $D(\lambda)$ *is equal to the rank of the corresponding eigenvalue.*

We saw above that the kernel $K(s, t)$ has (144) as its Fourier series with respect to the system of eigenfunctions $\varphi_n(t)$. On substituting this series for $K(s, t)$ in the right-hand side of (201), we find that the resolvent has a Fourier series of the following form:

$$\sum_{n=1}^{\infty} \frac{\varphi_n(s)\,\varphi_n(t)}{\lambda_n - \lambda}. \tag{202}$$

Using the fact that the series on the right-hand side of (201) is uniformly convergent, we can say that series (202) is uniformly convergent along with series (144), and if the latter is uniformly convergent, we have, in addition to expression (145), the formula

$$R(s, t; \lambda) = \sum_{n=1}^{\infty} \frac{\varphi_n(s)\,\varphi_n(t)}{\lambda_n - \lambda}. \tag{203}$$

The Fourier coefficients of the function $R(s, t; \lambda)$ are in fact easily obtained directly by multiplying both sides of (201) by $\varphi_n(t)$ and integrating with respect to t. Using the fact that $\varphi_n(t)$ is an eigenfunction of the kernel $K(s, t)$ and that the $\varphi_n(t)$ are orthogonal and normalized, we thus obtain the coefficients of series (202):

$$\int_a^b R(s, t; \lambda)\,\varphi_n(t)\,\mathrm{d}t = \frac{1}{\lambda_n - \lambda}\,\varphi_n(s).$$

This equation shows that the $\varphi_n(s)$ are eigenfunctions of the kernel $R(s, t; \lambda)$ corresponding to the eigenvalues $(\lambda_n - \lambda)$, where the real value of λ is fixed in an arbitrary manner. It is easily seen that this is a complete system of all the eigenfunctions of the real symmetric kernel $R(s, t; \lambda)$. For, let there be a further eigenfunction $\varphi_0(s)$. If it corresponds to an eigenvalue different from all the values $(\lambda_n - \lambda)$, it must be orthogonal to all the $\varphi_k(s)$. If $\varphi_0(s)$ corresponds to some eigenvalue $(\lambda_0 - \lambda)$, $\varphi_0(s)$, being a new eigenfunction, must be linearly independent of those of the $\varphi_k(s)$ which correspond to the same eigenvalue. On adding to $\varphi_0(s)$ a linear combination of the $\varphi_k(s)$ corresponding to this eigenvalue, we can select the coefficients of the linear combination in such a way that the eigenfunction obtained is orthogonal to all the $\varphi_k(s)$ just mentioned. By the theorem proved in [21],

this new eigenfunction will be also orthogonal to all the $\varphi_k(s)$ corresponding to the other eigenvalues. We can therefore simply assume that the new eigenfunction $\varphi_0(s)$ is orthogonal to all the functions $\varphi_k(s)$. Hence it is also orthogonal to the kernel $K(s, t)$ [24]. We multiply both sides of (201) by $\varphi_0(t)$, and obtain by integrating with respect to t:

$$\int_a^b R(s, t; \lambda)\, \varphi_0(t)\, \mathrm{d}t = 0,$$

whence it follows that $\varphi_0(t)$ cannot be an eigenfunction of the kernel $R(s, t; \lambda)$. The hypothesis is therefore false, that the $\varphi_k(s)$ do not form a complete system of eigenfunctions of the kernel $R(s, t; \lambda)$.

We can thus assert that, *if $R(s, t; \lambda)$ is taken as the new kernel, this kernel has the same set of eigenfunctions $\varphi_n(s)$ as the original kernel whilst the corresponding eigenvalues are $(\lambda_n - \lambda)$.* On applying expression (201) to the kernel $R(s, t; \lambda)$ and writing μ for the parameter appearing in the resolvent, it may be seen that the resolvent of this kernel is

$$\widetilde{R}(s, t, \lambda; \mu) = R(s, t; \lambda) + \mu \sum_{n=1}^{\infty} \frac{\varphi_n(s)\, \varphi_n(t)}{(\lambda_n - \lambda)(\lambda_n - \lambda - \mu)},$$

whilst we easily find, by expanding $R(s, t; \lambda)$ in accordance with. (201) and carrying out elementary transformations, that

$$\widetilde{R}(s, t, \lambda; \mu) = K(s, t) + (\lambda + \mu) \sum_{n=1}^{\infty} \frac{\varphi_n(s)\, \varphi_n(t)}{\lambda_n[\lambda_n - (\lambda + \mu)]} = R(s, t; \lambda + \mu),$$

i.e. *if $R(s, t; \lambda)$ is taken as a new kernel, its resolvent will be $R(s, t; \lambda + \mu)$*

Expansion (201) also holds for weakly polar kernels inasmuch as the iterated kernel expansion has been obtained and the uniform convergence of series (200) has been proved for such kernels. The convergence of series (45) with λ close to zero can be proved very simply, by using say the expression

$$K_n(s, t) = \int_a^b K_{n-2}(s, t_1)\, K_2(t_1, t)\, \mathrm{d}t_1$$

and the continuity of the kernels $K_p(s, t)$ with $p \geqslant 2$. If λ is different from the λ_n, (46) also holds.

31. Hermitian kernels. We have defined a symmetric kernel as a real kernel which remains the same when its arguments are interchanged. These kernels are analogous to the symmetric matrices of

linear algebra. These symmetric matrices are a particular case of Hermitian matrices, whose elements satisfy the condition $a_{ki} = \overline{a_{ik}}$ [III_1, 40]. In the same way, symmetric kernels are a particular case of *Hermitian kernels,* which are defined by the property that they change to the conjugate when the arguments are interchanged. In the one-dimensional case:

$$K(t, s) = \overline{K(s, t)}. \tag{204}$$

All the previous theory still holds for such kernels when they are continuous or weakly polar. As regards justifying Theorem I, all the eigenvalues are real, but the eigenfunctions may be complex, and system (139) becomes an orthonormal system of complex functions:

$$\int_a^b \varphi_p(s)\overline{\varphi_q(s)}\,\mathrm{d}s = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q. \end{cases}$$

Series (144) takes the form

$$\sum_k \frac{\varphi_k(s)\overline{\varphi_k(t)}}{\lambda_k}. \tag{144_1}$$

This is the Fourier series in functions $\varphi_k(s)$ of $K(s, t)$, regarded as a function of s. It can also be regarded as the Fourier series of $K(s, t)$, defined in k_0, in functions $\varphi_k(s)\overline{\varphi_1(t)}$ $(k = 1, 2, \ldots)$, which form an orthonormal system in k_0 [cf. 22]).

When the series is uniformly convergent in k_0, we have

$$K(s, t) = \sum_k \frac{\varphi_k(s)\overline{\varphi_k(t)}}{\lambda_k}. \tag{145_1}$$

Theorem 2 and expression (152) still hold. Theorem II also still holds. We obtain the expansions for iterated kernels:

$$K_n(s, t) = \sum_k \frac{\varphi_k(s)\overline{\varphi_k(t)}}{\lambda_k^n} \quad (n = 2, 3, \ldots), \tag{159_1}$$

which are regularly convergent in k_0. We have, instead of (161):

$$\sum_k \frac{1}{\lambda_k^2} = \int_a^b\int_a^b |K(s, t)|^2\,\mathrm{d}s\,\mathrm{d}t, \tag{161_1}$$

and instead of (162):

$$\lim_{n\to\infty} \int_a^b \left| K(s, t) - \sum_{k=1}^{k} \frac{\varphi_k(s)\overline{\varphi_k(t)}}{\lambda_k} \right|^2 \mathrm{d}t = 0. \tag{162_1}$$

Theorem 3 is fully preserved, given the previous definition (168) of orthogonality to the kernel. Formula (172) is replaced by

$$\int_a^b\int_a^b K(s,t)\overline{p(s)}\,p(t)\,\mathrm{d}s\,\mathrm{d}t = \sum_k \frac{|p_k|^2}{\lambda_k} \tag{172$_1$}$$

and further, the discussion relating to the classification of kernels, the extremal properties of the eigenvalues, and the treatment of [30] are retained, assuming that the eigenfunctions are suitably replaced by their conjugates. The entire theory also admits of natural generalization to weakly polar Hermitian kernels. As previously, the non-homogeneous equation (191) has solution (197).

Integral equations with Hermitian kernels are directly connected with integral equations having so-called *skew-symmetric kernels* [13]. A real kernel is said to be skew-symmetric if it satisfies

$$K(t,s) = -K(s,t). \tag{205}$$

Obviously, if $K(s, t)$ is skew-symmetric, $iK(s, t)$ is an Hermitian kernel. Therefore, if we have the integral equation with skew-symmetric kernel

$$\varphi(s) = f(s) + \lambda \int_a^b K(s,t)\,\varphi(t)\,\mathrm{d}t,$$

we obtain on replacing λ by λi an integral equation with Hermitian kernel:

$$\varphi(s) = f(s) + \lambda \int_a^b iK(s,t)\,\varphi(t)\,\mathrm{d}t.$$

Hence it follows that an equation with a skew-symmetric kernel must have eigenvalues, and all these eigenvalues are pure imaginary.

32. Equations reducible to symmetric equations. We now mention a class of integral equations often encountered in applications, which can be reduced by a simple transformation to equations with symmetric kernels. These are equations of the form

$$\varphi(s) = f(s) + \lambda \int_a^b K(s,t)\,p(t)\,\varphi(t)\,\mathrm{d}t, \tag{206}$$

where $K(s, t)$ is a real symmetric kernel and $p(t) > 0$ in the interval $[a, b]$. On multiplying both sides by $\sqrt{p(s)}$ and introducing instead of

$\varphi(s)$ the new required function $\psi(s) = \sqrt{p(s)}\,\varphi(s)$, we arrive at the integral equation

$$\psi(s) = f(s)\sqrt{p(s)} + \lambda \int_a^b L(s, t)\,\psi(t)\,\mathrm{d}t$$

with the symmetric kernel

$$L(s, t) = K(s, t)\sqrt{p(s)\,p(t)}.$$

Let λ_k and $\psi_k(s)$ be the eigenvalues and eigenfunctions of the corresponding homogeneous equation. We can assume as usual that the $\psi_k(s)$ are orthogonal and normalized, i.e.

$$\int_a^b \psi_p(s)\,\psi_q(s)\,\mathrm{d}s = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q. \end{cases}$$

On using the expression

$$\psi_k(s) = \varphi_k(s)\sqrt{p(s)},$$

we find that the eigenfunctions of the homogeneous equation

$$\varphi(s) = \lambda \int_a^b K(s, t)\,p(t)\,\varphi(t)\,\mathrm{d}t$$

are orthogonal and normalized with the weight $p(s)$:

$$\int_a^b p(s)\,\varphi_p(s)\,\varphi_q(s)\,\mathrm{d}s = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q. \end{cases}$$

We have for the second iterated kernel

$$L_2(s, t) = \int_a^b K(s, t_1)\,K(t_1, t)\,p(t_1)\sqrt{p(s)\,p(t)}\,\mathrm{d}t_1$$

the expansion

$$L_2(s, t) = \sum_{k=1}^{\infty} \frac{\psi_k(s)\,\psi_k(t)}{\lambda_k^2}.$$

Hence, on cancelling by the factor $\sqrt{p(s)\,p(t)}$, we obtain for the function defined by the equation

$$H_2(s, t) = \int_a^b K(s, t_1)\,K(t_1, t)\,p(t_1)\,\mathrm{d}t_1,$$

the expansion

$$H_2(s, t) = \sum_{k=1}^{\infty} \frac{\varphi_k(s)\,\varphi_k(t)}{\lambda_k^2}.$$

Similarly, by applying complete induction for the functions

$$H_p(s, t) = \int_a^b H_{p-1}(s, t_1) K(t_1, t) p(t_1) \, dt$$

we obtain

$$H_p(s, t) = \sum_{k=1}^{\infty} \frac{\varphi_k(s) \varphi_k(t)}{\lambda_k^p}.$$

We have, in addition:

$$K(s, t) = \sum_{k=1}^{\infty} \frac{\varphi_k(s) \varphi_k(t)}{\lambda_k},$$

if the series on the right is uniformly convergent with respect to one of the variable with any fixed value of the other.

Suppose that the function $f(s)$ is expressible in terms of the kernel $L(s, t)$, i.e.

$$f(s) = \int_a^b L(s, t) h(t) \, dt. \tag{207}$$

Then

$$f(s) = \sum_{k=1}^{\infty} f_k \psi_k(s), \tag{208}$$

where

$$f_k = \int_a^b f(s) \psi_k(s) \, ds = \int_a^b f(s) \sqrt{p(s)} \, \varphi_k(s) \, ds.$$

On cancelling both sides of (207) and (208) by $\sqrt{p(s)}$, we obtain for the function

$$F(s) = f(s) : \sqrt{p(s)} = \int_a^b K(s, t) \sqrt{p(t)} \, h(t) \, dt$$

the expansion

$$F(s) = \sum_{k=1}^{\infty} F_k \varphi_k(s) \tag{209}$$

in an absolutely and uniformly convergent series, where the coefficients are defined in accordance with the usual Fourier rule with the addition of a weight:

$$F_k = \int_a^b p(s) F(s) \varphi_k(s) \, ds.$$

We can reduce equation (206) at once to an equation with a symmetric kernel, by introducing instead of s and t the new variables x and y:

$$x = \int_a^s p(u)\,\mathrm{d}u; \qquad y = \int_a^t p(u)\,\mathrm{d}u,$$

where the new variables are increasing along with the previous ones, in view of the fact that $p(u) > 0$. After replacing the variables we get the new functions $f_1(x) = f(s)$; $\omega(x) = \varphi(s)$ and the new symmetric kernel $K_1(x, y) = K(s, t)$, whilst equation (206) becomes

$$\omega(x) = f_1(x) + \lambda \int_0^l K_1(x, y)\,\omega(y)\,\mathrm{d}y \qquad \left(l = \int_a^b p(u)\,\mathrm{d}u\right).$$

33. Examples. 1. Let us take the kernel of [1], where we put $l = 1$ to simplify the writing, i.e.

$$K(s, t) = \begin{cases} s(1-t) & \text{for } s \leqslant t \\ t(1-s) & \text{for } s \geqslant t \end{cases} \qquad \begin{pmatrix} 0 \leqslant s \leqslant 1 \\ 0 \leqslant t \leqslant 1 \end{pmatrix}. \tag{210}$$

We can find all the eigenvalues and eigenfunctions here in an explicit form. We have to use the second of expressions (210) when integrating from $t = 0$ to $t = s$ (i.e. with $t \leqslant s$) in the homogeneous integral equation

$$\varphi(s) = \lambda \int_0^1 K(s, t)\,\varphi(t)\,\mathrm{d}t \tag{211}$$

whilst we use the first of expressions (210) when integrating from $t = s$ to $t = 1$; thus the equation must be rewritten as

$$\varphi(s) = \lambda \int_0^s t(1-s)\,\varphi(t)\,\mathrm{d}t + \lambda \int_s^1 s(1-t)\,\varphi(t)\,\mathrm{d}t.$$

We differentiate both sides with respect to s:

$$\varphi'(s) = -\lambda \int_0^s t\varphi(t)\,\mathrm{d}t + \lambda s(1-s)\,\varphi(s) + \lambda \int_s^1 (1-t)\,\varphi(t)\,\mathrm{d}t - \lambda s(1-s)\,\varphi(s).$$

The non-integral terms on the right cancel, and we get by differentiating again with respect to s:

$$\varphi''(s) + \lambda\varphi(s) = 0. \tag{212}$$

Kernel (210) obviously satisfies the condition $K(0, t) = K(1, t) = 0$, and formula (211) gives $\varphi(0) = \varphi(1) = 0$, i.e. we can confine ourselves to the solutions of (212) that satisfy the boundary conditions $\varphi(0) = \varphi(1) = 0$. Equation (212) is soluble in elementary functions, and as we know from [II, 167],

our boundary value problem for it can only have non-zero solutions when $\lambda_n = n^2\pi^2$, these solutions being $\varphi_n(s) = \sqrt{2}\sin n\pi s$.

It is easily shown by direct substitution in equation (211) that the numbers and functions indicated are in fact eigenvalues and eigenfunctions of (211). This can be seen, however, by remarking that, given the boundary conditions in question, no extraneous solutions are brought in when carrying out the above-mentioned operations of differentiation of both sides of the equation. We have already encountered these eigenvalues and eigenfunctions when discussing the problem of the vibrations of a string fixed at its ends [II, 167]. This has a direct connection with the fact that kernel (210), as we proved in [1], gives the statical bending of a string in the presence of a concentrated force. We shall analyse this idea later for a wide class of problems of mathematical physics. For the present example, series (144) will be uniformly convergent, and we have the following formula:

$$\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin k\pi s\sin k\pi t}{k^2} = \begin{cases} s(1-t) & \text{for } s \leqslant t \\ t(1-s) & \text{for } s \geqslant t \end{cases} \qquad \begin{pmatrix} 0 \leqslant s \leqslant 1 \\ 0 \leqslant t \leqslant 1 \end{pmatrix}. \tag{213}$$

Suppose that some function $f(s)$ has continuous derivatives up to the second order and satisfies the boundary conditions $f(0) = f(1) = 0$. We have an expression in terms of a kernel for such a function, viz.

$$f(s) = -\int_0^1 K(s,t)f''(t)\,dt = -\int_0^s t(1-s)f''(t)\,dt - \int_s^1 s(1-t)f''(t)\,dt,$$

which is easily verified by integrating by parts, and which also follows from what was said in [1] in regard to finding the sag with a continuously distributed load, which must be taken equal to $f''(t)$ in the present case. Theorem II shows us therefore, that every function $f(s)$ satisfying the conditions indicated above can be expanded in the interval $[0, 1]$ as an absolutely and uniformly convergent Fourier series in functions $\sqrt{2}\sin k\pi s$. We shall see later that the conditions imposed on $f(s)$ can be substantially simplified. It may be remarked that (213) also represents the expansion of its right-hand side in a Fourier series.

This series can be regarded either as the Fourier series of the right-hand side considered as a function of s (t is a parameter) in functions $\sqrt{2}\sin k\pi s$ ($k = 1, 2, \ldots$) or as the Fourier series of the right-hand side considered as a function given in the square ($0 \leqslant s \leqslant 1$; $0 \leqslant t \leqslant 1$) in functions $2\sin k\pi s\sin l\pi t$ ($k, l = 1, 2, \ldots$), which form an orthonormal system in this square. The above treatment can be applied to a kernel of the form

$$K(s,t) = \begin{cases} ast + bs + ct + d & \text{for } s \leqslant t \\ ast + bt + cs + d & \text{for } s \geqslant t \end{cases}$$

(see I. I. Privalov, *Integral Equations* (Integral'nye uravneniya), 1935, p. 102).

2. Let us take the kernel $K(s, t)$ which is a function of the difference $s - t$:

$$K(s,t) = \omega(s-t),$$

where $\omega(x)$ is a continuous even function of period 2π. Since $\omega(x)$ is even this kernel is symmetric. We bring in the Fourier coefficients of the function $\omega(x)$:

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \omega(x) \cos kx \, dx \qquad (k = 0, 1, 2, \ldots);$$

here, since the function is even,

$$\int_{-\pi}^{\pi} \omega(x) \sin kx \, dx = 0.$$

We now consider the integral

$$\int_{-\pi}^{\pi} \omega(s - t) \cos kt \, dt.$$

On substituting $s - t = x$ and using the fact that $\omega(x)$ is even, we obtain

$$\int_{-\pi}^{\pi} \omega(s - t) \cos kt \, dt = \cos ks \int_{s-\pi}^{s+\pi} \omega(x) \cos kx \, dx,$$

or, taking into account that the length of the integration path is 2π, we finally have

$$\int_{-\pi}^{\pi} \omega(s - t) \cos kt \, dt = \pi c_k \cos ks.$$

Similarly:

$$\int_{-\pi}^{\pi} \omega(s - t) \sin kt \, dt = \pi c_k \sin ks.$$

We consider the homogeneous integral equation

$$\varphi(s) = \lambda \int_{-\pi}^{\pi} \omega(s - t) \varphi(t) \, dt.$$

If all the Fourier coefficients c_k of the function $\omega(x)$ differ from zero, it follows from the above working that this equation has the eigen values

$$\lambda_k = \frac{1}{\pi c_k} \qquad (k = 0, 1, 2, \ldots),$$

to which there corrresponds the following orthonormal system of eigenfunctions:

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}} \cos s, \quad \frac{1}{\sqrt{\pi}} \cos 2s, \ \ldots$$

$$\frac{1}{\sqrt{\pi}} \sin s, \quad \frac{1}{\sqrt{\pi}} \sin 2s, \ \ldots$$

Our kernel can have no other eigenfunctions, since the functions mentioned form a closed system [II, 155]. There are two eigenfunctions corresponding to the eigenvalue λ_k with $k \geqslant 1$. If say $c_1 = 0$, whilst the remaining c_k differ from zero, the two eigenfunctions $(1/\sqrt{\pi}) \cos s$ and $(1/\sqrt{\pi}) \sin s$ are missing from the system, and the kernel ceases to be complete.

Whatever the assumptions regarding the coefficients c_n, series (144) will now have the form

$$\frac{1}{2} c_0 + \sum_{k=1}^{\infty}{}' c_k (\cos ks \cos kt + \sin ks \sin kt) = \frac{1}{2} c_0 + \sum_{k=1}^{\infty}{}' c_k \cos k (s - t),$$

i.e. this will be the Fourier series of the function $\omega(s - t)$. We cannot in general assert its convergence. But if the Fourier coefficients c_k satisfy the condition $c_k \geqslant 0$, it follows at once from Mercer's theorem that the series is absolutely and uniformly convergent and yields $\omega(s - t)$. The same conclusion will hold if there is only a finite number of positive or negative coefficients among the c_k.

34. Kernels depending on a parameter. In our discussion of the general theory of integral equations we have so far considered the parameter λ mentioned only as a factor in the kernel. We in [30] an integral equation with kernel $R(s, t; \lambda)$, which is an analytic (meromorphic) function of the parameter.

When considering integral equations with kernels which are analytic functions of a parameter, we can come up against substantial deviations from the rules obtained above in the general theory. Let us take as an elementary example a type of homogeneous equation in which the kernel is a first degree polynomial in λ:

$$\varphi(s) = \int_a^b \left[K_0(s, t) + K_1(s, t) \lambda\right] \varphi(t)\, dt,$$

where

$$K_0(s, t) = \varrho(s) \varrho(t); \quad K_1(s, t) = \sigma(s) \varrho(t),$$

and

$$\int_a^b [\varrho(s)]^2 \, ds = 1; \quad \int_a^b \varrho(s) \sigma(s)\, ds = 0.$$

It is easily seen that the homogeneous equation has the solution for any λ:

$$\varphi(s) = \varrho(s) + \sigma(s) \lambda.$$

We now take the general case of a kernel $K(s, t; \lambda)$, under the following conditions: (1) $K(s, t; \lambda)$ is a continuous function of s, t, λ, when (s, t) belongs to the square k_0 and λ lies inside some domain B of the plane of the complex variable λ; (2) for all (s, t) belonging to this square, $K(s, t; \lambda)$ is a regular function of λ inside B.

We write an integral equation by introducing the auxiliary parameter μ in front of the integral sign:

$$\varphi(s) = f(s) + \mu \int_a^b K(s, t; \lambda) \varphi(t)\, dt.$$

We can repeat all the arguments of [5] and [7], by replacing the λ featured in the formulae of these sections by μ. We thus arrive at the resolvent for the equation:

$$R(s, t; \mu) = \frac{D(s, t, \lambda; \mu)}{D(\lambda; \mu)}.$$

The numerator and denominator of this fraction are power series in the variable μ, and the coefficients of the series are regular functions inside B. If λ lies in any closed domain B_1 contained in B, the series in question are absolutely and uniformly convergent with respect to λ [7] for any value of μ, so that the sums of these series are regular functions of λ inside B [III$_2$, 12]. Putting $\mu = 1$, we get the equation:

$$\varphi(s) = f(s) + \int_a^b K(s, t; \lambda)\, \varphi(t)\, dt. \tag{214}$$

Here, two cases are possible: (1) the function $D(\lambda; 1)$, regular inside B, is not identically zero; (2) $D(\lambda; 1) \equiv 0$. In the first case equation (214) has the resolvent

$$R_1(s, t; \lambda) = \frac{D(s, t, \lambda; 1)}{D(\lambda; 1)}$$

for all λ differing from the zeros of $D(\lambda, 1)$, and there can only be a finite number of such zeros in any closed domain B_1 lying inside B. The resolvent obviously satisfies the equations

$$\begin{aligned} R_1(s, t; \lambda) &= K(s, t; \lambda) + \int_a^b K(s, t_1; \lambda)\, R_1(t_1, t; \lambda)\, dt_1, \\ R_1(s, t; \lambda) &= K(s, t; \lambda) + \int_a^b K(t_1, t; \lambda)\, R_1(s, t_1; \lambda)\, dt_1, \end{aligned} \tag{215}$$

and if λ is not a zero of $D(\lambda; 1)$, equation (214) has a unique solution for any $f(s)$:

$$\varphi(s) = f(s) + \int_a^b R_1(s, t; \lambda)\, f(t)\, dt.$$

If $\lambda = \lambda_0$ is a zero of $D(\lambda, 1)$, it follows from this that the entire function $D(\lambda_0; \mu)$ has a zero $\mu = 1$, and the results of [8] imply that the homogeneous equation

$$\varphi(s) = \int_a^b K(s, t; \lambda)\, \varphi(t)\, dt \tag{216}$$

has non-zero solutions for $\lambda = \lambda_0$. Hence it follows, inter alia, that $\lambda = \lambda_0$ is a pole of $R_1(s, t; \lambda)$. For the resolvent $R_1(s, t; \lambda)$ would otherwise be regular at the point $\lambda = \lambda_0$ with any (s, t), and would satisfy equations (215), as may easilybe seen by a continuous displacement to the point $\lambda = \lambda_0$ from points closeto it at which equations (215) hold. But if equations (215) hold for $\lambda = \lambda_0$, it now follows that equation (214) has a unique solution for any $f(s)$ [6], so that the homogeneous equation (216) can only have a zero solution. In the

case $D(\lambda; 1) \equiv 0$ the homogeneous equation (216) obviously has a non-zero solution for any λ inside B, and the non-homogeneous equation (214) is not soluble whatever the function $f(s)$.

It follows from the foregoing discussion that, given our assumptions about the kernel $K(s, t; \lambda)$ and $D(\lambda; 1) \not\equiv 0$, the eigenvalues cannot be compressed inside B, i.e. there can only be a finite number of them in any closed domain B_1 inside B. If we assume that the kernel can have poles independent of s and t instead of being regular, it is possible for there to be an infinite set of eigenvalues in any small neighbourhood of each such pole. For instance, if the equation

$$\varphi(s) = f(s) + \lambda \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t$$

with continuous symmetric kernel has an infinite set of eigenvalues λ_n, then $|\lambda_n| \to +\infty$, and the eigenvalues λ_n^{-1} of the equation

$$\varphi(s) = f(s) + \frac{1}{\lambda} \int_a^b K(s, t)\, \varphi(t)\, \mathrm{d}t$$

with kernel $K(s, t; \lambda)$ and pole at $\lambda = 0$, tend to $\lambda = 0$ as $n \to \infty$.

It may happen, however, that the resolvent of a kernel which has poles has no singular points at all. For instance, let $R(s, t; \lambda)$ be the resolvent of an integral equation with a symmetric kernel. As we know, it is a meromorphic function of λ, the poles of which do not depend on s and t. We form the integral equation

$$\varphi(s) = f(s) - \lambda \int_a^b R(s, t; \lambda)\, \varphi(t)\, \mathrm{d}t$$

with kernel $R(s, t; \lambda)$ and parameter $\mu = -\lambda$. By what was said in [30], the resolvent of this equation is equal to

$$R(s, t; \lambda + \mu)\,|_{\mu=-\lambda} = R(s, t; 0) = K(s, t),$$

and it does not depend on λ.

Equations with kernels analytically dependent on a parameter have been considered in various works, including in particular: Miranda *(Circolo Matem. di Palermo*, vol. 608, 1937), Iglisch *(Mathem. Annal.*, Bd. 117, 1938) and Z. I. Khalilova *(Dokl. Akad. Nauk S. S. S. R.*, vol. 54, no. 7, 1946). The literature of the subject will be found in these works.

35. Space of continuous functions. We finally turn to the proof of Theorems I and II, which we stated in [21] and [22] and used in our discussion of the theory of integral equations with symmetric kernels. We shall use in the proof of these theorems the ideas, concepts and notation of present-day functional analysis. The relevant material will be considered in detail in Vol. V, and we shall confine ourselves here to the part relating to continuous functions. This is due to the fact

that the entire theory of integral equations has been described here for continuous functions and on the basis of the ordinary concept of the integral (without using Lebesgue integrals). We shall start with a description of the fundamental concepts and results of functional analysis for families, or as they are generally termed, spaces of continuous functions.

We take the set of all real functions, continuous in a given finite interval $[a, b]$. We shall call this set the *space* F. Every concrete real function continuous in $[a, b]$ is called an *element of this space.* We denote these elements in future by the last letters of the Greek alphabet, i.e. instead of $\sigma(s)$, $\tau(s)$, $\varphi(s)$, $\psi(s)$, ... we shall simply write σ, τ, φ, ψ, ... The *zero element* is defined as the identically zero function. We shall denote it by the number zero and shall use the first letters of the Latin alphabet $a, b, c, \ldots$ for the real numbers. If $\varphi_k(s)$ are continuous real functions and c_k are real numbers, the finite sum $c_1\varphi_1(s) + c_2\varphi_2(s) + \ldots + c_m\varphi_m(s)$ is also a continuous real function. The elements of space F can therefore be multiplied by real numbers and added, and further elements of F thus obtained. The linear independence of elements of F amounts to linear independence of the corresponding functions [3]. The element $(-\varphi)$ corresponds to the function $-\varphi(s)$.

The *scalar product* of two elements is defined as the integral of their product and is written in the usual way as

$$(\varphi, \psi) = \int_a^b \varphi(s)\psi(s)\,ds. \tag{217}$$

The scalar product of two elements is therefore a number. The following properties of the scalar product are immediate consequences of the elementary properties of the integral:

$$(c\varphi, d\psi) = cd(\varphi, \psi); \tag{218$_1$}$$

$$(\varphi_1 + \varphi_2, \psi_1 + \psi_2) = (\varphi_1, \psi_1) + (\varphi_2, \psi_1) + (\varphi_1, \psi_2) + (\varphi_2, \psi_2). \tag{218$_2$}$$

It is obvious, in addition, that

$$(\varphi, \psi) = (\psi, \varphi). \tag{219}$$

Furthermore:

$$(\varphi, \varphi) = \int_a^b [\varphi(s)]^2\,ds, \tag{220}$$

whence we see that $(\varphi, \varphi) \geqslant 0$, where the sign of equality only holds for the zero element.

The *norm of the element* φ is defined as the arithmetic value of the square root of (φ, φ). The notation $\|\varphi\|$ is used for the norm:

$$\|\varphi\| = \sqrt{(\varphi, \varphi)} = \sqrt{\int_a^b [\varphi(s)]^2 \, ds}. \tag{221}$$

We have $\|\varphi\| \geqslant 0$, where the sign of equality only holds for the zero element. Further, it follows from (218_1) with $c = d$ and $\psi = \varphi$ that

$$\|c\varphi\| = |c| \cdot \|\varphi\|. \tag{222}$$

We obtain by applying Buniakowski's inequality to integral (217):

$$|(\varphi, \psi)| \leqslant \|\varphi\| \cdot \|\psi\|. \tag{223}$$

It follows from (222) that

$$\|\psi - \varphi\| = \|\varphi - \psi\|. \tag{224}$$

We have further, by (218_2) and (219):

$$\begin{aligned} \|\varphi + \psi\|^2 = (\varphi + \psi, \varphi + \psi) &= (\varphi, \varphi) + (\psi, \psi) + 2(\varphi, \psi) \\ &\leqslant \|\varphi\|^2 + \|\psi\|^2 + 2\|\varphi\| \cdot \|\psi\|, \end{aligned}$$

whence the *triangle inequality* is obtained:

$$\|\varphi + \psi\| \leqslant \|\varphi\| + \|\psi\|. \tag{225}$$

Two elements φ and ψ are said to be *mutually orthogonal* or simply *orthogonal*, if their scalar product is zero. Let the elements $\psi_1, \psi_2, \ldots, \psi_m$ be pairwise orthogonal. By using (218_2) and definition (221), we get Pythagoras' theorem:

$$\|\psi_1 + \psi_2 + \cdots + \psi_m\|^2 = \|\psi_1\|^2 + \|\psi_2\|^2 + \cdots + \|\psi_m\|^2. \tag{226}$$

The concept of limit is introduced by using the concept of norm. An element φ is said to be the *limit of the sequence of elements* φ_n if $\|\varphi - \varphi_n\| \to 0$ on increase of the subscript n, or in other words, $\|\varphi - \varphi_n\|^2 \to 0$. We write in this case $\varphi_n \Rightarrow \varphi$. This is equivalent to the following:

$$\|\varphi - \varphi_n\|^2 = \int_a^b [\varphi(s) - \varphi_n(s)]^2 \, ds \to 0, \tag{227}$$

i.e. the convergence $\varphi_n \Rightarrow \varphi$ is defined as a convergence in the mean. We retain the previous notation for passage to the limit in the case

of numbers: $a_n \to a$. It is easy to show that a *limit is unique*. For, let $\varphi_n \Rightarrow \varphi$ and $\varphi_n \Rightarrow \psi$. We can write

$$\varphi - \psi = (\varphi - \varphi_n) + (\varphi_n - \psi),$$

whence, by (225),

$$\| \varphi - \psi \| \leqslant \| \varphi - \varphi_n \| + \| \varphi_n - \psi \|.$$

On indefinite increase of n the right-hand side tends to zero, whilst the left-hand side is independent of n, so that $\| \varphi - \psi \| = 0$ i.e. $\varphi - \psi$ is the zero element, i.e. the continuous functions $\varphi(s)$ and $\psi(s)$ are identically equal, i.e. the elements φ and ψ coincide.

We notice further that, if the sequence φ_n has a limit, $\| \varphi_m - \varphi_n \| \to 0$ on indefinite increase of m and n. For, it follows from

$$\varphi_m - \varphi_n = (\varphi_m - \varphi) + (\varphi - \varphi_m) \qquad (\varphi_n \Rightarrow \varphi)$$

that

$$\| \varphi_m - \varphi_n \| \leqslant \| \varphi_m - \varphi \| + \| \varphi - \varphi_n \|,$$

and the right-hand side tends to zero on indefinite increase of m and n. We now prove a theorem.

THEOREM 1. *The expressions $c\varphi$, $\varphi + \psi$ and (φ, ψ) are continuously dependent on the number c and the elements φ and ψ*, i.e. *if $c_n \to c$, $\varphi_n \Rightarrow \varphi$ and $\psi_n \Rightarrow \psi$, then*

$$c_n\varphi_n \Rightarrow c\varphi; \quad \varphi_n + \psi_n \Rightarrow \varphi + \psi; \quad (\varphi_n, \psi_n) \to (\varphi, \psi).$$

We write down the obvious equation

$$c\varphi - c_n\varphi_n = (c\varphi - c_n\varphi) + (c_n\varphi - c_n\varphi_n)$$

and apply the triangle inequality to the sum on the right:

$$\| c\varphi - c_n\varphi_n \| \leqslant | c - c_n | \cdot \| \varphi \| + | c_n | \cdot \| \varphi - \varphi_n \|,$$

whence it follows that $c_n \varphi_n \Rightarrow c\varphi$.

We have further:

$$(\varphi + \psi) - (\varphi_n + \psi_n) = (\varphi - \varphi_n) + (\psi - \psi_n),$$

whence

$$\| (\varphi + \psi) - (\varphi_n + \psi_n) \| \leqslant \| \varphi - \varphi_n \| + \| \psi - \psi_n \|,$$

so that

$$\| (\varphi + \psi) - (\varphi_n + \psi_n) \| \to 0, \quad \text{i. e.} \quad \varphi_n + \psi_n \Rightarrow \varphi + \psi.$$

We turn finally to the proof of the continuity of the scalar product. We have to show that $(\varphi_n, \psi_n) \to (\varphi, \psi)$. We write $\to$ because we are

concerned here with a convergence of numbers, not of elements of F. We put: $\varphi_n - \varphi = \sigma_n$ and $\psi_n - \psi = \tau_n$. The norms $\| \sigma_n \|$ and $\| \tau_n \|$ tend to zero, because $\varphi_n \Rightarrow \varphi$ and $\psi_n \Rightarrow \psi$.

We have:

$$(\varphi, \psi) - (\varphi_n, \psi_n) = (\varphi, \psi) - (\varphi + \sigma_n, \psi + \tau_n)$$
$$= - (\varphi, \tau_n) - (\sigma_n, \psi) - (\sigma_n, \tau_n),$$

whence, by (223):

$$| (\varphi, \psi) - (\varphi_n, \psi_n) | \leqslant \| \varphi \| \cdot \| \tau_n \| + \| \sigma_n \| \cdot \| \psi \| + \| \sigma_n \| \cdot \| \tau_n \| .$$

The right-hand side tends to zero on indefinite increase of n, so that $(\varphi_n, \psi_n) \to (\varphi, \psi)$.

COROLLARY. *If* $\varphi_n \Rightarrow \varphi$, *then* $\| \varphi_n \| \to \| \varphi \|$. For:

$$\| \varphi_n \| = \sqrt{(\varphi_n, \varphi_n)} \to \sqrt{(\varphi, \varphi)} = \| \varphi \| .$$

Let the elements φ_k $(k = 1, 2, \ldots, n)$ be mutually orthogonal and normalized, i.e.

$$(\varphi_p, \varphi_q) = \begin{cases} 0 & \text{for} \quad p \neq q \\ 1 & \text{for} \quad p = q, \end{cases} \tag{228}$$

and let φ be any element of F. The sum

$$\sum_{k=1}^{n} (\varphi, \varphi_k)\, \varphi_k$$

is an element of F; in fact, it is the Fourier series of the element φ in elements φ_k $(k = 1, 2, \ldots, n)$. The difference

$$\varphi - \sum_{k=1}^{n} (\varphi, \varphi_k)\, \varphi_k, \tag{229}$$

may easily be seen to be orthogonal to all the φ_k $(k = 1, 2, \ldots, n)$.

We notice further that, *if a sequence of functions* $\omega_n(s)$ *tends uniformly in* $[a, b]$ *to* $\omega(s)$, *we can pass to the limit under the integral sign in integral* (227), *and we have* $\omega_n \Rightarrow \omega$. But it does not follow from $\omega_n \Rightarrow \omega$ that the $\omega_n(s)$ tend uniformly to $\omega(s)$ [cf. II, 148].

The structure of the F space is exactly similar to that of the R_n space in linear algebra. We have confined ourselves here to real functions. The extension to complex functions presents no difficulties and will be given later.

Given the definition of limit in F, we can clearly consider an infinite series such as

$$\sum_{k=1}^{\infty} \psi_k,$$

where the ψ_k are elements of F. If, on indefinite increase of n,

$$\sigma_n = \sum_{k=1}^{n} \psi_k \Rightarrow \psi,$$

we say that the series is convergent and that its sum is ψ. In accordance with what has been said, we could also consider the infinite Fourier series:

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)\, \varphi_k.$$

No use will be made of these in future.

The convergence of the above series is a convergence in mean, i.e. the convergence of the series $\sum\limits_{k=1}^{\infty}$ to the element ψ means that

$$\int_a^b \left[\psi(s) - \sum_{k=1}^{n} \psi_k(s)\right]^2 ds \to 0.$$

36. Linear operators. Any definite law in accordance with which an element φ of F is associated with a definite element also of F is called an *operator* in F. We use the notation $A, B, \ldots$ for operators in F, so that the symbols $A\varphi, B\varphi, \ldots$ denote the elements with which φ is associated by the operators $A, B, \ldots$

The *distributive law* for an operator is given by

$$A(c_1\varphi_1 + c_2\varphi_2 + \ldots + c_m\varphi_m) = c_1 A\varphi_1 + c_2 A\varphi_2 + \ldots + c_m A\varphi_m. \quad (230)$$

An operator A is said to be *bounded* if a number N exists, such that, for any element φ,

$$\|A\varphi\| \leqslant N \|\varphi\|. \quad (231)$$

A distributive bounded operator is called a linear operator. An example of a linear operator is provided by the integral operator with real continuous or weakly polar kernel:

$$A\varphi = \int_a^b K(s, t)\, \varphi(t)\, dt. \quad (232)$$

The distributive property follows from the elementary properties o the integral, and the boundedness from Buniakowski's inequality

$$\left[\int_a^b K(s,t)\,\varphi(t)\,\mathrm{d}t\right]^2 \leqslant \int_a^b [K(s,t)]^2\,\mathrm{d}t \cdot \int_a^b [\varphi(t)]^2\,\mathrm{d}t,$$

provided the first factor on the right does not exceed, some definite number:

$$\int_a^b [K(s,t)]^2\,\mathrm{d}t \leqslant N_1,$$

for any s, as is the case for continuous and weakly polar kernels. On integrating both sides of the inequality with respect to s:

$$\left[\int_a^b K(s,t)\,\varphi(t)\,\mathrm{d}t\right]^2 \leqslant N_1 \int_a^b [\varphi(t)]^2\,\mathrm{d}t,$$

we obtain:

$$\| A\varphi \|^2 \leqslant (b-a)\,N_1 \,\| \varphi \|^2,$$

i.e. inequality (231) with $N = \sqrt{(b-a)N_1}$.

A further example of a linear operator is provided by the operator that associates any continuous function with the same function. This identity transformation operator is usually written as $\mathscr{E}$, so that $\mathscr{E}\varphi = \varphi$. We can take $N = 1$ in (231) for this operator.

Let us also consider the linear operator A which amounts to multiplying any element φ by some fixed number a, i.e. $A\varphi = a\varphi$. In this case we can take $N = |\, a \,|$ in (231). If $a = 0$, the operator A transforms any element φ to the zero element, i.e. multiplies any function $\varphi(s)$ by zero. We shall call this the *annihilation operator*. The characteristic of the annihilation operator is that $N = 0$ in (231). For, if $N = 0$, it follows from (231) that $\| A\varphi \| = 0$ for any φ, i.e. $A\varphi$ is the zero element for any φ, since the norm of the zero element alone is zero. Thus the N in (231) must be a positive number for every linear operator differing from the annihilation operator.

We shall be concerned only with linear operators in what follows, and these are to be understood when we speak of operators.

If ω is the zero element (i.e. $\omega(s) \equiv 0$), we can write $\omega = 0\varphi$, where φ is any element, in which case, by (230), $A\omega = A(0\varphi) = 0A\varphi = \omega$, i.e. every operator transforms the zero element to zero.

We return to inequality (231). If ω is the zero element, $A\omega$ is also the zero element, i.e. $\| A\omega \| = \| \omega \| = 0$. In this case (231) holds for any choice of N.

We can therefore assume that $\| \varphi \| > 0$ when considering (231). Let φ be a normalized element, i.e. $\| \varphi \| = 1$. We can now write (231) as

$$\| A\varphi \| \leqslant N \quad (\| \varphi \| = 1). \tag{233}$$

It is easily seen that, conversely, (231) follows from (233). For, let φ be any element different from zero. Now, by (222), $(1/\| \varphi \|)(\varphi)$ is a normalized element, and (233) gives

$$\left\| A \left(\frac{1}{\| \varphi \|} \varphi \right) \right\| \leqslant N,$$

whence, by (230):

$$\left\| \frac{1}{\| \varphi \|} A\varphi \right\| \leqslant N, \quad \text{or} \quad \frac{1}{\| \varphi \|} \| A\varphi \| \leqslant N, \quad \text{or} \quad \| A\varphi \| \leqslant N \| \varphi \|,$$

i.e. (231) in fact follows from (233). We can thus consider (233) instead of (231), and vice versa.

If (231) or (233) is fulfilled for some N, it is fulfilled all the more for all greater values of N. It is natural for us to seek the least possible N.

If φ is any normalized element of F, $\|A\varphi\|$ will be a set of non-negative numbers, and all the numbers of this set will not exceed N. The set has a strict upper bound [I, 42], which we denote by n_A:

$$n_A = \sup_{\| \varphi \| = 1} \| A\varphi \|. \tag{234}$$

Now, by the definition of strict upper bound, $\| A\varphi \| \leqslant n_A$ for $\| \varphi \| = 1$, but, given any positive ε, there will exist a normalized φ such that $\| A\varphi \| > n_A - \varepsilon$. Hence, n_A is the least possible value of N such in (231) and (233):

$$\| A\varphi \| \leqslant n_A \quad (\| \varphi \| = 1) \tag{235}$$

or

$$\| A\varphi \| \leqslant n_A \| \varphi \|. \tag{236}$$

Instead of (234), we can obviously define n_A by

$$n_A = \sup \frac{\| A\varphi \|}{\| \varphi \|} = \sup \left\| A \left(\frac{1}{\| \varphi \|} \varphi \right) \right\| \tag{237}$$

where φ is any non-zero element.

The number n_A is usually termed the *norm of the operator*. This norm is equal to zero for the annihilation operator and is positive for any other operator, as we have seen. Let us give another definition of the norm.

THEOREM 2. *The norm n_A is the strict upper bound of the numbers $|(A\varphi, \psi)|$ for $\|\varphi\| = \|\psi\| = 1$,* i.e.

$$n_A = \sup |(A\varphi, \psi)| \quad \text{for} \quad \|\varphi\| = \|\psi\| = 1. \tag{238}$$

If A is the annihilation operator, $(A\varphi, \psi) = 0$ for all φ and ψ, and the theorem is obvious, since $n_A = 0$ in this case. Suppose that A is not the annihilation operator. It follows from

$$|(A\varphi, \psi)| \leqslant \|A\varphi\| \cdot \|\psi\| = n_A \|\varphi\| \cdot \|\psi\| \tag{239}$$

that

$$|(A\varphi, \psi)| \leqslant n_A \quad \text{for} \quad \|\varphi\| = \|\psi\| = 1. \tag{240}$$

On the other hand, if we replace ψ by the normalized element $A\varphi/\|A\varphi\|$ in the scalar product $(A\varphi, \psi)$, where $A\varphi$ is not the zero element, we get $(A\varphi, \psi) = \|A\varphi\|$. By (234), we can choose the normalized element so that $\|A\varphi\|$, i.e. $|(A\varphi, \psi)|$, is as close as desired to n_A. This assertion, along with (240), in fact gives (238). We observe that, in the case of the integral operator (232), the scalar product $(A\varphi, \psi)$ is given by

$$(A\varphi, \psi) = \int_a^b \left[\int_a^b K(s, t)\, \varphi(t)\, dt \right] \psi(s)\, ds.$$

On changing the order of integration, as is possible for continuous or weakly polar kernels, we obtain

$$(A\varphi, \psi) = \int_a^b \left[\int_a^b K(s, t)\, \psi(s)\, ds \right] \varphi(t)\, dt. \tag{241}$$

If, along with operator (232), we introduce the operator A^* with adjoint kernel:

$$A^*\psi = \int_a^b K(s, t)\, \psi(s)\, ds, \tag{242}$$

we can write (241) as

$$(A\varphi, \psi) = (\varphi, A^*\psi). \tag{243}$$

In the case of a symmetric kernel, the operator A^* is the same as A, and (243) becomes

$$(A\varphi, \psi) = (\varphi, A\psi). \tag{244}$$

DEFINITION. *The operator A is described as self-conjugate if* (244) *holds for any elements φ and ψ.*

The integral operator with symmetric kernel is not the only self-conjugate operator. For instance, the operator of multiplication by a number is easily seen to be self-conjugate. We now prove a theorem of importance for what follows:

THEOREM 3. *The norm of a self-conjugate operator is the strict upper bound of the values of* $|(A\varphi, \varphi)|$ *for all possible normalized elements* φ.

Let
$$d = \sup_{\|\varphi\|=1} |(A\varphi, \varphi)|. \tag{245}$$

We have to show that $d = n_A$. If φ is any element different from zero, we can also write:

$$d = \sup\left|\left(A\frac{1}{\|\varphi\|}\varphi, \frac{1}{\|\varphi\|}\varphi\right)\right| = \sup\frac{|(A\varphi, \varphi)|}{\|\varphi\|^2},$$

whence
$$|(A\varphi, \varphi)| \leqslant d\|\varphi\|^2. \tag{246}$$

This relationship is obvious for the zero element φ. We can write:

$$\big(A(\varphi+\psi), \varphi+\psi\big) - \big(A(\varphi-\psi), \varphi-\psi\big) = 4(A\varphi, \psi), \tag{247}$$

this being obtained by applying formulae (218_2), (219), (230) and (244) to the left-hand side. On the other hand, by taking (246) into account, we obtain

$$\begin{aligned}
&|(A(\varphi+\psi), \varphi+\psi) - \big(A(\varphi-\psi), \varphi-\psi\big)| \leqslant \\
&\qquad \leqslant |\big(A(\varphi+\psi), \varphi+\psi\big)| + |\big(A(\varphi-\psi), \varphi-\psi\big)| \leqslant \\
&\qquad \leqslant d(\varphi+\psi, \varphi+\psi) + d(\varphi-\psi, \varphi-\psi) = \\
&\qquad = 2d(\|\varphi\|^2 + \|\psi\|^2),
\end{aligned}$$

whence, by (247):

$$2|(A\varphi, \psi)| \leqslant d(\|\varphi\|^2 + \|\psi\|^2), \tag{248}$$

so that
$$|(A\varphi, \psi)| \leqslant d \quad \text{with} \quad \|\varphi\| = \|\psi\| = 1.$$

On the other hand, by Theorem 2, n_A is the strict upper bound of the left-hand side of the last inequality, so that $d \geqslant n_A$. To prove the theorem, it remains to show that $d \leqslant n_A$.

We have by (238): $|(A\varphi, \varphi)| \leqslant n_A$ with $\|\varphi\| = 1$. But, by definition (245), d is the strict upper bound of the left-hand side of this inequality with $\|\varphi\| = 1$, whence it follows that $d \leqslant n_A$. Thus, for a self-conjugate operator:

$$n_A = \sup_{\|\varphi\|=1} |(A\varphi, \varphi)| = \sup\frac{|(A\varphi, \varphi)|}{\|\varphi\|^2}. \tag{249}$$

The quantities $(A\varphi, \psi)$ and $(A\varphi, \varphi)$ are analogous to bilinear and quadratic forms in linear algebra [III_1, 40]. These are sometimes called the bilinear and quadratic functionals corresponding to the operator A.

We discussed these quantities in [25] for an integral operator with symmetric kernel. The quadratic functional for the integral operator (232) has the form

$$(A\varphi, \varphi) = \int_a^b \int_a^b K(s, t)\, \varphi(s)\, \varphi(t)\, \mathrm{d}s\, \mathrm{d}t\,, \tag{250}$$

where the order of integration on the right-hand side is of no consequence in the case of a continuous or weakly polar kernel.

The proof of the fundamental theorems will be linked up with a discussion of the extremal values of the quadratic functional (250).

We developed this point of view in [26] by making use of Theorems I and II. Here we shall discuss these extremal values directly, without having recourse to the results obtained above in the theory of integral equations with symmetric kernels. We can only obtain a proof of Theorems I and II for a certain class of linear self-conjugate operators, and our next task will be to distinguish this class.

We first mention a property of any linear operator. Let $\varphi_n \Rightarrow \varphi$, i.e. $\| \varphi - \varphi_n \| \to 0$. We have:

$$\| A\varphi - A\varphi_n \| = \| A(\varphi - \varphi_n) \| \leqslant n_A \| \varphi - \varphi_n \|$$

so that $\| A\varphi - A\varphi_n \| \to 0$, i.e. $A\varphi_n \Rightarrow A\varphi$. Thus it follows from $\varphi_n \Rightarrow \varphi$ that $A\varphi_n \Rightarrow A\varphi$, i.e. *every linear operator is continuous.*

Let us introduce some new concepts. *A set G* of elements φ is described as *bounded in norm* or simply *bounded* if a number C exists such that, for all elements φ of G:

$$\| \varphi \| = \sqrt{\int_a^b [\varphi(s)]^2\, \mathrm{d}s} \leqslant C\,. \tag{251}$$

Further, a set G of elements φ is described as *compact in the sense of convergence in the mean,* or simply as *compact,* if a subsequence having a limit in the sense of convergence in the mean can be extracted from any sequence of elements of G.

As well as a set bounded in norm, we can consider *a set G of elements φ bounded in absolute value.* For this type of set there must exist a number C such that the absolute values of all the functions $\varphi(s)$ of G do not exceed C, i.e. instead of (251) we have

$$| \varphi(s) | \leqslant C\,. \tag{252}$$

Similarly, a set G of elements φ is said to be *compact in the sense of uniform convergence,* if a subsequence having a limit in the sense of

uniform convergence can be extracted from any sequence of elements of G. A set which is compact in the sense of uniform convergence will also be compact in the sense of convergence in the mean, since convergence in the mean is a consequence of uniform convergence.

The theorem proved in [16] can be stated as follows:

THEOREM. *If all the elements of G satisfy condition* (252), *and are equicontinuous, the set G is compact in the sense of uniform convergence.*

We now define the class of linear operators with which we shall be later concerned.

DEFINITION. *A linear operator A is described as completely continuous if it transforms any set bounded in norm into a set which is compact in the sense of convergence in the mean.*

In other words, if the elements φ satisfy condition (251), with a fixed number C, the set of elements $A\varphi$ must be compact in the sense of convergence in the mean.

If the set $A\varphi$ happens to be compact in the sense of uniform convergence, it is all the more compact in the sense of convergence in the mean. An operator possessing this property will be described as *rigorously completely continuous.*

DEFINITION. *A linear operator A is said to be rigorously completely continuous, if it transforms any set bounded in norm into a set compact in the sense of uniform convergence.*

As just mentioned, every rigorously completely continuous operator is also completely continuous.

We shall prove some fundamental theorems in the next section for all completely continuous self-conjugate operators. However, when considering integral operators and establishing the conditions in which they are completely continuous, we have to make use of the theorem of [16] and hence prove that the corresponding operator is rigorously completely continuous. The condition for compactness in the sense of convergence in the mean, which is bound up with the theory of functions of a real variable and the Lebesgue integral, will be proved in Vol. V. We also discuss in Vol. V the theory of integral equations on a more general and natural basis.

37. Existence of the eigenvalue. Let us consider a completely continuous self-conjugate operator A differing from the annihilation operator, and the homogeneous equation with parameter μ:

$$A\varphi = \mu\varphi; \tag{253}$$

this corresponds to writing the homogeneous integral equation in the form [cf. 2]:

$$\int_a^b K(s, t)\,\varphi(t)\,\mathrm{d}t = \mu\varphi(t). \tag{254}$$

The eigenvalues for equation (253) are therefore the reciprocals of the eigenvalues discussed when dealing with the theory of integral equations. By (249), a sequence of normalized elements ψ_n $(n=1, 2, \ldots)$ exists, such that

$$|(A\psi_n, \psi_n)| \to n_A \qquad (\|\psi_n\| = 1). \tag{255}$$

Since $n_A > 0$ (A is not the annihilation operator), the quantities $(A\psi_n, \psi_n)$ are non-zero for sufficiently large n, and an infinity of them is positive, or an infinity negative, or an infinity both positive and negative. Whatever the case, we can extract from the sequence of ψ_n a subsequence such that we can write, retaining the previous notation for the subscripts,

$$(A\psi_n, \psi_n) \to \mu_1, \tag{256}$$

where

$$\mu_1 = n_A \tag{257}$$

or

$$\mu_1 = -n_A. \tag{258}$$

We form the element

$$\tau_n = \mu_1\psi_n - A\psi_n, \tag{259}$$

and find the square of its norm:

$$\begin{aligned}\|\tau_n\|^2 &= (\mu_1\psi_n - A\psi_n,\ \mu_1\psi_n - A\psi_n)\\ &= \mu_1^2(\psi_n, \psi_n) - 2\mu_1(A\psi_n, \psi_n) + (A\psi_n, A\psi_n),\end{aligned} \tag{260}$$

or, on recalling that $\|\psi_n\| = 1$ and $\|A\psi_n\|^2 \leqslant n_A^2 = \mu_1^2$:

$$\|\tau_n\|^2 \leqslant 2\mu_1\left[\mu_1 - (A\psi_n, \psi_n)\right]. \tag{261}$$

By (256), the right-hand side tends to zero as $n \to \infty$, so that $\|\tau_n\| \to 0$ also, i.e.

$$\mu_1\psi_n - A\psi_n \Rightarrow 0. \tag{262}$$

We have not as yet used the fact that A is a completely continuous operator. We shall use it next.

The elements ψ_n are normalized, so that the set of them is bounded, i.e. the sequence $A\psi_n$ is compact. We can extract from this a subsequence having a limit element. On retaining the previous notation for the

subscripts, we can assume that the sequence $A\psi_n$ has a limit element. But now, by (262), the sequence ψ_n also has a limit element ($\mu_1 \neq 0$). Let $\psi_n \Rightarrow \varphi_1$. The limit element φ_1, like the ψ_n, is normalized, by Theorem 1 of [35]. On passing to the limit in (262) and taking the continuity of operator A into account, we get $\mu_1 \varphi_1 - A\varphi_1 = 0$, i.e.

$$A\varphi_1 = \mu_1\varphi_1. \tag{263}$$

Hence equation (253) has an eigenvalue μ_1 and the corresponding normalized eigenelement φ_1. It follows from (263) that

$$(A\varphi_1, \varphi_1) = \mu_1. \tag{264}$$

The above discussion leads to the following existence theorem for the eigenvalue:

THEOREM I. *If A is a completely continuous self-conjugate operator in space F, different from the annihilation operator, equation* (253) *has a non-zero eigenvalue μ_1, whose absolute value is equal to n_A.*

38. Sequences of eigenvalues and expansion theorem. We now take, instead of the set F of all continuous functions, the set F_2 of the elements φ (we put $F_1 = F$) which are orthogonal to φ_1, i.e. the set of the real continuous functions $\varphi(s)$ which satisfy the condition

$$(\varphi, \varphi_1) = \int_a^b \varphi(s)\, \varphi_1(s)\, ds = 0. \tag{265}$$

Some important facts regarding F_2 must be noted. If we form linear combinations of elements of F_2, we obtain further elements of F_2. For, if $(\omega_1, \varphi_1) = (\omega_2, \varphi_1) = 0$, then

$$(c_1\omega_1 + c_2\omega_2, \varphi_1) = c_1(\omega_1, \varphi_1) + c_2(\omega_2, \varphi_1) = 0.$$

Further, if ω_n belongs to F_2 and $\omega_n \Rightarrow \omega_0$, then ω_0 also belongs to F_2. For, a passage to the limit in $(\omega_n, \varphi_1) = 0$ gives us $(\omega_0, \varphi_1) = 0$. We show further that, if an element τ belongs to F_2, then $A\tau$ also belongs to F_2. For, $(\tau, \varphi_1) = 0$ by hypothesis, and we have

$$(A\tau, \varphi_1) = (\tau, A\varphi_1) = (\tau, \mu_1\varphi_1) = \mu_1(\tau, \varphi_1) = 0.$$

We can therefore regard the operator A as self-conjugate and completely continuous, and defined in F_2. It transforms elements of F_2 into elements of F_2. All our discussion of [35], [36] and [37] retains its force on replacing F by F_2.

The question arises as to the norm of the operator A in F_2. We write it as ν_2 ($\nu_1 = n_A$).

By Theorem 3 of [36], this norm is given by

$$\nu_2 = \sup_{\|\varphi\|=1} |(A\varphi, \varphi)|. \tag{266}$$

The norm n_A of the same operator in the wider space F was given by a similar expression (249), where φ runs over F instead of F_2. Hence ν_2 is the strict upper bound of a narrower set of numbers, and we can assert that $\nu_2 \leqslant n_A$. In particular, it may happen that $\nu_2 = 0$, i.e. that A is the annihilation operator in F_2. Suppose that this is not the case. On repeating the arguments of [37], we see that equation (253), regarded as an equation in F_2, now has an eigenvalue μ_2, and the corresponding normalized eigenelement φ_2 of F_2: $A\varphi_2 = \mu_2 \varphi_2$. Now, $|\mu_2| = \nu_2$ and $(A\varphi_2, \varphi_2) = \mu_2$. It follows from $\nu_2 \leqslant n_A$ that $|\mu_1| \geqslant |\mu_2|$.

We now construct the set F_3 of elements of F satisfying the conditions

$$(\varphi, \varphi_1) = (\varphi, \varphi_2) = 0. \tag{267}$$

We can say the same of F_3 as was proved above for F_2, and A can be regarded as a self-conjugate completely continuous operator in F_3. If it is not the annihilation operator, we obtain as above an eigenvalue μ_3 and the normalized eigenelement φ_3 of F_3. Now, $|\mu_3| = \nu_3$, where ν_3 is the norm of A as an operator in F_3. In this case $|\mu_1| \geqslant |\mu_2| \geqslant |\mu_3|$.

By proceeding in this way we get the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ and the corresponding mutually orthogonal and normalized elements $\varphi_1, \varphi_2, \ldots, \varphi_n$, where

$$|\mu_1| \geqslant |\mu_2| \geqslant \ldots \geqslant |\mu_n|, \tag{268}$$

and $|\mu_k|$ is the norm of A as an operator in F_k, so that

$$|(A\varphi, \varphi)| \leqslant |\mu_k| \cdot \|\varphi\|^2, \tag{269}$$

if

$$(\varphi, \varphi_1) = (\varphi, \varphi_2) = \ldots = (\varphi, \varphi_{k-1}) = 0.$$

Suppose that the process breaks off on constructing the next eigenvalue, i.e. A turns out to be the annihilation operator in the set F_{n+1}, given by the conditions

$$(\varphi, \varphi_1) = (\varphi, \varphi_2) = \ldots = (\varphi, \varphi_n) = 0. \tag{270}$$

Let ω be any element of F. We construct the element

$$\varphi = \omega - \sum_{k=1}^{n} (\omega, \varphi_k)\, \varphi_k, \tag{271}$$

satisfying (270), i.e. belonging to F_{n+1}. We have by hypothesis:

$$A\left[\omega - \sum_{k=1}^{n}(\omega, \varphi_k)\,\varphi_k\right] = 0$$

or, on removing the brackets and noting that $A\varphi_k = \mu_k\,\varphi_k$ ($k = 1, 2, \ldots, n$), we obtain

$$A\omega = \sum_{k=1}^{n}(\omega, \varphi_k)\,\mu_k\varphi_k,$$

i.e. every element of the form $A\omega$ may be expanded in eigenelements φ_k. It is easily shown that $(\omega, \varphi_k)\,\mu_k$ are the Fourier coefficients of the element $A\omega$:

$$(A\omega, \varphi_k) = (\omega, A\varphi_k) = (\omega, \mu_k\varphi_k) = \mu_k\,(\omega, \varphi_k).$$

Now suppose that the above process of constructing the non-zero eigenvalues μ_s continues indefinitely. We show first that the sequence of eigenvalues μ_s tends to zero. Suppose, on the contrary, that the non-increasing sequence of positive numbers μ_s^2 has a limit a greater than zero. Since all the eigenelements φ_s have a norm equal to unity, the sequence $A\varphi_s$ must be compact. On the other hand, on recalling that the φ are pairwise orthogonal, we have by Pythagoras' theorem:

$$\| A\varphi_m - A\varphi_n \|^2 = \| \mu_m\varphi_m - \mu_n\varphi_n \|^2 = \mu_m^2 + \mu_n^2$$

and on indefinite increase of m and n the last sum has a limit $2a$ greater than zero, whence it follows that the sequence $A\varphi$ cannot be compact. The contradiction obtained shows that $\mu_s \to 0$.

We again consider element (271) belonging to F_{n+1}. The norm of the operator A in F_{n+1} is equal to μ_{n+1}, so that

$$\left\| A\left(\omega - \sum_{k=1}^{n}(\omega, \varphi_k)\,\varphi_k\right)\right\|^2 \leqslant \mu_{n+1}^2 \left\|\omega - \sum_{k=1}^{n}(\omega, \varphi_k)\,\varphi_k\right\|^2. \qquad (272)$$

But, as may easily be shown [3]:

$$\left\|\omega - \sum_{k=1}^{n}(\omega, \varphi_k)\,\varphi_k\right\|^2 = \|\omega\|^2 - \sum_{k=1}^{n}[(\omega, \varphi_k)]^2 \leqslant \|\omega\|^2,$$

so that it follows from (272) that

$$\left\| A\left(\omega - \sum_{k=1}^{n}(\omega, \varphi_k)\,\varphi_k\right)\right\|^2 \leqslant \mu_{n+1}^2\,\|\omega\|^2,$$

and the right-hand side tends to zero as $n \to \infty$, whence

$$A\left[\omega - \sum_{k=1}^{n} (\omega, \varphi_k)\varphi_k\right] = A\omega - \sum_{k=1}^{n} (\omega, \varphi_k)\mu_k\varphi_k \Rightarrow 0\,,$$

i.e.

$$A\omega = \sum_{k=1}^{\infty} (\omega, \varphi_k)\mu_k\varphi_k\,, \tag{273}$$

where the convergence of the infinite series must be understood as a convergence in the mean of the segments of this series to $A\omega$.

We show finally that the φ_k are all linearly independent eigenelements of operator A, corresponding to non-zero eigenvalues.

The eigenelements corresponding to different eigenvalues are pairwise orthogonal. For, if $\mu' \neq \mu''$, and

$$A\varphi' = \mu'\varphi'; \quad A\varphi'' = \mu''\varphi'',$$

we obtain by forming the scalar product of the first equation with φ'' and of the second with φ':

$$(\mu' - \mu'')(\varphi', \varphi'') = (A\varphi', \varphi'') - (\varphi', A\varphi'') = (A\varphi', \varphi'') - (A\varphi', \varphi'') = 0,$$

i.e. $(\varphi', \varphi'') = 0$. The eigenelements corresponding to the same eigenvalue can be orthogonalized. Hence, if there is an eigenelement τ linearly independent of the φ_s, corresponding to the non-zero eigenvalue μ, it can be assumed orthogonal to all the φ_k [cf. 21], i.e. $(\tau, \varphi_k) = 0$.

On substituting $\omega = \tau$ in (273) and using the fact that $A\tau = \mu\tau$, we get $\mu\tau = 0$, i.e. τ is the zero element ($\mu \neq 0$), which is absurd, since τ is an eigenelement by hypothesis. The φ_k therefore yield a complete system of linearly independent eigenelements, corresponding to the non-zero eigenvalues. Formula (273) gives the expansions of any element of the form $A\omega$ in eigenelements φ_k. We thus obtain, for the integral self-conjugate completely continuous operator, the expansion of any element expressible in terms of the kernel:

$$F(s) = \int_a^b K(s, t)\, h(t)\, dt,$$

in eigenfunctions $\varphi_n(s)$, the convergence being understood as convergence in the mean. But we have seen that the Fourier series of $F(s)$ in the $\varphi_n(s)$ is uniformly convergent in $[a, b]$ for continuous and weakly polar kernels. Let $\varphi(s)$ be its sum (a continuous function).

The uniform convergence implies convergence in the mean, i.e. the Fourier series $F(s)$ in the $\varphi_n(s)$ is also convergent in the mean to $\varphi(s)$.

But the limit in the mean is unique, and we have shown above that the Fourier series of $F(s)$ is convergent in the mean to $F(s)$. Consequently $\varphi(s)$ coincides with $F(s)$, i.e. the Fourier series of $F(s)$ in the $\varphi_n(s)$ is uniformly convergent to $F(s)$. This proves Theorem II of [22].

In the theory of integral equations we wrote instead of (253): $\varphi = \lambda A\varphi$, i.e. $\lambda = 1/\mu$. By what has been proved, $\mu_n \to 0$, so that $\lambda_n = 1/\mu_n \to \infty$, as we have already seen above.

The results of the present section may be stated as follows:

THEOREM II. *All the eigenvalues of a completely continuous self-conjugate operator A, different from the annihilation operator, have finite rank, and there is a finite number of them outside any interval $[-\varepsilon, \varepsilon]$. Every element of the form A can be expanded in a Fourier series in eigen-elements φ_k, the convergence being understood as a convergence in the mean.*

39. Space of complex continuous functions. Starting from [35], we could have developed the entire theory for complex continuous functions. Let H be the space of complex functions $\omega(s) = \omega_1(s) + \omega_2(s)i$, continuous in $[a, b]$. We can now use complex coefficients when forming linear combinations of functions. The scalar product is defined by

$$(\varphi, \psi) = \int_a^b \varphi(s)\overline{\psi(s)}\, ds. \tag{217_1}$$

We have, instead of (218_1):

$$(c\varphi, d\psi) = c\bar{d}(\varphi, \psi)$$

and instead of (219):

$$(\psi, \varphi) = \overline{(\varphi, \psi)}. \tag{219_1}$$

The norm is defined by

$$\|\varphi\| = \sqrt{(\varphi, \varphi)} = \sqrt{\int_a^b |\varphi(s)|^2\, ds}. \tag{221_1}$$

A self-conjugate operator is defined by (244) as previously. The quantity $(A\varphi, \varphi)$ is always real for a self-conjugate operator, since $(A\varphi, \varphi) = (\varphi, A\varphi) = \overline{(A\varphi, \varphi)}$, and a number equal to its conjugate is real. A self-conjugate operator can only have real eigenvalues, since $A\varphi = \mu\varphi$ implies $(A\varphi, \varphi) = \mu \|\varphi\|^2$, whence μ must be real.

A modification is needed in the proof of Theorem 3 of [36]. We have now, instead of (248):

$$2\,|\mathcal{R}(A\varphi, \psi)| \leqslant d\left[\|\varphi\|^2 + \|\psi\|^2\right], \tag{248_1}$$

where $\mathcal{R}$ is the symbol for the real part. We show that this leads us to the same inequality for the modulus $(A\varphi, \psi)$, i.e.

$$2\,|(A\varphi, \psi)| \leqslant d\left[\|\varphi\|^2 + \|\psi\|^2\right]. \tag{248}$$

Let $(A\varphi, \psi) = re^{i\beta}$, where r is the modulus and β the amplitude of $(A\varphi, \psi)$.

The element φ in inequality (248_1) is arbitrary and it can be replaced by $e^{-i\beta}\varphi$. We get:

$$2\,|\mathcal{R}e^{-i\beta}(A\varphi, \psi)| \leqslant d\left[\|e^{-i\beta}\varphi\|^2 + \|\psi\|^2\right].$$

On observing that $|e^{-i\beta}| = 1$ and $(A\varphi, \psi) = re^{i\beta}$, we can rewrite the last inequality as

$$2\,|\mathcal{R}r| \leqslant d\left[\|\varphi\|^2 + \|\psi\|^2\right].$$

But the real part of r is r itself, i.e.

$$2r \leqslant d\left[\|\varphi\|^2 + \|\psi\|^2\right] \quad \text{or} \quad 2\,|(A\varphi, \psi)| \leqslant d\left[\|\varphi\|^2 + \|\psi\|^2\right],$$

which is what we wished to prove. All the rest of the discussion is as before.

This theory of completely continuous self-conjugate operators in complex continuous function space yields, as above, an existence theorem for the eigenvalues and an expansion theorem for integral equations with Hermitian kernels.

40. Completely continuous integral operators. We consider the integral operator

$$\psi(s) = \int_a^b K(s, t)\,\varphi(t)\,dt \tag{274}$$

and discuss the conditions in which it becomes rigorously completely continuous in space F. It is necessary, first of all, that the integral written should have a meaning for any choice of continuous function $\varphi(t)$, and also that $\psi(s)$ be a continuous function. This is clearly true if $K(s, t)$ is a continuous or polar kernel [17].

We form the difference

$$\psi(s+h) - \psi(s) = \int_a^b \left[K(s+h, t) - K(s, t)\right]\varphi(t)\,dt$$

and apply Buniakowski's inequality:

$$|\psi(s+h) - \psi(s)|^2 \leqslant \int_a^b |K(s+h, t) - K(s, t)|^2 \, \mathrm{d}t \cdot \int_a^b |\varphi(t)|^2 \, \mathrm{d}t$$

whilst similarly, from (274):

$$|\psi(s)|^2 \leqslant \int_a^b |K(s, t)|^2 \, \mathrm{d}t \cdot \int_d^b |\varphi(t)|^2 \, \mathrm{d}t.$$

Suppose that we have a set of functions $\varphi(t)$ bounded in norm, i.e.

$$\int_a^b |\varphi(t)|^2 \, \mathrm{d}t \leqslant C^2, \tag{275}$$

whence, by virtue of the above inequalities:

$$|\psi(s)|^2 \leqslant C^2 \int_a^b |K(s, t)|^2 \, \mathrm{d}t; \tag{276}$$

$$|\psi(s+h) - \psi(s)|^2 \leqslant C^2 \int_a^b |K(s+h, t) - K(s, t)|^2 \, \mathrm{d}t. \tag{277}$$

We now let the kernel $K(s, t)$ satisfy the following two conditions: (1) there exists a number M^2 such that

$$\int_a^b |K(s, t)|^2 \, \mathrm{d}t \leqslant M^2 \qquad (a \leqslant s \leqslant b); \tag{278}$$

(2) given any positive Σ, there exists a positive η such that

$$\int_a^b |K(s+h, t) - K(s, t)|^2 \, \mathrm{d}t \leqslant \sum \text{ with } |h| \leqslant \eta. \tag{279}$$

Now, by (276) and (277), the set of functions $\psi(s)$ will be a set of functions which are bounded in modulus and equicontinuous, i.e. a compact set in the sense of uniform convergence, whilst operator (274) will be rigorously completely continuous.

Conditions (278) and (279) are obviously satisfied when the kernel $K(s, t)$ is continuous. Condition (278) holds also for a weakly polar kernel [28]. It is easily shown that condition (279) likewise holds for a weakly polar kernel. To prove this, we need only repeat the arguments which were used in [17] as regards integral (106), except that, instead of the inequality

$$|K(s+h, t) - K(s, t)| \leqslant |K(s+h, t)| + |K(s, t)|,$$

we have to use the inequality

$$|K(s+h,t)-K(s,t)|^2 \leqslant \frac{1}{2}\left[|K(s+h,t)|^2+|K(s,t)|^2\right].$$

Thus, *integral operators with continuous and weakly polar kernels are rigorously completely continuous operators in F.*

If, in addition, the kernel is Hermitian (or real and symmetric), the operator is self-conjugate, and the whole of the above theory applies.

We show further that, if a continuous or polar kernel of the type indicated in [17] is not identically zero in k_0, operator (274) is not the annihilation operator.

Suppose that the continuous kernel $K(s,t)$ is real and non-zero at some point (s_0, t_0).

Let $K(s_0, t_0) > 0$. Since the kernel is continuous, there exists a positive number δ such that $K(s_0, t) > 0$ for $|t-t_0| \leqslant \delta$. Further, there exists a continuous function $\varphi(t)$, positive for $|t-t_0| < \delta$ and zero for $|t-t_0| \geqslant \delta$. On substituting this function in (274) and putting $s = s_0$, we obtain

$$\psi(s_0) = \int_a^c K(s_0,t)\,\varphi(t)\,\mathrm{d}t = \int_{|t-t_0|\leqslant\delta} K(s_0,t)\,\varphi(t)\,\mathrm{d}t > 0\,,$$

so that the continuous function $\psi(s)$ is not identically zero, and operator (274) is not the annihilation operator. The arguments are the same as regards a polar kernel if we take into account the continuity of $L(s,t)$. The discussion is also just the same for complex kernels.

It can be shown, by using the concept of the Lebesgue integral, that a sufficient condition for operator (274) to be completely continuous is that the integral

$$\int_a^b\int_a^b |K(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t$$

be finite.

41. Normal operators. Let A_1 and A_2 be two linear operators in space H. If φ is any element, on applying operator A_2 to the element $A_1\varphi$, we get the element $A_2(A_1, \varphi)$, so that the result of successive application of operators A_1 and A_2 is also an operator, which is usually denoted by the symbol $A_2 A_1$ [cf. III_1, 21]:

$$(A_2A_1)\,\varphi = A_2\,(A_1\varphi). \tag{280}$$

It is easily seen that $A_2 A_1$ is a linear operator.

The distributive property follows at once from the distributive properties of A_1 and A_2, whilst the boundedness follows from the boundedness of A_1 and A_2:

$$\| A_2 (A_1 \varphi) \| \leqslant n_{A_2} \| A_1 \varphi \| \leqslant n_{A_2} n_{A_1} \| \varphi \| .$$

We can also define an operator $A_1 A_2$, as the successive application, firstly of A_2, then of A_1:

$$(A_1 A_2) \varphi = A_1 (A_2 \varphi). \tag{281}$$

In general, the linear operator $A_1 A_2$ is different from $A_2 A_1$. Let A_1 and A_2 be integral operators:

$$\psi (s) = \int_a^b K_1 (s, t) \varphi (t) \, dt , \tag{A_1}$$

$$\psi (s) = \int_a^b K_2 (s, t) \varphi (t) \, dt. \tag{A_2}$$

We obtain the following expressions for operators $A_2 A_1$ and $A_1 A_2$:

$$\psi (s) = \int_a^b K_2 (s, t_1) \left[\int_a^b K_1 (t_1, t) \varphi (t) \, dt \right] dt_1, \tag{A_2A_1}$$

$$\psi (s) = \int_a^b K_1 (s, t_1) \left[\int_a^b K_2 (t_1, t) \varphi (t) \, dt \right] dt_1 \tag{A_1A_2}$$

or, if we can change the order of integration:

$$\psi (s) = \int_a^b \left[\int_a^b K_2 (s, t_1) K_1 (t_1, t) \, dt_1 \right] \varphi (t) \, dt, \tag{A_2A_1}$$

$$\psi (s) = \int_a^b \left[\int_a^b K_1 (s, t_1) K_2 (t_1, t) \, dt_1 \right] \varphi (t) \, dt, \tag{A_1A_2}$$

i.e. operators $A_2 A_1$ and $A_1 A_2$ are integral operators with kernels

$$K_{21} (s, t) = \int_a^b K_2 (s, t_1) K_1 (t_1, t) \, dt_1 \quad \text{for} \quad A_1 A_1,$$

$$K_{12} (s, t) = \int_a^b K_1 (s, t_1) K_2 (t_1, t) \, dt_1 \quad \text{for} \quad A_1 A_2.$$

If operators $A_2 A_1$ and $A_1 A_2$ coincide, operators A_1 and A_2 are said to *commute*. In the case of integral operators, where it is possible to change the order of integration as indicated, the coincidence of the kernels $K_{21}(s, t)$ and $K_{12}(s, t)$ guarantees the commutation of operators A_1 and A_2 (it can be shown that this is in fact necessary for the commutation of say continuous or polar kernels).

If c_1 and c_2 are constants, operators $c_1 A_1$ and $c_1 A_1 + c_2 A_2$ can be defined by the expressions

$$(c_1 A_1) \varphi = c_1 (A_1 \varphi) \, ; \quad (c_1 A_1 + c_2 A_2) \varphi = c_1 A_1 \varphi + c_2 A_2 \varphi .$$

In the case of integral operators, $c_1 A_1 + c_2 A_2$ is the integral operator with kernel $c_1 K_1(s, t) + c_2 K_2(s, t)$.

Let A be an integral operator with continuous or weakly polar kernel:

$$\psi(s) = \int_a^b K(s, t) \varphi(t) \, dt, \tag{A}$$

and let us construct the so-called conjugate operator A^* with kernel $\overline{K^*(s, t)} = \overline{K(t, s)}$, which is also continuous or weakly polar:

$$\psi(s) = \int_a^b \overline{K(t, s)} \varphi(t) \, dt. \tag{A^*}$$

We have for any elements φ and ψ (cf. [36]):

$$(A\varphi, \psi) = (\varphi, A^*\psi). \tag{282}$$

We construct the operators

$$A_1 = \frac{1}{2} A + \frac{1}{2} A^*; \qquad A_2 = \frac{1}{2i} A - \frac{1}{2i} A^*. \tag{283}$$

These are integral operators with continuous or weakly polar kernels of the form

$$\frac{1}{2}\left[K(s, t) + \overline{K(t, s)}\right] \quad \text{and} \quad \frac{1}{2i}\left[K(s, t) - \overline{K(t, s)}\right].$$

It is easily seen that these are Hermitian kernels, so that A_1 and A_2 are self-conjugate rigorously completely continuous operators in space H of complex continuous functions. Operators A and A^* are given in terms of A_1 and A_2 by

$$A = A_1 + iA_2 \qquad A^* = A_1 - iA_2. \tag{284}$$

We now distinguish a class of operators for which we can introduce the theory described above for self-conjugate completely continuous operators.

DEFINITION. *An integral operator A is said to be normal if it commutes with A^**, i.e. *if $A^*A = AA^*$.*

Using what has been said above, we can write the (sufficient) condition for an integral operator to be normal:

$$\int_a^b K(s, t_1) \overline{K(t, t_1)} \, dt_1 = \int_a^b \overline{K(t_1, s)} K(t_1 t) \, dt_1, \tag{285}$$

on the natural assumption that the order of integration can be changed when forming operators AA^* and A^*A. This will be the case if the kernel is continuous or weakly polar.

If A commutes with A^*, it follows at once from (283) that A_1 and A_2 also commute. Let $K(s, t)$ be such that A and A^* are completely continuous operators and A commutes with A^*, i.e. A is a normal operator. We shall not use the fact that A is an integral operator in what follows. The only points of importance

for us are that (282) holds for any φ and ψ, that A and A^* commute, and that both these operators are completely continuous. It now follows from (283) that the self-conjugate operators A_1 and A_2 also commute.

Using only (282), let us show that say A_1 is self-conjugate. It follows from (282) that $(A^* \varphi, \psi) = (\varphi, A\psi)$. Further:

$$(A_1\varphi, \psi) = \left(\frac{1}{2} A\varphi + \frac{1}{2} A^*\varphi,\ \psi\right) = \frac{1}{2}(A\varphi, \psi) + \frac{1}{2}(A^*\varphi, \psi)$$
$$= \frac{1}{2}(\varphi, A^*\psi) + \frac{1}{2}(\varphi, A\psi) = \left(\varphi, \frac{1}{2} A^*\psi + \frac{1}{2} A\psi\right) = (\varphi, A_1\psi),$$

i.e. $(A_1 \varphi, \psi) = (\varphi, A_1\psi)$, whence A_1 must be self-conjugate.

Let μ_k and φ_k be the eigenvalues and pairwise orthogonal normalized eigen-elements of the self-conjugate completely continuous operator A_1:

$$A_1\varphi_k = \mu_k\varphi_k. \tag{286}$$

On applying operator A_2 to both sides and remembering that A_1 and A_2 commute, we obtain

$$A_1(A_2\varphi_k) = \mu_k(A_2\varphi_k),$$

whence it is evident that $A_2\varphi_k$ is an eigenelement of operator A_1 corresponding to the eigenvalue μ_k, or else $A_2\varphi_k$ is the zero element.

Suppose that μ is an eigenvalue of rank h and that $\mu_k = \mu_{k+1} = \ldots = \mu_{k+h-1}$. We must now have, from what has been said:

$$A_2\varphi_p = \sum_{q=k}^{k+h-1} c_{pq}\varphi_q \qquad (p = k,\ k+1,\ \ldots,\ k+h-1)$$

and

$$c_{pq} = (A_2\varphi_p, \varphi_q) = (\varphi_p, A_2\varphi_q) = \overline{(A_2\varphi_q, \varphi_p)} = \overline{c_{qp}},$$

i.e. the c_{pq} form an Hermitian matrix.

We can perform any unitary transformation U [III$_1$, 28] on the φ_p $(p = k, k+1, \ldots, k+h-1)$ and again obtain an orthogonal and normalized system of eigenelements of operator A_1, corresponding to the eigenvalue $\mu = \mu_k$. Further, we can choose this unitary transformation in such a way that the Hermitian matrix c_{pq} reduces to the diagonal form.

Let $\nu_k, \nu_{k+1}, \ldots, \nu_{k+h-1}$ be the diagonal elements of this diagonal matrix (they are real). We therefore have, retaining the previous notation for the eigen-elements:

$$A_1\varphi_p = \mu_p\varphi_p\,; \qquad A_2\varphi_p = \nu_p\varphi_p$$
$$(p = k,\ k+1, \ldots,\ k+h-1\,; \qquad \mu_k = \mu_{k+1} = \ldots \mu_{k+h-1}),$$

where some or even all of the numbers ν_p may vanish. We carry out this operation for all the non-zero eigenvalues. It is possible that not all the non-zero eigenvalues of operators A_2 are obtained by this means. We take the non-zero eigenvalues of A_2 which have not been obtained and perform an operation analogous to the above, departing from A_2 and passing to A_1. We finally obtain a finite or infinite number of elements φ_k $(k = 1, 2, \ldots)$, which are mutually orthogonal and normalized, and such that

$$A_1\varphi_k = \mu_k^{(1)}\varphi_k. \qquad A_2\varphi_k = \mu_k^{(2)}\varphi_k, \tag{287}$$

where at least one of the two real numbers $\mu_k^{(1)}$ and $\mu_k^{(2)}$ differs from zero, and every eigenelement of A_1 corresponding to a non-zero eigenvalue is linearly expressible in terms of a finite number of φ_k, and similarly for A_2. It folows at once from (287) and (283) that

$$A\varphi_k = \left(\mu_k^{(1)} + \mu_k^{(2)}\, i\right)\varphi_k\,; \qquad A^*\varphi_k = \left(\mu_k^{(1)} - \mu_k^{(2)}\, i\right)\varphi_k \qquad (k = 1, 2, \ldots),$$

i.e. the complex numbers $\sigma_k = \mu_k^{(1)} + \mu_k^{(2)}\, i$ are eigenvalues of the normal operator A and the φ_k are the corresponding eigenelements.

The expansion theorem which we proved in [38] for a self-conjugate completely continuous operator also follows readily for a normal operator A. We have by the previous theorem, with any choice of element ω:

$$A_1\omega = \sum_k (\omega, \varphi_k)\, \mu_k^{(1)}\, \varphi_k\,; \qquad A_2\, \omega = \sum_k (\omega, \varphi_k)\, \mu_k^{(2)}\, \varphi_k \tag{288}$$

and it follows from (284) that

$$A\omega = \sum_k (\omega, \varphi_k)\, (\mu_k^{(1)} + \mu_k^{(2)}\, i)\, \varphi_k, \tag{289}$$

the convergence of the series being understood as convergence in the mean. As in [38], the coefficients of this series may be shown to be the Fourier coefficients of $A\omega$. We remark that, if certain $\mu_m^{(1)} = 0$, the term in φ_m will be absent in the expansion of $A_1\, \omega$, and similarly for $A_2\, \omega$.

We have now proved the existence theorem for the eigenvalues and the expansion theorem for normal operators.

As we saw in [38], given our assumptions regarding the kernel $K(s, t)$ (continuity or weak polarity), series (273) is convergent absolutely and uniformly in $[a, b]$. The same can therefore be said of the series (289), and we can assert as in [38] that the sum of the series is equal to $A\omega$.

We now consider the function expressible in terms of the kernel:

$$\int_a^b K\ (s, t_1)\, \overline{K\ (t, t_1)}\, \mathrm{d}t_1.$$

Its Fourier coefficients with respect to the system of functions $\varphi_k(s)$ are equal to $(\sigma_k)^2\, \overline{\varphi_k(t)}$, so that we have

$$\int_a^b K\ (s, t_1)\, \overline{K\ (t, t_1)}\, \mathrm{d}t_1 = \sum_k{}' \mid \sigma_k \mid^2 \varphi_k\, (s)\, \overline{\varphi_k\, (t)},$$

the series being uniformly convergent in $[a, b]$. Putting $t = s$ and integrating with respect to s, we get

$$\sum_k \mid \sigma_k \mid^2 = \int_a^b \int_a^b \mid K\ (s, t) \mid^2 \mathrm{d}t\, \mathrm{d}s$$

or, in the earlier notation:

$$\sum_k{}' \frac{1}{\mid \lambda_k \mid^2} = \int_a^b \int_a^b \mid K\ (s, t) \mid^2 \mathrm{d}t\, \mathrm{d}s. \tag{290}$$

We have thus proved for normal kernels. a formula analogous to (161). It can be shown that, if the kernel is not normal, formula (290) no longer holds (I. A. Gol'dfain, *Uchenye zapiski Moskovskogo universiteta*, 1946). We remark that the Hermitian kernel is a particular case of the normal kernel, since here $A^* = A$, and the commutation of A and A^* is obvious.

42. The case of functions of several variables. We have defined the space F as the set of real or complex functions continuous in a finite interval $[a, b]$. We might similarly have defined F as the set of functions $\varphi(M)$, continuous in some finite closed domain B on a plane, on a surface or in three-dimensional space. The whole of the above theory is still applicable in this case. The integrations must always be carried out over the domain B. As in the case of a single independent variable, integral operators with continuous or weakly polar kernels are rigorously completely continuous operators in F.

43. Volterra's equation. We turn to a discussion of the second order Volterra equation in the one-dimensional case:

$$\varphi(s) = f(s) + \lambda \int_0^s K(s, t)\, \varphi(t)\, dt. \tag{291}$$

As already mentioned, this equation is a particular case of the Fredholm equation, when $K(s, t) = 0$ for $t > s$, i.e. when the kernel vanishes in the half of the square k_0 lying to one side of its diagonal $s = t$. Let $f(s)$ be a continuous function in the interval $a \leqslant s \leqslant b$ and let $K(s, t)$ be continuous for $a \leqslant s \leqslant b$, $a \leqslant t \leqslant s$ and $K(s, t) = 0$ for $t > s$. Hence, the kernel has discontinuities of the first kind on the diagonal $s = t$, if $K(s, s) \neq 0$. All the fundamental theorems and treatment of [5—11] are fully retained [14].

As earlier, we seek the solution as the series

$$\varphi(s) = \varphi_0(s) + \varphi_1(s)\lambda + \varphi_2(s)\lambda^2 + \dots \tag{292}$$

We obtain the expressions for the functions $\varphi_n(s)$:

$$\varphi_0(s) = f(s); \quad \varphi_n(s) = \int_a^s K(s, t)\, \varphi_{n-1}(t)\, dt \qquad (n = 1, 2, \dots).$$

We can write for continuous functions in a finite interval or square:

$$|f(s)| \leqslant m; \qquad |K(s, t)| \leqslant M$$

and obtain the following inequalities in turn for the $\varphi_n(s)$:

$$|\varphi_0(s)| \leqslant m; \qquad |\varphi_1(s)| \leqslant \int_a^s |K(s,t)||\varphi_0(t)|\,\mathrm{d}t \leqslant mM(s-a),$$

$$|\varphi_2(s)| \leqslant \int_a^s |K(s,t)||\varphi_1(t)|\,\mathrm{d}t \leqslant mM^2 \int_a^s (t-a)\,\mathrm{d}t = mM^2 \frac{(s-a)^2}{2!}$$

and in general:

$$|\varphi_n(s)| \leqslant m \frac{[M(s-a)]^n}{n!}.$$

When s varies in the finite interval $[a, b]$ the moduli of the terms of series (292) do not exceed the positive numbers

$$m \frac{[|\lambda| M(b-a)]^n}{n!},$$

which form a convergent series whatever the λ, so that series (292) is absolutely and uniformly convergent in $[a, b]$, and its sum $\varphi(s)$ is a continuous function and satisfies equation (291).

Precisely as in [5], we can form the resolvent

$$R(s, t; \lambda) = \sum_{n=0}^{\infty} K_{n+1}(s,t)\lambda^n, \tag{293}$$

where

$$K_1(s,t) = K(s,t); \qquad K_n(s,t) = \int_a^s K_{n-1}(s,t_1) K(t_1, t)\,\mathrm{d}t_1 \tag{294}$$

$$(n = 2, 3, \ldots),$$

the implication of these latter expressions being that $K_n(s, t) = 0$ for $t > s$. For, if $t > s$, then $t_1 < t$ and $K(t_1, t) = 0$.

As above, series (293) can be shown to be absolutely and uniformly convergent for all λ. The resolvent for the Volterra equation (291) is therefore an entire function, and the equation has a unique solution for any λ, given by [6]:

$$\varphi(s) = f(s) + \lambda \int_a^s R(s, t; \lambda) f(t)\,\mathrm{d}t. \tag{295}$$

We can thus say that the Volterra equation has no eigenvalues, i.e. the homogeneous equation

$$\varphi(s) = \lambda \int_a^s K(s, t)\,\varphi(t)\,\mathrm{d}t$$

only has zero solutions for any λ. Apropos of this, if we were to construct the Fredholm determinant $D(\lambda)$ for equation (291), it would prove to have no zeros at all [8].

It can be shown that, if the kernel has the form

$$K(s, t) = \frac{L(s, t)}{(s - t)^{\alpha}} \qquad (s > t),$$

where $L(s, t)$ is a continuous function and $0 < \alpha < 1$, equation (291) has a unique solution as before, and this solution can be found by the method of successive approximations indicated above. As from a certain n, the kernels $K_n(s, t)$ are now continuous. Even the kernel $K_2(s, t)$ will be continuous with $\alpha < 1/2$ [28].

The method of successive approximations is similarly applicable to the system of equations

$$\varphi_i(s) = f_i(s) + \lambda \sum_{k=1}^{m} \int_a^s K_{ik}(s, t)\, \varphi_k(t)\, \mathrm{d}t. \tag{296}$$

It is a characteristic feature of the Volterra equation that, given our assumptions, the series obtained by the method of successive approximations is convergent for all values of λ in the interval. If the continuity condition is observed for all $s \geqslant a$, we obtain a solution for all $s \geqslant a$.

Let us consider the equation with two variable limits:

$$\varphi(s) = f(s) + \lambda \int_{\omega(s)}^{s} K(s, t)\, \varphi(t)\, \mathrm{d}t \tag{297}$$

or the equation

$$\varphi(s) = f(s) + \lambda \int_a^{\omega(s)} K(s, t)\, \varphi(t)\, \mathrm{d}t. \tag{298}$$

Suppose that s varies in an interval $[a, b]$, with the usual continuity conditions for $f(s)$ and $K(s, t)$, and suppose further that $a \leqslant \omega(s) \leqslant s$ in this interval. There obviously exist positive numbers N and M such that, for $a \leqslant s,\ t \leqslant b$:

$$|f(s)| \leqslant N; \qquad |K(s, t)| \leqslant M.$$

We replace $f(s)$ and $K(s, t)$ in (297) or (298) by the larger positive numbers N and M, and $(\omega(s), s)$ or $(a, \omega(s))$ by the wider interval of integration (a, s):

$$\varphi(s) = N + \lambda M \int_a^s \varphi(t)\, \mathrm{d}t. \tag{299}$$

Application of the method of successive approximations to the last equation leads, as may readily be shown, to a power series in λ, the coefficients of which are positive and not less than the absolute values of the coefficients of the power series obtained by solving equation (297) or (298). Equation (299) has the usual form, the role of $K(s, t)$ being played by the constant M for $t \leqslant s$, and the corresponding power series is uniformly convergent with respect to s in the interval $[a, b]$ for any λ. The same can be said all the more for the series obtained on solving equation (297) or (298), and this series gives the solution of the corresponding equation. We remark that the solution of equation (299) is expressible in an explicit form, viz.

$$\varphi(s) = N e^{\lambda M(s-a)}.$$

We also remark that equation (297), for instance, can be written in the ordinary form (291), the kernel being subject to the condition: $K(s, t) = 0$ for $t < \omega(s)$.

We can interchange the limits in the integral appearing in equation (291), whilst simultaneously changing the sign of the kernel. Thus the fact that the upper limit is variable is not essential to the theory. Similarly, we could have laid down the condition $s \leqslant a$ instead of $s \geqslant a$. One case is transformed to the other by means of the simple substitutions $s' = -s$ and $t' = -t$. Similarly, we could have taken $s \leqslant \omega(s) \leqslant b$ in equation (297), for instance, instead of the above inequalities for $\omega(s)$.

We further consider the equation

$$\varphi(s) = f(s) + \lambda \int_{-s}^{s} K(s, t)\, \varphi(t)\, dt, \tag{300}$$

where $f(s)$ is defined and continuous for $-b \leqslant s \leqslant b$, and the kernel $K(s, t)$ is defined for $-b \leqslant s \leqslant b$; $-b \leqslant t \leqslant b$. On splitting the interval of integration into two parts $(-s, 0)$, and $(0, s)$, and replacing the variable of integration t in the first case by $(-t)$, we get:

$$\varphi(s) = f(s) + \lambda \int_0^s K(s, -t)\, \varphi(-t)\, dt + \lambda \int_0^s K(s, t)\, \varphi(t)\, dt;$$

we replace s by $(-s)$ and t by $(-t)$:

$$\varphi(-s) = f(-s) - \lambda \int_0^s K(-s, t)\, \varphi(t)\, dt - \lambda \int_0^s K(-s, -t)\, \varphi(-t)\, dt.$$

We obtain on taking $0 \leqslant s \leqslant b$ and $0 \leqslant t \leqslant b$:

$$\varphi(s) = \varphi_1(s); \quad \varphi(-s) = \varphi_2(s); \quad f(s) = f_1(s); \quad f(-s) = f_2(s);$$
$$K(s,t) = K_{11}(s,t); \quad K(s,-t) = K_{12}(s,t); \quad K(-s,t) = -K_{21}(s,t),$$
$$K(-s,-t) = -K_{22}(s,t).$$

Integral equation (300) is reduced to the system of equations of the usual form:

$$\varphi_1(s) = f_1(s) + \lambda \int_0^s K_{11}(s,t)\,\varphi_1(t)\,dt + \lambda \int_0^s K_{12}(s,t)\,\varphi_2(t)\,dt,$$

$$\varphi_2(s) = f_2(s) + \lambda \int_0^s K_{21}(s,t)\,\varphi_1(t)\,dt + \lambda \int_0^s K_{22}(s,t)\,\varphi_2(t)\,dt.$$

If we solve this system, we get two functions $\varphi_1(s)$ and $\varphi_2(s)$, continuous in the interval $0 \leqslant s \leqslant b$. We now find the solution $\varphi(s)$ of (300) from the formulae: $\varphi(s) = \varphi_1(s)$ for $0 \leqslant s \leqslant b$; $\varphi(s) = \varphi_2(-s)$ for $-b \leqslant s \leqslant 0$. Either of the two formulae is applicable for $s = 0$, so that $\varphi_1(0) = f_1(0) = f(0)$ and $\varphi_2(0) = f_2(0) = f(0)$. This shows, that the solution of equation (300) thus obtained is continuous at the point $s = 0$.

The above method of successive approximations is also applicable to the case of several independent variables. For instance, in the case of two independent variables we have the equation

$$\varphi(x,y) = f(x,y) + \lambda \int_a^x \int_c^y K(x,y;\ s,t)\,\varphi(s,t)\,ds\,dt,$$

to which everything that has been said above can be applied. An expansion in the parameter λ, convergent for all values of λ, is also possible for more general equations in which when the right-hand side contains single integrals as well as a double integral:

$$\varphi(x,y) = f(x,y) + \lambda \int_a^x K_1(x,y;\ s)\,\varphi(s,y)\,ds +$$
$$+ \lambda \int_c^y K_2(x,y;s)\,\varphi(x,s)\,ds + \lambda^2 \int_a^x \int_c^y K_3(x,y;s,t)\,\varphi(s,t)\,ds\,dt.$$

The parameter λ is introduced here only for convenience in carrying out the method of successive approximations. The proof of the existence and uniqueness of the solution of the equation

$$\varphi(x,y) = f(x,y) + \lambda \int_{\omega_2(y)}^{y} \int_{\omega_1(x)}^{x} K(x,y;\ s,t)\,\varphi(s,t)\,ds\,dt,$$

where $a \leqslant \omega_1(x) \leqslant x$ and $c \leqslant \omega_2(y) \leqslant y$, is exactly similar to the above.

We might also have taken the function ω_2 as dependent on x instead of on y, and the function ω_1 as dependent on y. The uniqueness of the solution of equations (297) and (298) is easy to prove.

44. Laplace transformation. We shall be concerned below with Volterra equations in the special case when the kernel $K(s, t)$ depends only on the difference $(s - t)$. We have to investigate as a preliminary an integral transformation closely connected with the Fourier transformation, namely, the Laplace transformation. This will be needed for the solution of certain problems of mathematical physics, as well as for investigating the Volterra equation with a kernel depending on a difference.

We recall that, if a function $f(x)$ is defined in the interval $-\infty < x < +\infty$, is continuous, satisfies the Dirichlet conditions in any finite interval, and is such that the integral

$$\int_{-\infty}^{+\infty} |f(x)|\, dx \tag{301}$$

exists, the Fourier transform of $f(x)$ is defined as

$$f_1(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{axi}\, dx, \tag{302}$$

and the following inversion formula holds [II, 160]:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(a)\, e^{-axi}\, da; \tag{303}$$

this is equivalent to Fourier's formula, the integral being understood as an integral in the sense of the principal value, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \lim_{M \to +\infty} \int_{-M}^{M} f_1(a)\, e^{-axi}\, da.$$

Suppose that, in addition to integral (301), the integral

$$\int_{-\infty}^{+\infty} e^{\beta x} |f(x)|\, dx \tag{304}$$

also has a finite value for $-m < \beta < m$. The function $f_1(a)$ is now defined, not only for real, but also for complex $a = a_1 + a_2 i$ satisfying the condition $-m < a_2 < m$, since

$$|f(x)\,e^{axi}| = |f(x)|\,e^{-a_2 x},$$

and this integral has a meaning by hypothesis for $-m < a_2 < m$. The a is replaced by the purely imaginary $a = si$ in the Laplace transformation, and in addition — though this is not essential — the factor $(\sqrt{2\pi})^{-1}$ falls out.

We now make a detailed study of the Laplace transformation. A precisely similar investigation is possible for the Fourier transform (302).

Suppose that the function $\varphi(x)$ is continuous in $(-\infty, +\infty)$ except for points of discontinuity of the first kind, the number of these in any bounded piece of the interval being finite. Further, let the function have a simple derivative, or left and right-hand derivatives, at every point, these latter derivatives being understood at points of discontinuity as the limits of the ratios

$$\frac{\varphi(c-h) - \varphi(c-0)}{-h} \quad \text{and} \quad \frac{\varphi(c+h) - \varphi(c+0)}{h}$$

as $h \to +0$.

We suppose further that the integral

$$\int_{-\infty}^{+\infty} e^{-\sigma x}\varphi(x)\,dx \tag{305}$$

is absolutely convergent when σ satisfies the inequality

$$\alpha < \sigma < \beta, \tag{306}$$

where α and β are certain fixed real numbers which may be equal to $(-\infty)$ or $(+\infty)$. The usual convergence equation for a Dirichlet integral and Fourier's formula [cf. II, 152, 160] are now applicable to the function $e^{-\alpha x}\varphi(x)$.

We consider the function of a complex variable $s = \sigma + \tau i$, defined by the equation

$$f(s) = \int_{-\infty}^{+\infty} e^{-sx}\varphi(x)\,dx. \tag{307}$$

Inequality (306) defines a strip parallel to the imaginary axis (or a half-plane if one of the numbers α or β is infinity) on the plane of the complex variable $s = \sigma + \tau i$ (or it may even define the entire plane).

Let B be a finite closed domain inside the strip (306). We can take a point $s_0 = \sigma_0 + \tau_0 i$ lying to the left of B but inside (306), i.e. such that $\sigma > \sigma_0$ for all points $s = \sigma + \tau i$ of B, and a further point $s_1 = \sigma_1 + \tau_1 i$ to the right of B. We now have the inequalities, for all points s of B and for all real x:

$$|e^{-sx}\varphi(x)| \leqslant e^{-\sigma_0 x}|\varphi(x)| \quad \text{for} \quad x \geqslant 0;$$
$$|e^{-sx}\varphi(x)| \leqslant e^{-\sigma_1 x}|\varphi(x)| \quad ,, \quad x \leqslant 0.$$

But, by hypothesis, the functions on the right-hand sides of these inequalities are integrable over the intervals $(0, +\infty)$ and $(-\infty, 0)$. It follows from this that integral (307) is convergent in the domain B absolutely and uniformly with respect to s, so that the function $f(s)$ is regular in the domain B [III_2, 70], and hence, in view of the arbitrary choice of B, $f(s)$ is regular inside strip (306).

We now prove a theorem which gives us an expression for the original function $\varphi(s)$ in terms of the transformed function $f(s)$. In general (307) represents a functional transformation of the function $\varphi(x)$ with the above-mentioned properties, the result of the transformation being a function of a complex variable which is regular in the strip in question.

THEOREM 1. *Given the above assumptions regarding $\varphi(x)$, we have the inversion formula:*

$$\varphi(x) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{sx} f(s)\, ds, \tag{308}$$

in which the integral is taken over any straight line parallel to the imaginary axis and lying inside the strip (306), *and the integral has to be understood in the sense of the principal value.*

The product $e^{-ax}\varphi(x)$ satisfies the above-mentioned conditions for $\varphi(x)$, and in particular, integral (305) is absolutely convergent, so that Fourier's formula can be applied to $e^{-ax}\varphi(x)$:

$$e^{-\sigma x}\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-axi}\, da \int_{-\infty}^{+\infty} e^{-(\sigma - ai)t}\varphi(t)\, dt =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-axi} f(\sigma - ai)\, da.$$

On introducing the new variable of integration $s = \sigma - ai$ instead of a, we in fact get (308).

The function $f(s)$, defined inside strip (306) by (307), behaves in a definite way when the point s moves upwards or downwards to in-

finity in the strip; in fact, by using the absolute convergence of the integral, it can easily be shown that $f(s)$ tends to zero as the point moves to infinity in any strip J_ε defined by the inequality $\alpha + \varepsilon < \sigma < \beta - \varepsilon$, where ε is any given positive number. We assume that $f(s)$ is regular inside (306). Further, given any strip J_ε, let there exist a function $\omega(\varrho)$, defined for $\varrho > 0$, which takes only positive values, satisfies the condition $\omega(\varrho) \to 0$ as $\varrho \to \infty$, has a convergent integral:

$$\int_0^\infty \omega(\varrho)\, d\varrho$$

and is such that we have in J_ε:

$$|f(s)| \leqslant \omega(|\tau|) \qquad (s = \sigma + \tau i). \tag{309}$$

We now prove a theorem analogous to Theorem 1.

THEOREM 2. *Given our assumptions, formula* (308) *yields a function* $\varphi(x)$ *which is defined throughout the real axis, is continuous and does not depend on the choice of* σ. *The original function* $f(s)$ *is now defined in terms of the transformed function* $\varphi(x)$ *by formula* (307), *where the integral is to be understood in the sense of the principal value.*

On putting $s = \sigma + \tau i$ in the right-hand side of (308), we obtain

$$\varphi(x) = \frac{e^{x\sigma}}{2\pi} \int_{-\infty}^{+\infty} f(\sigma + \tau i)\, e^{\tau x i}\, d\tau. \tag{310}$$

Whatever the choice of x, the modulus of the integrand does not exceed the function $\omega(\varrho)$, which has a convergent integral; the integral of (310) is therefore convergent absolutely and uniformly with respect to x. We thus see that $\varphi(x)$ is defined for any real x and is a continuous function [II, 84].

We now show that this function is independent of the choice of σ. We take any rectangle $ABCD$, bounded by the straight lines $\sigma = \sigma_1$; $\sigma = \sigma_2$; $\tau = \pm T$ (Fig. 1). By Cauchy's theorem, the integral of $f(s)e^{xs}$ along the contour of this rectangle vanishes. We consider the value of the integral along the sides $\tau = \pm T$, parallel to the real axis. For instance, we have for the side $\tau = T$:

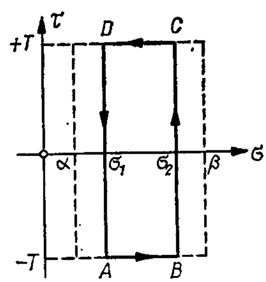

FIG. 1

$$\int_{\sigma_2}^{\sigma_1} f(\sigma + iT)\, e^{x(\sigma + iT)}\, d\sigma. \tag{311}$$

By (309), we can write for this last integral:

$$\left| \int_{\sigma_2}^{\sigma_1} f(\sigma + iT)\, e^{x(\sigma+iT)}\, d\sigma \right| \leqslant e^{\max |x\sigma_k|}\, \omega(T)\, (\sigma_2 - \sigma_1)$$

where $k = 1$ or 2. Hence, since $\omega(\varrho) \to 0$ as $\varrho \to \infty$, it is clear that integral (311) tends to zero as $T \to \infty$. A similar result is found for the integral along $\tau = -T$. On applying the Cauchy theorem mentioned, we can say that the integral of $f(s)e^{xs}$ along the straight line $\sigma = \sigma_1$ from top to bottom differs only in sign from the integral along $\sigma = \sigma_2$ taken from bottom to top, i.e. both integrals are equal if both are taken from bottom to top. Since the straight lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$ are chosen arbitrarily, we can say that the integral of $f(s)e^{xs}$ along $\sigma = \sigma_0$ has the same value whatever the choice of the straight line inside the strip, i.e. whatever the choice of σ_0, provided it satisfies $a < \sigma_0 < \beta$.

It still remains to prove that $f(s)$ is given in terms of $\varphi(x)$ by (307). Putting $s = \sigma - \tau i$ in (308), we have:

$$e^{-x\sigma}\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\sigma - \tau i)\, e^{-\tau x i}\, d\tau .$$

We multiply both sides by e^{xui} and integrate with respect to x from $(-\infty)$ to $(+\infty)$. Since Fourier's formula is applicable to $f(\sigma - \tau i)$, regarded as a function of the real variable τ:

$$f(\sigma - ui) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{xui}\, dx \int_{-\infty}^{+\infty} f(\sigma - \tau i)\, e^{-\tau x i}\, d\tau,$$

we have:

$$f(\sigma - ui) = \int_{-\infty}^{+\infty} \varphi(x)\, e^{-(\sigma - ui)x}\, dx,$$

and this gives us (307), since u is arbitrary. Formulae (307) and (308) are inversions of each other, in the sense indicated in Theorems 1 and 2.

We show that, if $\beta = +\infty$ in Theorem 2, i.e. if the given function $f(s)$ is regular in the half-plane $\sigma > a$ and satisfies the remaining conditions in it, the function $\varphi(x)$ defined by (308) vanishes for $x < 0$. We observe that a function $\omega(\varrho)$ with the above-mentioned properties must exist in the present case in the half-plane $\sigma \geqslant a + \varepsilon$, whatever the positive ε. Let us now show that $\varphi(x) = 0$ for $x < 0$. On writing

the usual inequality for the integral and using (309), we get

$$|\varphi(x)| \leqslant e^{x\sigma} \frac{1}{\pi} \int_0^\infty \omega(\varrho)\, d\varrho .$$

If x is a fixed negative number, the right-hand side tends to zero as $\sigma \to +\infty$, whilst the left-hand side is independent of the choice of σ, so that in fact $\varphi(x) = 0$ for $x < 0$. In the present case the inversion formula for transform (308) has the following form instead of (307):

$$f(s) = \int_0^\infty e^{-sx} \varphi(x)\, dx . \tag{312}$$

Conversely, if we take $\varphi(x)$ as given, (307) is usually termed the *two-sided Laplace transform*, whilst (312) is the *one-sided Laplace transform*. This latter transform is obviously a particular case of the former and is obtained from it when the given $\varphi(x)$ vanishes for negative x. In the case of the one-sided Laplace transform we have to impose on $\varphi(x)$ the condition that integral (312) is absolutely convergent in the half-plane $\sigma > a$. If B is a finite closed domain lying inside this half-plane, we can take a straight line $\sigma = \sigma_0 > a$ inside the half-plane to the left of B. By hypothesis, the integral

$$\int_0^\infty e^{-\sigma_0 x} |\varphi(x)|\, dx$$

is convergent, and recalling that the variable of integration $x > 0$, we have for s belonging to B:

$$|e^{-sx}\varphi(x)| < e^{-\sigma_0 x} |\varphi(x)| ,$$

i.e. integral (312) is absolutely and uniformly convergent with respect to s for all s belonging to B, and gives a function $f(s)$ which is regular in B, i.e. regular in the half-plane $\sigma > a$.

The following proposition is an immediate consequence of the above inequalities: if integral (312) is absolutely convergent at the point $s_0 = \sigma_0 + \tau_0 i$, it is absolutely and uniformly convergent in the half-plane $\sigma \geqslant \sigma_0$. The inverse of (312) is (308). We remark that the theorems stated above can also be proved under more general assumptions regarding $\varphi(x)$ and $f(s)$. The right-hand sides of (312) and (307) are very often written as $L_1(\varphi)$ and $L_2(\varphi)$:

$$L_1(\varphi) = \int_0^\infty e^{-sx} \varphi(x)\, dx; \quad L_2(\varphi) = \int_{-\infty}^{+\infty} e^{-sx}\varphi(x)\, dx .$$

Transformations $L_1(\varphi)$ and $L_2(\varphi)$ are distributive, i.e.

$$L_i(c_1\varphi) = c_1 L_i(\varphi); \quad L_i(c_1\varphi_1 + c_2\varphi_2) = c_1 L_i(\varphi_1) + c_2 L_i(\varphi_2),$$

where c_1 and c_2 are arbitrary constants and $\varphi_i(x)$ are functions satisfying the conditions indicated above. If we introduce the new variable $u = e^{-x}$ in place of x and put $\varphi(x) = \psi(u)$, transforms (307) and (308) become

$$f(s) = \int_0^\infty u^{s-1}\psi(u)\,du; \quad \psi(u) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} u^{-s} f(s)\,ds.$$

If we were to discuss in the same way the Fourier transform (302), instead of a vertical strip in which $f(s)$ is a regular function we should have a horizontal strip (parallel to the real axis) in which $f_1(a)$ is regular $(a = si)$. For the rest, the results would be the same apart from a constant factor in front of the integral.

45. Convolution of functions. Let $\varphi(x)$ and $\varphi_2(x)$ be two continuous functions defined for $x \geqslant 0$. The *convolution of these two functions* is defined as the function $\varphi_3(x)$ given by

$$\varphi_3(x) = \int_0^x \varphi_1(t)\,\varphi_2(x-t)\,dt. \tag{313}$$

This function is defined for $x \geqslant 0$ and is also continuous. On introducing the new variable of integration $\tau = x - t$ in place of t, we can write $\varphi_3(x)$ as

$$\varphi_3(x) = \int_0^x \varphi_1(x-\tau)\,\varphi_2(\tau)\,d\tau. \tag{314}$$

The convolution of two functions is generally denoted by the symbol

$$\varphi_3 = \varphi_1^* \varphi_2,$$

whilst it follows at once from (313) and (314) that the convolution is independent of the order of the functions, i.e. $\varphi_2^*\varphi_1 = \varphi_1^*\varphi_2$. The operation of obtaining the convolution is also called *convolution* (folding, faltung).

Let functions $\varphi_1(x)$ and $\varphi_2(x)$ be subjected to transformation (312), which is absolutely convergent in a half-plane $\sigma > a$. We show that (312) is also convergent for $\varphi_3(x)$ in the same half-plane, and that the following expression holds:

$$L_1(\varphi_1^*\varphi_2) = L_1(\varphi_1)\,L_1(\varphi_2), \tag{315}$$

i.e. *convolution in the domain of functions* $\varphi_k(x)$ *corresponds to simple multiplication in the domain of the transformed functions:*

$$f_k(s) = \int_0^\infty e^{-sx} \varphi_k(x)\,dx. \tag{316}$$

To prove this, we write down the product on the right-hand side of (315):

$$\int_0^\infty e^{-su} \varphi_1(u)\,du \int_0^\infty e^{-sv} \varphi_2(v)\,dv, \tag{317}$$

where the variables of integration have been denoted by u and v. We can write this product as an absolutely convergent double integral over the first quadrant of the (u, v) plane:

$$\int_0^\infty e^{-su}\varphi_1(u)\,du \cdot \int_0^\infty e^{-sv}\varphi_2(v)\,dv = \int_0^\infty \int_0^\infty e^{-s(u+v)}\varphi_1(u)\,\varphi_2(v)\,du\,dv.$$

The fact that (317) can be written in this way follows at once from the absolute convergence of the integrals appearing in it. We can see this simply by performing the integration over a finite interval $(0, m)$ in these integrals, transforming the product into a double integral, then letting m tend to infinity and making use of the usual definition of improper double integral [II, 86]. We introduce the new variables of integration $x = u + v$ and $t = v$ into the double integral. We arrive at the absolutely convergent double integral

$$\int_B\int e^{-sx}\varphi_1(x-t)\,\varphi_2(t)\,dt\,dx,$$

for which the domain of integration is defined in the old variables by $u \geqslant 0$; $v \geqslant 0$, or in the new variables by $t \geqslant 0$; $x - t \geqslant 0$, i.e. the domain of integration on the (t, x) plane is the part of the first quadrant lying above the bisector $t = x$. We obtain on reducing the double integral to two quadratures:

$$\int_0^\infty e^{-su}\varphi_1(u)\,du \cdot \int_0^\infty e^{-sv}\varphi_2(v)\,dv = \int_0^\infty e^{-sx}\Big[\int_0^x \varphi_1(x-t)\,\varphi_2(t)\,dt\Big]\,dx,$$

which proves (315).

We can write for the function $\varphi_3(x)$:

$$|\varphi_3(x)| \leqslant \int_0^x |\varphi_1(x-t)|\,|\varphi_2(t)|\,dt,$$

which implies the inequality

$$\int_0^m e^{-\sigma x} |\varphi_3(x)| \, dx \leqslant \int_0^m dx \int_0^x e^{-\sigma x} |\varphi_1(x-t)| \, |\varphi_2(t)| \, dt,$$

or, on carrying out a Dirichlet transformation [II, 79]:

$$\int_0^m e^{-\sigma x} |\varphi_3(x)| \leqslant \int_0^m |\varphi_2(t)| \, dt \int_t^m e^{-\sigma x} |\varphi_1(x-t)| \, dx,$$

On replacing x on the right-hand sides by the new variable of integration $\tau = x - t$, we get:

$$\int_0^m e^{-\sigma x} |\varphi_3(x)| \, dx \leqslant \int_0^m e^{-\sigma t} |\varphi_2(t)| \, dt \cdot \int_0^{m-t} e^{-\sigma \tau} |\varphi_1(\tau)| \, d\tau,$$

or, all the more:

$$\int_0^m e^{-\sigma x} |\varphi_3(x)| \, dx \leqslant \int_0^\infty e^{-\sigma t} |\varphi_2(t)| \, dt \cdot \int_0^\infty e^{-\sigma \tau} |\varphi_1(\tau)| \, d\tau,$$

i.e. the absolute convergence of integrals (317) in the half-plane $\sigma > a$ implies the absolute convergence of the same integral for $\varphi_3(x)$. We observe that the reduction of the double integral over a quadrant to two quadratures is easily justified in the usual way, by first considering a finite part of the quadrant above the bisector $t = x$, then passing to the limit. The assertion that (315) holds is usually called the *convolution theorem.*

We can introduce the concept of convolution and prove a convolution theorem for two-sided Laplace transformations in exactly the same way, i.e. the following proposition holds: if $\varphi_1(x)$ and $\varphi_2(x)$ are continuous functions defined in the infinite interval $(-\infty, +\infty)$, and the integrals $L_2(\varphi_1)$ and $L_2(\varphi_2)$ are absolutely convergent in a strip $a < \sigma < \beta$, the integral

$$\varphi_3(x) = \int_{-\infty}^{+\infty} \varphi_1(t) \, \varphi_2(x-t) \, dt \tag{318}$$

is absolutely convergent for any real x. The Laplace transform of the function $\varphi_3(x)$ will be absolutely convergent in the strip mentioned, and the convolution formula holds:

$$L_2(\varphi_3) = L_2(\varphi_1) \cdot L_2(\varphi_2). \tag{319}$$

46. Volterra equation of special type. Let us take the Volterra equation with a kernel depending only on the difference of the two arguments:

$$\varphi(x) = f(x) + \int_0^x K(x-t)\,\varphi(t)\,\mathrm{d}t. \tag{320}$$

Suppose that the continuous functions $f(x)$ and $K(x)$ tend to zero as $x \to +\infty$ and satisfy for large x:

$$|f(x)| \leqslant A\mathrm{e}^{-ax}; \quad |K(x)| \leqslant B\mathrm{e}^{-bx}, \tag{321}$$

where the constants A and $B > 0$, whilst constants a and $b \geqslant 0$. Let f_0 and K_0 be the upper bounds of $|f(x)|$ and $|K(x)|$ for $x \geqslant 0$. On applying the method of successive approximations to equation (320) [43], we obtain for $\varphi(x)$ with $x \geqslant 0$ the inequality $|\varphi(x)| \leqslant f_0\,\mathrm{e}^{K_0 x}$. It is clear from this that a one-sided Laplace transformation with $\sigma > \max(a, b, K_0)$ is applicable to $\varphi(x)$, $f(x)$ and $K(x)$, and we obtain the transforms

$$\Phi(s) = L_1(\varphi); \quad F(s) = L_1(f); \quad L(s) = L_1(K), \tag{322}$$

which are regular in the half-plane $\sigma > K_0$. On applying a one-sided Laplace transformation to both sides of (320) and using the convolution formula, we have:

$$\Phi(s) = F(s) + L(s)\,\Phi(s),$$

whence

$$\Phi(s) = \frac{F(s)}{1 - L(s)}. \tag{323}$$

We saw above that the function $\varphi(s)$ must be regular in the half-plane $\sigma > K_0$. Hence it follows, in view of the complete independence of $L(s)$ and $F(s)$, that the denominator in the fraction written above cannot have zeros inside the half-plane. We obtain by inversion of the first of expressions (322):

$$\varphi(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s)\,\mathrm{e}^{sx}\,\mathrm{d}s \quad (\sigma > K_0). \tag{324}$$

Thus, on defining functions $F(s)$ and $L(s)$ by formulae (322) and $\varphi(s)$ by (323), we obtain from (324) the solution of equation (320) in the explicit form. We remark that, when determining the function $\varphi(x)$ in the finite interval $(0, l)$, we make use in equation (320) of the values of $f(x)$ and $K(x)$ in this interval and can therefore continue these functions outside the interval in any

manner, and in particular, in such a way that they satisfy the conditions indicated above. We can further suppose them to be identically zero for sufficiently large positive x.

We show that all the iterated kernels also depend only on the difference $(x - t)$ for equation (320). We have [43]:

$$K_2(x, t) = \int_t^x K(x - t_1) K(t_1 - t)\, dt_1.$$

We bring in the new variable of integration $\tau = t_1 - t$ in place of t_1:

$$K_2(x, t) = \int_0^{x-t} K(x - t - \tau) K(\tau)\, d\tau,$$

whence it follows at once that $K_2(x, t)$ is a function of the difference $(x - t)$, i.e. $K_2(x, t) = K_2(x - t)$.

The proof is similar for the succeeding iterated kernels. We can therefore say, by virtue of (293) with $\lambda = 1$, that the resolvent of equation (320) depends only on the above difference. On writing the resolvent as $R(x - t)$, we can use (295) to write the solution of equation (320) as

$$\varphi(x) = f(x) + \int_0^x R(x - t) f(t)\, dt. \tag{325}$$

On applying the Laplace transformation to both sides of this equation and introducing, together with (322), the notation

$$M(s) = L_1(R), \tag{326}$$

we obtain

$$\Phi(s) = F(s) + M(s) F(s).$$

By using (323), we can find $M(s)$ in terms of the known function $L(s)$:

$$M(s) = \frac{L(s)}{1 - L(s)}, \tag{327}$$

and inversion of (326) gives us the resolvent $R(x)$:

$$R(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} M(s)\, e^{sx}\, ds. \tag{328}$$

The solution follows on substituting in (325).

This method of solving (320) is also applicable to a system of Volterra equations:

$$\varphi_i(x) = f_i(x) + \sum_{k=1}^{p} \int_0^x K_{ik}(x - t)\, \varphi_k(t)\, dt \qquad (i = 1, 2, \ldots, p).$$

Application of the Laplace transformation to both sides gives us

$$\Phi_i(s) = F_i(s) + \sum_{k=1}^{p} L_{ik}(s)\,\Phi_k(s) \qquad (i = 1,\ 2,\ \ldots,\ p).$$

We obtain the $\varphi_i(s)$ on solving this system of first degree equations, and the solution of the original system is obtained as

$$\varphi_i(x) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \Phi_i(s)\, e^{sx}\, ds.$$

We observe that conditions (321) for kernel $K(x)$ and the function $f(x)$ can be considerably weakened. It is sufficient to require that a positive constant c exist such that $f(x)e^{-cx}$ and $K(x)e^{-cx}$ are bounded in absolute value for $x > 0$. Expressions (324) and (328) now hold for all sufficiently large σ. This is proved simply by multiplying both sides of (320) by e^{-cx} and introducing the new required function $\varphi_1(x) = \varphi(x)e^{-cx}$, the function $f_1(x) = f(x)e^{-cx}$ and the kernel $K_1(x) = K(x)e^{-cx}$.

47. Volterra equation of the first kind. We have so far been concerned exclusively with integral equations of the second kind. As we shall now see, Volterra equations of the first kind can easily be transformed into equations of the second kind, given certainly supplementary conditions. Let us take the Volterra equation of the first kind:

$$\int\limits_a^x K(x, t)\, \varphi(t)\, dt = f(x), \tag{329}$$

where it follows directly from the actual form of the equation that the given function $f(x)$ must satisfy the condition $f(a) = 0$. On differentiating the equation with respect to x and dividing by $K(x,x)$, we arrive at the following equation of the second kind:

$$\varphi(x) + \int\limits_a^x \frac{K_x(x, t)}{K(x, x)}\, \varphi(t)\, dt = \frac{f'(x)}{K(x, x)}, \tag{330}$$

where we assume that $f'(x)$ is continuous and $K(x, x) \neq 0$. A discussion of the general case can be found in H. Muntz's *Integral Equations*.

By using the condition $f(a) = 0$, we can easily return from equation (330) to equation (329), i.e. these equations are equivalent, so that (329) has a unique solution. We now take an equation of the first kind with a kernel of the form

$$K(x, t) = \frac{H(x, t)}{(x - t)^{1-\alpha}} \qquad (0 < \alpha < 1),$$

where $H(x, t)$ is a continuous function having a continuous derivative with respect to x. Abel's equation, which we have already considered, belongs to this type. We shall discuss the integral equation

$$\int_0^x \frac{H(x,t)}{(x-t)^{1-\alpha}} \varphi(t)\, dt = f(x), \tag{331}$$

where the lower limit of integration has been taken as zero, as in Abel's equation. On multiplying both sides of this equation by $(z-x)^{-\alpha}$, integrating with respect to x from $x = 0$ to $x = z$ and applying Dirichlet's formula [II, 79], we arrive at the integral equation:

$$\int_0^z \varphi(t)\, dt \int_t^z \frac{H(x,t)}{(z-x)^{\alpha}(x-t)^{1-\alpha}}\, dx = \int_0^z \frac{f(x)}{(z-x)^{\alpha}}\, dx, \tag{332}$$

the kernel of which is given by

$$K_1(z,t) = \int_t^z \frac{H(x,t)}{(z-x)^{\alpha}(x-t)^{1-\alpha}}\, dx.$$

This kernel is no longer singular, as may easily be seen by a transformation of the variables of integration: instead of x we introduce a new variable θ defined by

$$x = \frac{z+t}{2} + \frac{z-t}{2}\cos\theta,$$

whence we obtain

$$K_1(z,t) = \int_0^\pi \frac{H\left(\frac{z+t}{2} + \frac{z-t}{2}\cos\theta, t\right)\sin\theta}{(1+\cos\theta)^{1-\alpha}(1-\cos\theta)^{\alpha}}\, d\theta, \tag{333}$$

whilst this in turn shows that the kernel $K_1(z, t)$ is continuous, in view of the continuity of the kernel $H(x, t)$ and the uniform convergence with respect to z and t of the integral written above. By using expressions from the theory of gamma functions [III_2, 71, 72], we can write

$$\int_0^\pi (1+\cos\theta)^{\alpha-1}(1-\cos\theta)^{-\alpha}\sin\theta\, d\theta = \frac{\pi}{\sin \pi\alpha},$$

and (333) gives us

$$K_1(z,z) = H(z,z)\frac{\pi}{\sin \pi\alpha}.$$

The new continuous kernel $K_1(z, t)$ will therefore satisfy the condition $K_1(z, z) \neq 0$ provided the corresponding condition is satisfied by $H(x, t)$, i.e. $H(x, x) \neq 0$. It also follows directly from (333) that

$K_1(z, t)$ has a continuous derivative with respect to z, provided the continuous derivative $H_x(x, t)$ exists. In the same way, when the continuous derivative $f'(x)$ exists, it follows at once from

$$f_1(z) = \int_0^z \frac{f(x)}{(z-x)^a}\,\mathrm{d}x = \int_0^z \frac{(z-x)^{1-a} f'(x)}{1-a}\,\mathrm{d}x$$

that the right-hand side of equation (332) has a continuous derivative:

$$f_1'(z) = \int_0^z \frac{f'(x)}{(z-x)^a}\,\mathrm{d}x.$$

Thus, given our assumptions, equation (332) has a solution $\varphi(x)$. It remains to prove that this function in fact satisfies the original equation (331). We substitute $\varphi(x)$ in the original equation and form the difference

$$\omega(x) = f(x) - \int_0^x \frac{H(x, t)}{(x-t)^{1-a}}\,\varphi(t)\,\mathrm{d}t.$$

On multiplying both sides by $(z - x)^{-a}$, integrating with respect to x between the limits $0 \leqslant x \leqslant z$ and applying Dirichlet's formula [II, 79], we obtain by (332):

$$\int_0^z \frac{\omega(x)}{(z-x)^a}\,\mathrm{d}x = 0.$$

On multiplying both sides by $(u - z)^{a-1}$, integrating with respect to z from $z = 0$ to $z = u$ and changing the order of the integration, we have with any u:

$$\int_0^u \omega(x)\,\mathrm{d}x = 0,$$

whence it follows at once that $\omega(x) \equiv 0$.

Now let the function $K(x, t)$ featured in (329) depend only on the difference $(x - t)$, i.e. we consider an integral equation of the first kind:

$$\int_0^x K(x-t)\,\varphi(t)\,\mathrm{d}t = f(x). \tag{334}$$

We multiply both sides by e^{-sx} and integrate with respect to x from $x=0$ to $x=\infty$. On introducing the one-sided Laplace transforms of the given functions $f(x)$ and $K(x)$ and the required function $\varphi(x)$:

$$\Phi(s) = L_1(\varphi); \quad F(s) = L_1(f); \quad L(s) = L_1(K), \tag{335}$$

we obtain by the convolution theorem:

$$L(s)\,\Phi(s) = F(s). \tag{336}$$

Suppose that the kernel $K(x, t)$ satisfies the condition $K(x, x) \neq 0$ mentioned earlier, which in the present case becomes $K(0) \neq 0$. This guarantees the existence of a solution of equation (334). We can further suppose, as above, that $f(x)$ and $K(x)$ vanish for large positive x. Using the fact that $f(x)$ is arbitrary, we can say as in [46] that $L(s)$ does not vanish for values of s with sufficiently large real parts. Expression (336) gives us $\Phi(s)$, and the solution of (334) is obtained in the finite form by applying the inversion formula to the first of equations (335):

$$\varphi(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\,\Phi(s)\,ds. \tag{337}$$

The method indicated above is applicable also to equation (331) if $H(x, t)$ depends only on the difference $(x-t)$; it is easy to justify for this case the use of the Laplace transformation and of the convolution theorem if $0 < a < 1$.

48. Examples. 1. We take the equation

$$\varphi(x) = f(x) + \int_0^x (x-t)\,\varphi(t)\,dt. \tag{338}$$

Here, $K(x) = x$ and

$$L(s) = \int_0^\infty e^{-sx} x\,dx = \frac{1}{s^2},$$

where the real part of s is assumed positive. Formula (327) gives

$$M(s) = \frac{1}{s^2 - 1},$$

and, by (328), the resolvent is given by

$$R(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{sx}}{s^2-1}\,ds \qquad (x > 0), \tag{339}$$

where σ is any sufficiently large real number.

We consider the integral over the closed contour of the $s = \sigma + \tau i$ plane, consisting of the straight segment $\sigma = \sigma_0$, where $\sigma_0 > 1$, and the semicircle lying to the left of this straight line and having as its centre the intersection of the straight line with the real axis. On introducing the new variable of integration s_1 into integral (339) in accordance with the formula $s - \sigma_0 = is_1$, we obtain

a contour of integration on the s_1 plane consisting of a segment of the real axis and a semicircle with centre at the origin. On using Jordan's lemma [III$_2$ 60] and the fact that $x > 0$, the integral over the semicircle is seen to tend to zero as the radius tends to infinity, whence it follows at once that integral (339) with $\sigma > 1$ is equal to the sum of the residues of the integrand at the points $s = \pm 1$, i.e.

$$R(x) = \frac{1}{2}(e^x - e^{-x}),$$

and, by (325), the solution of equation (338) can be written as

$$\varphi(x) = f(x) + \frac{1}{2} e^x \int_0^x e^{-t} f(t)\, dt - \frac{1}{2} e^{-x} \int_0^x e^t f(t)\, dt.$$

2. In the case of the equation

$$\varphi(x) = f(x) + \int_0^x e^{x-t} \varphi(t)\, dt \tag{340}$$

we have $K(x) = e^x$, and consequently:

$$L(s) = \int_0^\infty e^{(1-s)x}\, dx = \frac{1}{s-1},$$

whence

$$M(s) = \frac{1}{s-2} \text{ and } R(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{sx}}{s-2}\, ds.$$

On applying the residue theorem as in the previous example, we obtain

$$R(x) = e^{2x},$$

and the solution of (340) becomes

$$\varphi(x) = f(x) + e^{2x} \int_0^x e^{-2t} f(t)\, dt.$$

3. We have already had the following formula, containing the Bessel function $J_0(x)$ [III$_2$, 53]:

$$\int_0^\infty e^{-kz} J_0(k\varrho)\, dk = \frac{1}{\sqrt{\varrho^2 + z^2}},$$

whence it follows that

$$\int_0^\infty e^{-sx} J_0(x)\, dx = \frac{1}{\sqrt{1 + s^2}}. \tag{341}$$

On making use of the asymptotic inequality for Bessel functions [III$_2$, 113], we can assert that (341) holds if the real part of s is positive.

Let us take the integral equation

$$\varphi(x) = f(x) + \lambda \int_0^x J_0(x-t)\,\varphi(t)\,\mathrm{d}t. \tag{342}$$

Here, $K(x) = \lambda J_0(x)$, and, by (341):

$$L(s) = \frac{1}{\sqrt{1+s^2}} \quad \text{and} \quad M(s) = \frac{\lambda}{\sqrt{1+s^2}-\lambda},$$

so that the resolvent is given by

$$R(x) = \frac{\lambda}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{e}^{sx}}{\sqrt{1+s^2}-\lambda}\,\mathrm{d}s$$

or

$$R(x) = \frac{\lambda}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\sqrt{1+s^2}-s}{1-\lambda^2+s^2}\,\mathrm{e}^{sx}\,\mathrm{d}s + \frac{\lambda}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{s+\lambda}{1-\lambda^2+s^2}\,\mathrm{e}^{sx}\,\mathrm{d}s.$$

The second of these last integrals can be evaluated by the residue theorem, as above. Let us transform the first integral. Along with (341), it can be shown that, for a positive integer n:

$$\int_0^\infty \mathrm{e}^{-ax} J_n(x)\,\mathrm{d}x = \frac{\left(\sqrt{1+a^2}-a\right)^n}{\sqrt{1+a^2}}$$

and integration of this last equation with respect to a from $a = s$ to $a = +\infty$ gives us

$$\int_0^\infty \mathrm{e}^{-sx} \frac{J_n(x)}{x}\,\mathrm{d}x = \frac{\left(\sqrt{1+s^2}-s\right)^n}{n}.$$

On the other hand, we obtain by using the residue theorem:

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{1-\lambda^2+s^2}\,\mathrm{e}^{sx}\,\mathrm{d}s = \frac{1}{\sqrt{1-\lambda^2}} \sin\left(\sqrt{1-\lambda^2}\,x\right).$$

We can therefore write

$$L_1\left(\frac{J_1(x)}{x}\right) = \sqrt{1+s^2}-s; \quad L_1\left(\frac{\sin\sqrt{1-\lambda^2}x}{\sqrt{1+\lambda^2}}\right) = \frac{1}{1-\lambda^2+s^2}$$

Application of the convolution theorem gives

$$L_1\left(\frac{\sin\left(\sqrt{1-\lambda^2}x\right)}{\sqrt{1-\lambda^2}} * \frac{J_1(x)}{x}\right) = \frac{\sqrt{1+s^2}-s}{1-\lambda^2+s^2}.$$

so that we obtain for the integral appearing in $R(x)$:

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\sqrt{1+s^2}-s}{1-\lambda^2+s^2}\,\mathrm{e}^{sx}\,\mathrm{d}s = \frac{1}{\sqrt{1-\lambda^2}} \int_0^x \sin\left[\sqrt{1-\lambda^2}\,(x-t)\right] \cdot \frac{J_1(t)}{t}\,\mathrm{d}t,$$

and the resolvent of equation (342) becomes

$$R(x) = \frac{\lambda}{\sqrt{1-\lambda^2}} \int_0^x \sin[\sqrt{1-\lambda^2}(x-t)] \cdot \frac{J_1(t)}{t} \, dt + $$
$$+ \lambda \cos(\sqrt{1-\lambda^2}x) + \frac{\lambda^2}{\sqrt{1-\lambda^2}} \sin(\sqrt{1-\lambda^2}x).$$

4. Let us take the equation of the first kind:

$$\int_0^x e^{x-t} \varphi(t) \, dt = x. \tag{343}$$

On applying a one-sided Laplace transformation to both sides, we obtain

$$\frac{\Phi(s)}{s-1} = \frac{1}{s^2}, \quad \text{i.e.} \quad \Phi(s) = \frac{s-1}{s^2},$$

and

$$\varphi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{s-1}{s^2} e^{sx} \, ds = 1 - x.$$

5. We take the equation

$$\int_0^x J_0(x-t) \varphi(t) \, dt = \sin x. \tag{344}$$

On taking into account that

$$L_1[J_0(x)] = \frac{1}{\sqrt{s^2+1}} \text{ and } L_1(\sin x) = \frac{1}{s^2+1}, \tag{345}$$

we obtain

$$\frac{1}{\sqrt{s^2+1}} \Phi(s) = \frac{1}{s^2+1}, \quad \text{i.e.} \quad \Phi(s) = \frac{1}{\sqrt{s^2+1}},$$

so that

$$\varphi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{sx}}{\sqrt{s^2+1}} \, ds,$$

or, on using the first of formulae (345):

$$\varphi(x) = J_0(x),$$

i.e. substitution of this solution in (344) gives us

$$\int_0^x J_0(x-t) J_0(t) \, dt = \sin x.$$

6. We take a kernel which becomes infinite at $t = x$:

$$\varphi(x) = f(x) + \lambda \int_0^x \frac{1}{(x-t)^\alpha} \varphi(t) \, dt \qquad (0 < \alpha < 1)$$

and construct the corresponding resolvent, though without justifying the application of the above method to this singular case.

We evaluate $L(s)$ and $M(s)$:

$$L(s) = \lambda \int_0^\infty \frac{e^{-sx}}{x^a}\,dx = \lambda\Gamma(1-a)\,s^{a-1},$$

$$M(s) = \frac{\lambda\Gamma(1-a)\,s^{a-1}}{1-\lambda\Gamma(1-a)\,s^{a-1}}$$

and obtain for the resolvent:

$$R(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx}\,\frac{\lambda\Gamma(1-a)\,s^{a-1}}{1-\lambda\Gamma(1-a)\,s^{a-1}}\,ds \qquad (x>0),$$

where σ is a sufficiently large positive number.

Series expansion gives us

$$\frac{\lambda\Gamma(1-a)\,s^{a-1}}{1-\lambda\Gamma(1-a)\,s^{a-1}} = \sum_{n=1}^{\infty} \lambda^n [\Gamma(1-a)]^n s^{n(a-1)}$$

and the problem reduces to evaluating the integral

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx} s^{n(a-1)}\,ds.$$

We substitute $sx = \tau$, suitably modify the contour and use (154) of [III_2, 74]; this gives us

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx} s^{n(a-1)}\,ds = \frac{x^{n(1-a)-1}}{\Gamma[n(1-a)]},$$

whence

$$R(x) = \sum_{n=1}^{\infty} \frac{[\lambda\Gamma(1-a)\,x^{1-a}]^n}{x\Gamma[n(1-a)]}.$$

49. Weighted integral equations. We have employed the usual concept of integral in our discussion of the theory of integral equations with continuous kernels. By starting from a different concept of integral, we can repeat the entire theory or at any rate part of it. We have already referred to the possibility of constructing a theory of integral equations on the basis of the Lebesgue integral. The essential point is that the integral concerned in constructing the theory should have all the properties already utilized above. We shall mention in this section a new concept of integral, on the basis of which the whole of the theory which we described at the start of this chapter can be retained. The results below are due to Kneser.†

† Kneser, *Rendiconti del Circolo Mat. di Palermo*. 38, 1914 and *Die Integral Gleichungen und ihre Anwendung in der mathem. Physik*, 1922, p. 117.

We shall confine ourselves to an elementary case. Let $f(x)$ be continuous in the finite interval $[a, b]$, let x_p $(p = 1, 2, \ldots, m)$ be fixed points of the interval, and a_p positive numbers. We define the integral of $f(x)$ over $[a, b]$ as the sum of the ordinary integral and the sum of the products of the values of $f(x)$ at the points $x = x_p$ with the numbers a_p. We shall distinguish this integral from the ordinary one by a stroke over the integral sign. The above definition gives the formula

$$\int_a^{\bar{b}} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \sum_{p=1}^m a_p f(x_p). \tag{346}$$

The following usual properties of the integral follow at once:

$$\int_a^{\bar{b}} [f_1(x) + f_2(x)]\,\mathrm{d}x = \int_a^{\bar{b}} f_1(x)\,\mathrm{d}x + \int_a^{\bar{b}} f_2(x)\,\mathrm{d}x;$$

$$\int_a^{\bar{b}} cf(x)\,\mathrm{d}x = c\int_a^{\bar{b}} f(x)\,\mathrm{d}x.$$

Further, we can change the order of successive integrations over the interval $[a, b]$, i.e.

$$\int_a^{\bar{b}} \Big[\int_a^{b} F(s,t)\,\mathrm{d}t\Big]\,\mathrm{d}s = \int_a^{\bar{b}} \Big[\int_a^{\bar{b}} F(s,t)\,\mathrm{d}s\Big]\,\mathrm{d}t.$$

For, direct use of definition (346) shows that both sides of this last equation are expressible as

$$\int_a^b\int_a^b F(s,t)\,\mathrm{d}s\,\mathrm{d}t + \sum_{p=1}^m \int_a^b [F(s,x_p) + F(x_p,s)]\,\mathrm{d}s + \sum_{p,q=1}^m F(x_p,x_q).$$

We have not so far used the fact that the coefficients a_p are positive. This becomes important for the next property. If $f(x) \geqslant 0$, integral (346) also has a non-negative value, and can only vanish when $f(x) \equiv 0$. Precisely the same property holds for an iterated integral. Hence follows the usual Buniakowski inequality for the new integral. If $|f(x)| \leqslant m$, a positive k exists such that

$$\Big|\int_a^{\bar{b}} f(x)\,\mathrm{d}x\Big| \leqslant km.$$

With $a_p > 0$, we can obviously take $k = (b - a) + a_1 + \ldots + a_p$.

It follows from the last property that, as usual [I, 145], a uniformly convergent series can be integrated term by term. Our new integral

can be used in a word-for-word repetition of the entire theory of integral equations with continuous kernels. If $K(s, t) = K(t, s)$, we obviously have

$$\int_a^{\bar{b}} K(s, t)\,\varphi(t)\,\mathrm{d}t = \int_a^{\bar{b}} K(t, s)\,\varphi(t)\,\mathrm{d}t.$$

The integral equation

$$\varphi(s) = f(s) + \lambda \int_a^{\bar{b}} K(s, t,)\,\varphi(t)\,\mathrm{d}t \tag{347}$$

is clearly equivalent to the following equation with the ordinary integral:

$$\varphi(s) = f(s) + \lambda \int_a^{b} K(s, t)\,\varphi(t)\,\mathrm{d}t + \lambda \sum_{p=1}^{m} a_p K(s, x_p)\,\varphi(x_p). \tag{348}$$

The eigenvalues and eigenfunctions are determined as usual from the homogeneous equation:

$$\varphi(s) = \lambda \int_a^{\bar{b}} K(s, t)\,\varphi(t)\,\mathrm{d}t.$$

The eigenvalues can be assumed orthogonal in the case of a symmetric kernel:

$$\int_a^{\bar{b}} \varphi_1(s)\,\varphi_2(s)\,\mathrm{d}s = 0,$$

or

$$\int_a^{b} \varphi_1(s)\,\varphi_2(s)\,\mathrm{d}s + \sum_{j=1}^{n} a_p \varphi_1(x_p)\,\varphi_2(x_p) = 0.$$

The Hilbert–Schmidt and Mercer theorems clearly remain in force. Equations of type (347) are called *weighted integral equations*.

Let us consider an example. We take the symmetric kernel $K(s, t)$, equal to s with $s \leqslant t$ and equal to t with $s \geqslant t$, $[0, 1]$ being taken as the basic interval. Suppose that $m = 1$ in (346) and that the one extra term on the right-hand side is taken with $x = 1$, i.e.

$$\int_a^{\bar{b}} f(x)\,\mathrm{d}x = \int_a^{b} f(x)\,\mathrm{d}x + af(1) \quad (a > 0).$$

The homogeneous equation

$$\varphi(s) = \lambda \int_0^{\bar{1}} K(s, t)\,\varphi(t)\,\mathrm{d}t$$

can be rewritten as

$$\varphi(s) = \lambda \int_0^s t\,\varphi(t)\,\mathrm{d}t + \lambda s \int_s^1 \varphi(t)\,\mathrm{d}t + \lambda s a\,\varphi(1). \tag{349}$$

Differentiation with respect to s gives

$$\varphi'(s) = \lambda \int_s^1 \varphi(t)\,\mathrm{d}t + \lambda a\,\varphi(1), \tag{350}$$

and further differentiation leads to

$$\varphi''(s) + \lambda\varphi(s) = 0. \tag{351}$$

The two boundary conditions: $\varphi(0) = 0$ and $\varphi'(1) = \lambda a \varphi(1)$ follow from (349) and (350). Conversely, the solution of (351) satisfying these conditions is easily seen to be a solution of integral equation (349). If $a = 0$, we get an ordinary integral equation, and the boundary conditions $\varphi(0) = \varphi'(1) = 0$ do not contain the parameter λ. On putting $\lambda = \mu^2$, we have by virtue of the first boundary condition: $\varphi(s) = C \sin \mu s$, whilst the second condition yields an equation for μ, viz. $\cos \mu = a\mu \sin \mu$.

Equations of a more general type have been discussed by Lichtenstein.†

Let B be a domain on the plane and l its contour. Lichtenstein considered equations of the form

$$\varphi(M) + \lambda \iint_B K_1(M, N)\,\varphi(N)\,\mathrm{d}\sigma_N +$$

$$+ \lambda \int_l K_2(M, N)\,\varphi(N)\,\mathrm{d}s_N + \lambda \sum_{k=1}^{m} K_3(M, P_k)\,\varphi(P_k) = f(M), \tag{352}$$

where P_k are fixed points belonging to the closed domain B. This equation can be written in the ordinary form if a new kernel and new differential are introduced: let M belong to the closed domain B and let N be different from the P_k. We put:

$$K(M, N) = \begin{cases} K_1(M, N) & \text{if } N \text{ is inside } B \\ K_2(M, N) & \text{if } N \text{ is on } l \end{cases} \qquad \mathrm{d}\omega_N = \begin{cases} \mathrm{d}\sigma_N & \text{in } B \\ \mathrm{d}s_N & \text{on } l, \end{cases}$$

and let

$$K(M, N) = K_3(M, P_k) \text{ and } \mathrm{d}\omega_N = 1,$$

if N coincides with P_k. Equation (352) now becomes

$$\varphi(M) + \lambda \int_{B+l} K(M, N)\,\varphi(N)\,\mathrm{d}\omega_N = f(M)$$

† *Studia Mathematica*, vol. III, 1931.

and the entire Fredholm theory can be repeated. We merely remark that the solution of the adjoint equation

$$\psi(M) + \lambda \int_{B+l} K(N, M)\,\psi(N)\,d\omega_N = f(M),$$

with continuous $f(M)$ will in general now have breaks in its continuity on reaching the contour l and at the points P_k. The same can be said of the solutions of the homogeneous adjoint equation.

The previous results also hold in three-dimensional space. Another metod of investigating weighted integral equations has been given by N. M. Günther.†

50. Integral equation of the first kind with Cauchy kernel. We now start an investigation of certain elementary integral equations in the one-dimensional case in which the integral is understood in the sense of the principal value [III_2; 26]. We shall make use here of the results previously obtained, relevant to the principal value of an integral of Cauchy's type [III_2; 26, 27, 28]. The foundations of the theory of such singular integral equations were laid in the works of Poincaré and Hilbert. The theory has since been substantially developed by Soviet mathematicians. A systematic treatment of the entire theory in the one-dimensional case may be found in N. I. Muskhelishvili's *Singular Integral Equations* (Singulyarnye integral'nye uravneniya), Moscow, 1946, and in N. P. Vekua's *Systems of Singular Integral Equations and Certain Boundary Value Problems* (Sistemy singulyarnykh integral'nykh uravnenii i nekotorye granichnye zadachi), Moscow, 1950. A work surveying a wide field is S. G. Mikhlin's *Singular Integral Equations* (Singulyarnye integral'nye uravneniya) (Uspekhi matemat. nauk, III, sec. 3 (25), 1948).

We shall assume in future, when speaking of a smooth contour, that its equations are: $x = x(s)$, $y = y(s)$, where s is the arc length and the functions $x(s)$, $y(s)$ have continuous derivatives up to the second order.

We start with an integral equation of the first kind with the Cauchy kernel:

$$\frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\tau - \xi}\,d\tau = f(\xi), \tag{353}$$

where L is a smooth closed contour, and $f(\xi)$ is a function given on L satisfying a Lipschitz condition.

† *Studia Mathematica*, vol. IV, 1932.

We shall assume that the required function $\omega(\tau)$ also satisfies the Lipschitz condition.

We obtained in [III$_2$; 28] the formula

$$\frac{1}{2\pi i}\int_L \frac{1}{\xi-\eta}\left[\frac{1}{2\pi i}\int_2 \frac{\omega(\tau)}{\tau-\xi}\,d\tau\right]d\xi = \frac{1}{4}\,\omega(\eta), \tag{354}$$

from which it follows at once that the function

$$\omega(\tau) = \frac{1}{\pi i}\int_L \frac{f(\xi)}{\xi-\tau}\,d\xi \tag{355}$$

satisfies equation (353). It is easily seen that the solution of this equation is unique. For, on multiplying both sides of (353) by $(1/\pi i)\cdot 1/(\xi-3)$, integrating with respect to ξ and taking (354) into account, we obtain (355). In brief, (354) and (355) are consequences of each other, by virtue of (354). We remark that it follows directly from (355) that $\omega(\tau)$ satisfies the Lipschitz condition provided this condition is satisfied by $f(\xi)$ [III$_2$; 27].

51. Boundary value problems for analytic functions. Before proceeding to the solution of integral eqations with Cauchy kernels, let us consider some boundary value problems for analytic functions. We first of all introduce a new concept and prove an auxiliary theorem.

Let a function $f(z)$ be regular in the neighbourhood of $z = \infty$. We say that *it is of finite order at infinity* if its expansion in the neighbourhood of $z = \infty$ has the form

$$f(z) = z^m\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right) \qquad (a_0 \neq 0), \tag{356}$$

and the integer m (positive, negative or zero) is called the *order of $f(z)$ at infinity*. If $m \leqslant 0$, $f(z)$ is regular at the point $z = \infty$, whilst if $n > 0$, $z = \infty$ is a pole of $f(z)$. When $m < 0$, we have $f(\infty) = 0$.

THEOREM. *If $f(z)$ is regular throughout the z plane and is of finite order at infinity, $f(z)$ is a polynomial.*

In the present case expansion (356) holds throughout the z plane and the expansion cannot contain negative powers of z because $z = 0$ must be a point at which $f(z)$ is regular. Hence, with $m > 0$ the function $f(z)$ will be a polynomial, whilst with $m = 0$ it is constant (a polynomial of zero degree). In particular, the constant can be zero. A function identically zero is also taken to be of finite order at infinity. Its order

is taken equal to zero, as in the case of a non-zero constant. The present theorem is in essence a generalization of Liouville's theorem [III_2, 9].

Let L be a smooth closed contour. We shall solve the following three boundary value problems.

PROBLEM 1. To find the function $\varphi^+(z)$, regular inside L, and the function $\varphi^-(z)$, regular outside L and of finite order at infinity, such that both functions are continuous up to L, and satisfy the relationship on L:

$$\varphi^+(\tau) - \varphi^-(\tau) = f(\tau) \qquad (\tau \text{ on } L), \tag{357}$$

where $f(\tau)$ is a complex function given on L, satisfying the Lipschitz condition.

The formula:

$$\varphi_0(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\,d\tau \tag{358}$$

defines the function $\varphi_0^+(z)$, regular inside L, and $\varphi_0^-(z)$, regular outside L and vanishing at infinity. By the formulae for the boundary values of a Cauchy type integral [III_2, 28]:

$$\left.\begin{aligned} \varphi_0^+(\tau) &= \frac{1}{2}f(\tau) + \frac{1}{2\pi i}\int_L \frac{f(\xi)}{\xi - \tau}\,d\xi, \\ \varphi_0^-(\tau) &= -\frac{1}{2}f(\tau) + \frac{1}{2\pi i}\int_L \frac{f(\xi)}{\xi - \tau}\,d\xi, \end{aligned}\right\} \tag{359}$$

we see that $\varphi_0^+(z)$ and $\varphi_0^-(z)$ satisfy (357), i.e. (358) gives a solution of Problem 1. It is easily seen that $\varphi_0^+(z)$ and $\varphi_0^-(z)$ also satisfy the Lipschitz condition [III_2, 27]. It is obvious that

$$\varphi(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\,d\tau + P(z), \tag{360}$$

where $P(z)$ is an arbitrary polynomial, also gives a solution of Problem 1, where $\varphi^+(\tau)$ and $\varphi^-(\tau)$ satisfy the Lipschitz condition.

We show that this formula gives all the solutions of Problem 1. Let $\varphi^+(z)$ and $\varphi^-(z)$ be any solution of the problem. It follows from (357) and the same relationship for $\varphi_0(z)$ that

$$\varphi^+(\tau) - \varphi_0^+(\tau) = \varphi^-(\tau) - \varphi_0^-(\tau) \qquad (\tau \text{ on } L),$$

i.e. the differences

$$\varphi^+(z) - \varphi_0^+(z); \quad \varphi^-(z) - \varphi_0^-(z)$$

have the same values on L, i.e. they define a function which is regular throughout the plane [III_2, 24] and is of finite order at infinity. By the theorem just proved, the differences must be equal to the same polynomial $P(z)$, whence (360) follows.

If we lay down the condition $\varphi^-(\infty) = 0$, we must put $P(z) = 0$ in (360). We now state a second problem, first considered by Hilbert; it will be assumed in future that $z = 0$ lies inside L.

PROBLEM 2 (Hilbert's homogeneous problem). *To find $\varphi^+(z)$ and $\varphi^-(z)$ under the previous conditions, such that the condition*

$$\varphi^+(\tau) = g(\tau)\,\varphi^-(\tau) \qquad (\tau \text{ on } L) \tag{361}$$

is fulfilled instead of (357), *where $g(\tau)$ is a complex function given on L which is non-zero and satisfies the Lipschitz condition.*

Let k be an integer equal to the increment of the amplitude of $g(\tau)$ when the point τ makes a circuit round L, divided by 2π:

$$k = \frac{1}{2\pi}[\arg g(\tau)]_L. \tag{362}$$

The amplitude of the function

$$g_0(\tau) = \tau^{-k} g(\tau) \tag{363}$$

does not receive an increment when τ makes a circuit of L, and $\log g_0(\tau)$ is a continuous function on L. We now assign some definite value to the logarithm.

It is easily shown that $\log g_0(\tau)$, like $g_0(r)$, satisfies the Lipschitz condition, and the proof will be omitted. We form the function:

$$\psi_0(z) = e^{\omega_0(z)}, \tag{364}$$

where

$$w_0(z) = \frac{1}{2\pi i}\int\limits_L \frac{\log g_0(\tau)}{\tau - z}\,d\tau. \tag{365}$$

These expressions define functions which differ in general for z lying inside and outside L:

$$\psi_0^+(z) = e^{\omega_0^+(z)}; \quad \psi_0^-(z) = e^{\omega_0^-(z)}, \tag{366}$$

and we can show directly, by using expression (359) for the boundary values of the Cauchy integral, that functions (366) satisfy the relationship on L:

$$\psi_0^+(\tau) = g_0(\tau)\,\psi_0^-(\tau). \tag{367}$$

We introduce the new functions, regular inside and outside L:

$$\varphi_0^+(z) = \psi_0^+(z); \quad \varphi_0^-(z) = z^{-k}\psi_0^-(z). \tag{368}$$

It will be seen by using (363) and (367) that $\varphi_0^+(z)$ and $\varphi_0^-(z)$ are a solution of the homogeneous Hilbert problem. It follows from (365) and (366) that $\omega_0(\infty) = 0$ and $\psi^0(\infty) = 1$, and we can assert, by (368), that the order of $\varphi_0^-(z)$ at infinity is $(-k)$.

We remark further that $\varphi_0^+(z)$ does not vanish anywhere, whilst $\varphi^-(z)$ can only vanish with $z = \infty$. If $P(z)$ is any polynomial, the functions

$$\psi^+(z) = P(z)\,\varphi_0^+(z); \quad \psi^-(z) = P(z)\,\varphi_0^-(z) \tag{369}$$

are also a solution of the homogeneous Hilbert problem. If m is the degree of $P(z)$, the order of φ_0^- at infinity is equal to $(m - k)$. In this case, $\varphi^+(z)$ and $\varphi^-(z)$ satisfy the Lipschitz condition, as in Problem 1.

We show further that expressions (369) yield all the solutions of the problem. For, let $\varphi^+(z)$ and $\varphi^-(z)$ be any solution. The ratios

$$\frac{\varphi^+(z)}{\varphi_0^+(z)}, \quad \frac{\varphi^-(z)}{\varphi_0^-(z)} \tag{370}$$

are regular inside and outside L respectively, and the second is of finite order at infinity. In addition, these ratios coincide on L. Consequently, the ratios define a function which is regular throughout the plane and has a finite order at infinity, so that the function is a polynomial by the above theorem, and expressions (369) now follow for $\varphi^+(z)$ and $\varphi^-(z)$.

Problem 3 (Hilbert's non-homogeneous problem). *To find $\varphi^+(z)$ and $\varphi^-(z)$ under the above conditions such that the condition*

$$\varphi^+(\tau) = g(\tau)\,\varphi^-(\tau) + f(\tau) \qquad (\tau \text{ on } L) \tag{371}$$

is satisfied instead of (361), *where $g(\tau)$ and $f(\tau)$ are given functions on L satisfying the Lipschitz condition, and $g(\tau) \neq 0$.*

Let $\varphi_0^+(z)$ and $\varphi_0^-(z)$ be the non-zero solution constructed above of Problem 2. It follows from (361) that $g(\tau) = \varphi_0^+(\tau) : \varphi_0^-(\tau)$, and by substituting in (371), we can rewrite this condition as

$$\frac{\varphi^+(\tau)}{\varphi_0^+(\tau)} - \frac{\varphi^-(\tau)}{\varphi_0^-(\tau)} = \frac{f(\tau)}{\varphi_0^+(\tau)}, \tag{372}$$

i.e. we have arrived at the first problem for the ratio $\varphi(z)/\varphi_0(z)$, whence, by (361):

$$\frac{\varphi(z)}{\varphi_0(z)} = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\varphi_0^+(\tau)(\tau - z)}\,d\tau + P(z) \tag{373}$$

where $P(z)$ is any polynomial, and finally:

$$\varphi(z) = \frac{\varphi_0(z)}{2\pi i}\int_L \frac{f(\tau)}{\varphi_0^+(\tau)(\tau - z)}\,d\tau + P(z)\,\varphi_0(z). \tag{374}$$

We have to take $\varphi_0^+(z)$ instead of $\varphi_0(z)$ for domains lying inside L, and $\varphi_0^-(z)$ for outside domains. Expression (374) gives the general solution of Problem 3. The functions $\varphi_0^+(z)$ and $\varphi_0^-(z)$ are given by (368), and the number k by (362). The first term on the right-hand side is of order $(-k-1)$ at infinity, and the second of order $(m-k)$, where m is the degree of polynomial $P(z)$. As in the previous problems, $\varphi^+(\tau)$ and $\varphi^-(\tau)$ satisfy the Lipschitz condition.

We now consider the question of the solutions of Problem 3 that vanish at infinity. In other words, we seek the solutions of negative order at infinity. We take the cases: $k > 0$, $k = 0$ and $k < 0$. If $k > 0$, the first term of the right-hand side of (374) is of negative order at infinity, whilst the second will be of negative order when and only when $m < k$, i.e. in the case $k > 0$, (374) gives the general form of the solutions of Problem 3 that vanish at infinity if $P(z)$ is taken as any polynomial of degree less than k. In this case we have an infinite set of solutions of Problem 3 that vanish at infinity. The general solution contains k arbitrary constants (the coefficients of $P(z)$).

If $k = 0$, the first term is of negative order at infinity as before, whilst we have to take $P(z) \equiv 0$ in the second term. The solution of Problem 3 is now obviously unique. If $k > 0$, it is easily seen, in view of what has been said regarding the orders of the terms on the right-hand side of (374) that we have to take $P(z) \equiv 0$, and in addition, terms containing $z^{-k-1}, z^{-k-2}, \ldots, z^0$ must be absent in the first term, i.e. there must be no terms containing $z^{-1}, z^{-2}, \ldots, z^{-k}$ in the expansion of the integral

$$\frac{1}{2\pi i}\int\limits_L \frac{f(\tau)}{\varphi_0^+(\tau)(\tau-z)}\,d\tau = -\frac{z^{-1}}{2\pi i}\int\limits_L \frac{f(\tau)}{\varphi_0^+(\tau)}\,d\tau - \frac{z^{-2}}{2\pi i}\int\limits_L \frac{\tau f(\tau)}{\varphi_0^+(\tau)}\,d\tau + \ldots,$$

which holds for sufficient large $|z|$. This leads to the following necessary and sufficient condition for Problem 3 to have a solution vanishing at infinity:

$$\int\limits_L \frac{\tau^s f(\tau)}{\varphi_0^+(\tau)}\,d\tau = 0 \qquad (s = 0, 1, \ldots, k-1). \tag{375}$$

When this condition is fulfilled, the solution vanishing at infinity is unique and is given by (374) with $P(z) \equiv 0$.

52. Integral equations of the second kind with Cauchy kernels. We consider the equation

$$A(\xi)\varphi(\xi) + \frac{B(\xi)}{\pi i}\int\limits_L \frac{\varphi(\tau)}{\tau-\xi}\,d\tau = f(\xi), \tag{376}$$

where $A(\xi)$, $B(\xi)$ and $f(\xi)$ are given functions on L satisfying the Lipschitz condition, and we assume that

$$A(\xi) + B(\xi) \neq 0 \quad \text{and} \quad A(\xi) - B(\xi) \neq 0 \qquad (\xi \text{ on } L). \tag{377}$$

The solution $\varphi(\xi)$ will be sought among functions satisfying the Lipschitz condition. We introduce the function

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} \, dt. \tag{378}$$

It follows from (359), with the previous notation, that

$$\varphi(\xi) = \Phi^+(\xi) - \Phi^-(\xi), \tag{379}$$

$$\frac{1}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - \xi} d\tau = \Phi^+(\xi) + \Phi^-(\xi). \tag{380}$$

Substitution in (376) gives us:

$$[A(\xi) + B(\xi)]\,\Phi^+(\xi) - [A(\xi) - B(\xi)]\,\Phi^-(\xi) = f(\xi) \tag{381}$$

or

$$\Phi^+(\xi) = \frac{A(\xi) - B(\xi)}{A(\xi) + B(\xi)} \Phi^-(\xi) + \frac{f(\xi)}{A(\xi) + B(\xi)}, \tag{382}$$

i.e. $\Phi(z)$ must be a solution of Problem 3 that vanishes at $z = \infty$ and satisfies condition (375). Conversely, suppose that such a $\Phi(z)$ exists. On defining $\varphi(\xi)$ in accordance with (379), we have (378) [51] for $\Phi(z)$, from which (380) follows. After defining $\Phi^+(\xi)$ and $\Phi^-(\xi)$ from (379) and (380) and substituting in (382), we get (376). The solution of equation (376) is therefore equivalent to the solution of Problem 3 with boundary condition (382). The function $\varphi(\xi)$ is now given by (379). It remains to turn back to the results of [51], in order to obtain the complete solution of the problem. In accordance with (362), we bring in the integer

$$k = \frac{1}{2\pi} \left[\arg \frac{A(\xi) - B(\xi)}{A(\xi) + B(\xi)}\right]_L, \tag{383}$$

which is called the index of equation (381).

Let $\Phi_0(z)$ be the non-zero solution of Problem 2, under the condition

$$\Phi^+(\xi) = \frac{A(\xi) - B(\xi)}{A(\xi) + B(\xi)} \Phi^-(\xi),$$

which we obtained in [51]. We take the three cases:

(1) $k > 0$. We now have:

$$\Phi(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_L \frac{f(\tau)}{[A(\tau) + B(\tau)]\,\Phi_0^+(\tau)(\tau - z)} \mathrm{d}\tau + P_{k-1}(z)\,\Phi_0(z), \quad (384)$$

where $P_{k-1}(z)$ is an arbitrary polynomial of degree $(k - 1)$.

(2) $k = 0$. The solution is obtained from (384) with $P_{k-1}(z) \equiv 0$, i.e.

$$\Phi(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_L \frac{f(\tau)}{[A(\tau) + B(\tau)]\,\Phi_0^+(\tau)(\tau - z)} \mathrm{d}\tau. \quad (385)$$

(3) $k < 0$. The necessary and sufficient condition for Problem 3 to be soluble is that

$$\int\limits_L \frac{\tau^s f(\tau)}{[A(\tau) + B(\tau)]\,\Phi_0^+(\tau)} \mathrm{d}\tau = 0 \quad (s = 0, 1, \ldots, k-1), \quad (386)$$

and if this condition is satisfied, the solution is given by (385).

By using (374) and (379), we can now obtain the solution $\varphi(\xi)$ of equation (376). Here we have to make use of the formula for the jump of a Cauchy type integral. We have, with $k \geqslant 0$:

$$\varphi(\xi) = \frac{\Phi_0^+(\xi) + \Phi_0^-(\xi)}{2[A(\xi) + B(\xi)]\Phi_0^+(\xi)} f(\xi) + \left[\Phi_0^+(\xi) - \Phi_0^-(\xi)\right] P_{k-1}(\xi) + \\ + \frac{\Phi_0^+(\xi) - \Phi_0^-(\xi)}{2\pi i} \int\limits_L \frac{f(\tau)}{[A(\tau) + B(\tau)]\,\Phi_0^+(\tau)(\tau - \xi)} \mathrm{d}t, \quad (387)$$

where $P_{k-1}(z) \equiv 0$ with $k = 0$. When $k < 0$, if condition (386) is fulfilled, we get the same result, though with only $P_{k-1}(\xi) \equiv 0$.

It follows at once from this that the homogeneous equation

$$A(\xi)\,\varphi(\xi) + \frac{B(\xi)}{\pi i} \int\limits_L \frac{\varphi(\tau)}{\tau - \xi} \mathrm{d}\tau = 0 \quad (388)$$

has the general solution, with $k > 0$:

$$\varphi(\xi) = [\Phi_0^+(\xi) - \Phi_0^-(\xi)]\, P_{k-1}(\xi), \quad (389)$$

whilst with $k \leqslant 0$ equation (388) has only a zero solution. Expression (389) yields k linearly independent solutions of (388):

$$\varphi(\xi) = \left[\Phi_0^+(\xi) - \Phi_0^-(\xi)\right] \xi^s \quad (s = 0, 1, 2, \ldots, k-1). \quad (390)$$

Thus, with $k > 0$ non-homogeneous equation (376) is soluble for any $f(\xi)$, and homogeneous equation (388) has k linearly independent solutions. With $k = 0$, equation (376) is soluble for any $f(\xi)$ and has a unique solution, whilst homogeneous equation (388) has only a zero

solution. With $k < 0$ we have $(-k)$ conditions (386) for (376) to be soluble and if these are fulfilled (376) has a unique solution. The homogeneous equation has in this case only a zero solution. The results obtained here are different from those that we had when solving ordinary Fredholm equations.

We remark that the equation of the first kind:

$$\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\varphi - \xi}\,d\tau = f(\xi)$$

is obtained as a particular case of equation (376) with $A(\xi) = 0$ and $B(\xi) = 1/2$. In this particular case $k = 0$.

53. Boundary value problems for the case of a segment. We now consider the problems of [51] for the case when we have a segment $[a, b]$ of the real axis instead of a closed contour L. In future we shall always denote by $\Phi(z)$ a function which is regular outside $[a, b]$, is of finite order at infinity, is continuous up to $[a, b]$ from above and below, excepting possibly at the ends, and satisfies close to the ends the condition

$$|\Phi(z)| \leqslant \frac{A}{|z-c|^\alpha}, \tag{391}$$

where A and α are constants, $0 \leqslant \alpha < 1$ and c is one of the ends, i.e. $c = a$ or $c = b$. We shall write $\Phi^+(\xi)$ and $\Phi^-(\xi)$ for the boundary values of $\Phi(z)$ from above and below on $[a, b]$.

PROBLEM 1. *To find $\Phi(z)$ such that, with $a < \xi < b$, where $f(\xi)$ is a given function satisfying the Lipschitz condition on the closed segment $[a, b]$.*

$$\Phi^+(\xi) - \Phi^-(\xi) = f(\xi).$$

As in [51], the expression

$$\Phi(z) = \frac{1}{2\pi i}\int_a^b \frac{f(\tau)}{\tau - z}\,d\tau + P(z), \tag{392}$$

where $P(z)$ is an arbitrary polynomial, gives a solution of the problem. Condition (391) is immediately seen to be satisfied from the fact that, close to the ends, $\varphi(z)$ has the form [III_2, 27]:

$$\Phi(z) = \pm\frac{f(c)}{2\pi i}\log\frac{1}{z-c} + F(z),$$

where $F(z)$ has a finite limit as $z \to c$. It can be shown that (392) gives all the solutions of the problem. The proof is as follows. Let $\Phi_1(z)$ and $\Phi_2(z)$ be two solutions of the problem. It is sufficient to show that the difference $\omega(z) = \Phi_2(z) - \Phi_1(z)$ is a polynomial. As in [51], this

difference is regular throughout the plane, excepting possibly at $z = a$ and $z = b$, and has a finite order at infinity. It remains to show that $\omega(z)$ is regular at $z = a$ and $z = b$ also.

We use the fact that $\omega(z)$ satisfies inequality (391) close to $z = c$. It is easily shown that, given this fact, $\omega(z)$ is also regular at $z = c$. To see this, we only need to repeat the proof of the theorem of [III_2, 10]: if $f(z)$ is regular and single-valued in the neighbourhood of $z = a$ and bounded in modulus, it is also regular at $z = a$ itself. Here, the condition $| f(z) | \leqslant N$ can be replaced without detriment to the proof by condition (391), i.e. $| f(z) | \leqslant C/\varrho^{a}$ $(0 \leqslant a < 1)$.

The solution of Problem 1 satisfying the condition $\Phi(\infty) = 0$, is obtained from (392) with $P(z) = 0$.

We shall consider the following problem in the particular case when $g(\xi) = -1$.

Problem 2. *To find the $\Phi(z)$ such that, with $a < \xi < b$:*

$$\Phi^{+}(\xi) + \Phi^{-}(\xi) = 0. \tag{393}$$

On taking into account the fact that $\sqrt{z-c}$ changes sign when z makes a circuit round c, we can write the following solution of Problem 2:

$$\Phi_0(z) = \frac{1}{\sqrt{(z-a)(z-b)}}, \tag{394}$$

where the value of the radical is fixed in any manner. This solution differs from zero throughout the finite plane and $\Phi(\infty) = 0$.

A further solution will be

$$\Phi(z) = \frac{P(z)}{\sqrt{(z-a)(z-b)}}, \tag{395}$$

where $P(z)$ is an arbitrary polynomial. This formula gives all the solutions. For, if $\Phi(z)$ is any solution of the problem, it is readily shown as in Problem 1 that the ratio $\Phi(z) : \Phi_0(z)$ is a polynomial, whence (395) follows.

Problem 3. *To find $\Phi(z)$ such that, with $a < \xi < b$:*

$$\Phi^{+}(\xi) + \Phi^{-}(\xi) = f(\xi), \tag{396}$$

where $f(\xi)$ is a given function satisfying the Lipschitz condition on the closed segment $[a, b]$.

Since $\Phi_0(z)$ satisfies condition (393), we can rewrite (396) as

$$\frac{\Phi^{+}(\xi)}{\Phi_0^{+}(\xi)} - \frac{\Phi^{-}(\xi)}{\Phi_0^{-}(\xi)} = \frac{f(\xi)}{\Phi_0^{+}(\xi)}, \tag{397}$$

i.e. we have Problem 1 for the function $\Phi(z) : \Phi_0(z)$.

We defined the value of the radical in (394) so that e.g. the expansion of $\Phi_0(z)$ in the neighbourhood of $z = \infty$ starts with z^{-1}. The radical $\sqrt{(\xi - a)(\xi - b)}$ will now be pure imaginary with a positive factor for i on the upper side of the cut $[a, b]$. On taking this value for the radical, we can rewrite condition (397) in the form

$$\frac{\Phi^+(\xi)}{\Phi_0^+(\xi)} - \frac{\Phi^-(\xi)}{\Phi_0^-(\xi)} = f(\xi)\sqrt{(\xi - a)(\xi - b)}. \tag{398}$$

We show that the function

$$\sqrt{(\xi - a)(b - \xi)} \tag{399}$$

satisfies the Lipschitz condition with index half on the closed segment $[a, b]$. We make use here of the obvious inequality

$$\sqrt{\alpha + \beta} - \sqrt{\alpha} \leqslant \sqrt{|\beta|} \text{ for } \alpha \geqslant 0, \qquad \alpha + \beta \geqslant 0. \tag{400}$$

Let ξ and η belong to $[a, b]$. On putting

$$\alpha = (\eta - a)(b - \eta); \quad \alpha + \beta = (\xi - a)(b - \xi),$$

we obtain from (400):

$$\begin{aligned}\sqrt{(\xi - a)(b - \xi)} - \sqrt{(\eta - a)(b - \eta)} &\leqslant \sqrt{|(a + b)\xi - \xi^2 - (a + b)\eta + \eta^2|} \\ &= \sqrt{|[(a + b) - (\xi + \eta)](\xi - \eta)|},\end{aligned}$$

whence, on observing that $\xi + \eta \geqslant 2a$, we have:

$$\sqrt{(\xi - a)(b - \xi)} - \sqrt{(\eta - a)(b - \eta)} \leqslant \sqrt{b - a}\sqrt{|\eta - \xi|}.$$

We could have shown in exactly the same way that

$$\sqrt{(\eta - a)(b - \eta)} - \sqrt{(\xi - a)(b - \xi)} \leqslant \sqrt{b - a}\sqrt{|\eta - \xi|},$$

i.e.

$$\left|\sqrt{(\eta - a)(b - \eta)} - \sqrt{(\xi - a)(b - \xi)}\right| \leqslant \sqrt{b - a}\sqrt{|\eta - \xi|},$$

whence it follows that function (399) in fact satisfies the Lipschitz condition with index $\alpha = 1/2$ on $[a, b]$. Consequently, the whole of the right-hand side of (398) also satisfies the Lipschitz condition $[\mathrm{III}_2, 27]$. On solving Problem 1 for $\Phi(z) : \Phi_0(z)$ with the boundary condition (398), we obtain

$$\Phi(z) = \frac{1}{2\pi i\sqrt{(z - a)(b - z)}} \int_a^b \frac{f(\tau)\sqrt{(\tau - a)(b - \tau)}}{\tau - z}\, d\tau + \frac{P(z)}{\sqrt{(z - a)(b - z)}}, \tag{401}$$

where, as usual, $P(z)$ is an arbitrary polynomial. On using the inequality for the Cauchy integral close to the ends of the segment, it is easily shown that function (401) satisfies condition (391). If we wanted to obtain a solution satisfying the condition $\Phi(\infty) = 0$, we should have to put $P(z)$ constant in (401):

$$\Phi(z) = \frac{1}{2\pi i \sqrt{(z-a)(z-b)}} \int_a^b \frac{f(\tau)\sqrt{(\tau-a)(\tau-b)}}{\tau - z} \mathrm{d}\tau + \\ + \frac{C}{\sqrt{(z-a)(z-b)}} . \tag{402}$$

A supplementary condition can be laid down in the solution of Problem 3, that $\Phi(z)$ be bounded in the neighbourhood of the ends of the segment. Now, we should have to take, instead of the solution (394) of Problem 2:

$$\Phi_0(z) = \sqrt{(z-a)(z-b)} , \tag{403}$$

and we obtain, instead of (401):

$$\Phi(z) = \frac{\sqrt{(z-a)(z-b)}}{2\pi i} \int_a^b \frac{f(\tau)}{\sqrt{(\tau-a)(\tau-b)}(\tau-z)} \mathrm{d}\tau + \\ + P(z)\sqrt{(z-a)(z-b)}. \tag{404}$$

To obtain a solution satisfying the condition $\Phi(\infty) = 0$, we have to put $P(z) \equiv 0$, and in addition, the condition must be fulfilled:

$$\int_a^b \frac{f(\tau)}{\sqrt{(\tau-a)(\tau-b)}} \mathrm{d}\tau = 0. \tag{405}$$

If we demand that the solution be bounded only at the end $z = a$, we must take instead of (403):

$$\varphi(z) = \sqrt{\frac{z-a}{z-b}},$$

and we obtain, instead of (404):

$$\Phi(z) = \frac{1}{2\pi i}\sqrt{\frac{z-a}{z-b}} \int_a^b \sqrt{\frac{\tau-b}{\tau-a}} + \frac{f(\tau)}{\tau-z} \mathrm{d}\tau + \\ + P(z)\sqrt{\frac{z-a}{z-b}} . \tag{406}$$

In this present case, given any $f(\xi)$ there is a unique solution satisfying the condition $\Phi(\infty) = 0$:

$$\Phi(z) = \frac{1}{2\pi i} \sqrt{\frac{z-a}{z-b}} \int_a^b \sqrt{\frac{\tau-b}{\tau-a}} \frac{f(\tau)}{\tau-z} \mathrm{d}\tau. \tag{407}$$

We shall not dwell on the proof of (404) and (406). It may be found in the work already cited by N. I. Muskhelishvili, which we have in fact followed in these last sections.

54. Inversion of a Cauchy type integral. We now consider the problem of inverting the integral

$$\frac{1}{\pi i} \int_a^b \frac{\varphi(\tau)}{\tau-\xi} \mathrm{d}\tau = f(\xi) \qquad (a < \xi < b). \tag{408}$$

We proceed as in [51]. After introducing the function

$$\Phi(z) = \frac{1}{2\pi i} \int_a^b \frac{\varphi(\tau)}{\tau-z} \mathrm{d}\tau,$$

satisfying the condition $\Phi(\infty) = 0$, we obtain:

$$\varphi(\xi) = \Phi^+(\xi) - \Phi^-(\xi), \tag{409}$$

$$\Phi^+(\xi) + \Phi^-(\xi) = \frac{1}{\pi i} \int_a^b \frac{\varphi(\tau)}{\tau-\xi} \mathrm{d}\tau. \tag{410}$$

Equation (408) is therefore equivalent to the following:

$$\Phi^+(\xi) + \Phi^-(\xi) = f(\xi) \qquad (a < \xi < b). \tag{411}$$

This is Problem 3 of [53] with the supplementary condition $\Phi(\infty) = 0$. Having constructed the solution of this problem, we obtain $\varphi(\xi)$ in accordance with (409). After using (402) and the formula for the boundary values of a Cauchy type integral, we finally obtain

$$\varphi(\xi) = \frac{1}{\pi i \sqrt{(\xi-a)(\xi-b)}} \int_a^b \frac{f(\tau)\sqrt{(\tau-a)(\tau-b)}}{\tau-\xi} \mathrm{d}\tau + \frac{C}{\sqrt{(\xi-a)(\xi-b)}} \cdot \tag{412}$$

We remark that this function satisfies the Lipschitz condition only on any closed interval lying inside $[a, b]$, and not on $[a, b]$ itself, and it may increase indefinitely as ξ approaches either a or b. If condition

(405) is fulfilled, a solution of (408) may be obtained which is bounded at both ends:

$$\varphi(\xi) = \sqrt{(\xi - a)(\xi - b)} \int_a^b \frac{f(\tau)}{\sqrt{(\tau - a)(\tau - b)}(\tau - \xi)} d\tau. \qquad (413)$$

A detailed treatment of this inversion problem may be found in Muskhelishvili's work.

55. Fourier's integral equation. We turn now to a study of certain other singular integral equations for which the earlier fundamental theorems on Fredholm equations may prove to be invalid. These are equations with an infinite domain of integration (or interval of integration, in the one-dimensional case).

The construction of a more general theory of integral equations embracing examples of such singular equations requires a departure from the class of continuous functions and the employment of a more general concept of integral. We shall do this in Vol. V. We confine ourselves here to brief indications of the practical side of the subject, and leave the precise statement of the final theorems to Vol. V.

We start with the Fourier transformation for the case of an even function, and we shall be chiefly concerned in what follows with a kernel depending on a difference.

We recall Fourier's formula, proved earlier in [II; 160]. If $f(s)$ is continuous and absolutely integrable in the interval $0 \leqslant s < \infty$, and has only a finite number of intervals of increase and decrease in any finite part of this interval, by constructing the function

$$f_1(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt,$$

we can express $f(s)$ in terms of $f_1(s)$ by the formula:

$$f(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_1(t) \cos st \, dt.$$

On adding the last two formulae, we obtain

$$f(s) + f_1(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty [f(t) + f_1(t)] \cos st \, dt,$$

i.e. whatever the choice of function $f(s)$ satisfying the above-mentioned conditions. the function $\varphi(s) = f(s) + f_1(s)$ is an eigenfunction of the integral equation

$$\varphi(s) = \lambda \int_0^\infty \varphi(t) \cos st \, dt, \qquad (414)$$

corresponding to the eigenvalue $\lambda = \sqrt{2} : \sqrt{\pi}$.

By using the fact that $f(s)$ is arbitrary, it can be shown that, given the value of λ indicated, homogeneous equation (414) has an infinite set of linearly independent eigenfunctions. This special feature is due to the fact that the interval of integration is infinite.

Let us prove this. If we put $f(s) = e^{-ps}$ $(p > 0)$, it is easily shown that $f_1(s) = \sqrt{2}\, p \sqrt{\pi}\, (s^2 + p^2)$. With the given λ, we get a solution of equation (414) containing an arbitrary parameter p, which can take any positive values. The integral equation with kernel $\sin st$ can be considered in precisely the same way in the interval $[0, \infty]$.

56. Equations in the case of an infinite interval. By using the two-sided Laplace transformation or the Fourier transformation, we can apply the method of [46] to the case of a Fredholm equation of the form

$$\varphi(x) = f(x) + \int_{-\infty}^{+\infty} K(x-t)\, \varphi(t)\, dt. \tag{415}$$

All the previous formulae remain valid here, and all we have to do is replace the one-sided by the two-sided Laplace transformation everywhere. In addition, we have to offer a justification of the transformations used. Apart from the conditions imposed on $f(x)$ and $K(x)$, we have to indicate the class of functions in which the solution $\varphi(x)$ of equation (415) is sought.

We shall assume that $f(x)$ and $K(x)$ are continuous and that the integral exists:

$$\int_{-\infty}^{+\infty} |K(x)|\, dx = A. \tag{416}$$

We show that equation (415) can have only one bounded solution if the constant A satisfies the condition $A < 1$. If two bounded solutions were to exist, the homogeneous equation

$$\psi(x) = \int_{-\infty}^{+\infty} K(x-t)\, \psi(t)\, dt \tag{417}$$

would have to have a bounded solution different from zero. This leads to a contradiction. For, let δ be the strict upper bound of $|\psi(x)|$ in $-\infty < x < +\infty$. Whatever the small positive ε, an x exists such that $|\psi(x)| > \delta - \varepsilon$. Substitution of such an x in equation (417) gives

$$\delta - \varepsilon < \int_{-\infty}^{+\infty} |K(x-t)|\, |\psi(t)|\, dt \leqslant \delta \int_{-\infty}^{+\infty} |K(x)|\, dx = \delta A,$$

i.e. $\delta - \varepsilon < \delta A$. This contradicts the fact that $A < 1$ and that ε can be taken as small as desired. If $f(x)$ is a bounded function, i.e. $|f(x)| \leqslant M$, it can be shown, on the assumption that $A < 1$, that a bounded solution exists, by using the usual method of successive approximations. If we introduce a parameter into the equation:

$$\varphi(x) = f(x) + \lambda \int_{-\infty}^{+\infty} K(x-t)\, \varphi(t)\, dt,$$

the above condition ($A < 1$) becomes

$$|\lambda| < A^{-1}. \tag{418}$$

In addition to the bounded solution, there can clearly exist continuous solutions $\varphi(x)$ which are unbounded in $-\infty < x < +\infty$. Instead of the condition that the solution be bounded, the condition is sometimes laid down that the integral of $|\varphi(x)|$ or of $|\varphi(x)|^2$ exists over the infinite interval $-\infty < x < +\infty$. The condition for the existence and uniqueness of the solution in this sense, using the Lebesgue integral, may be found e.g. in Titchmarsh, *Introduction to the Theory of Fourier Integrals* (p. 304).

We now take the homogeneous equation:

$$\varphi(x) = \int_{-\infty}^{+\infty} K(x-t)\,\varphi(t)\,\mathrm{d}t \tag{419}$$

and seek its solution in the form

$$\varphi(x) = \mathrm{e}^{ax}, \tag{420}$$

where a is a constant. We substitute in (419):

$$\mathrm{e}^{ax} = \int_{-\infty}^{+\infty} K(x-t)\,\mathrm{e}^{at}\,\mathrm{d}t.$$

We replace t by a new variable of integration: $t_1 = x - t$. On cancelling by e^{ax} and replacing the letter t_1 by the letter t, we obtain an equation for a:

$$\int_{-\infty}^{+\infty} K(t)\,\mathrm{e}^{-at}\,\mathrm{d}t = 1. \tag{421}$$

If this equation has a root of multiplicity r, differentiation of the equation with respect to a gives:

$$\int_{-\infty}^{+\infty} K(t)\,\mathrm{e}^{-at}\,t^k\,\mathrm{d}t = 0 \quad (k = 1, 2, \ldots, r-1). \tag{422}$$

It follows from (421) and (422), with the aid of the previous change of variables, that the functions

$$\mathrm{e}^{ax},\ x\mathrm{e}^{ax},\ \ldots,\ x^{r-1}\mathrm{e}^{ax} \tag{423}$$

as well as function (420) are solutions of equation (419). It is obviously assumed here that a is such that the integral on the right-hand side of (419) has a meaning. It can be shown that these functions exhaust all the solutions of equation (419) belonging to the given class of functions. A strict statement of this result may be found in Titchmarsh's work (p. 305).

It may be mentioned that the solution e^{ax} will be bounded in the interval $-\infty < x < +\infty$ when and only when a is pure imaginary.

57. Examples. Let us elucidate the above with the aid of some examples.

1. Suppose that, in equation (415),

$$f(x) = \mathrm{e}^{-|x|}; \quad K(x) = \begin{cases} \lambda \mathrm{e}^{x} & \text{for } x \leqslant 0 \\ 0 & \text{,, } x > 0, \end{cases} \tag{424}$$

i.e. the equation has the form

$$\varphi(x) = e^{-|x|} + \lambda e^{x} \int_{x}^{\infty} e^{-t} \varphi(t)\, dt. \tag{425}$$

We find the Laplace transforms for functions (424):

$$F(s) = \int_{-\infty}^{+\infty} e^{-sx} e^{-|x|}\, dx = \int_{-\infty}^{0} e^{(1-s)x}\, dx + \int_{0}^{+\infty} e^{-(1+s)x}\, dx$$

$$= \frac{1}{1-s} + \frac{1}{1+s} = \frac{1}{1-s^2},$$

$$L(s) = \lambda \int_{-\infty}^{0} e^{(1-s)x}\, dx = \frac{\lambda}{1-s}.$$

We have assumed when determining $F(s)$ that the complex variable $s = \sigma + \tau i$ belongs to the strip $-1 < \sigma < 1$, and the inequality $\sigma < 1$ was required when determining $L(s)$. In accordance with (323):

$$\Phi(s) = \frac{2}{(1+s)(1-\lambda-s)},$$

so that

$$\varphi(x) = \frac{2}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{sx}}{(1+s)(1-\lambda-s)}\, ds. \tag{426}$$

It follows from (310) or by writing an inequality directly for the last integral that a bounded solution is obtained if we take $\sigma = 0$ for the line of integration, i.e. the imaginary axis. We assume here that the point $(1-\lambda)$ does not lie on the imaginary axis. Suppose that the point lies to the right of the imaginary axis, i.e. that the real part of $(1-\lambda)$ is positive. With $x > 0$ we apply the residue theorem precisely as in [44], the pole $s = -1$ being to the left of the line of integration:

$$\varphi(x) = \frac{2}{2-\lambda} e^{-x} \quad (x > 0). \tag{427}$$

With $x < 0$, in order to arrive at Jordan's lemma, we have to describe the semicircle to the right instead of the left of the line of integration [cf. III_2, 61]. After finding the residue at the point $s = 1 - \lambda$, we obtain

$$\varphi(x) = \frac{2}{2-\lambda} e^{(1-\lambda)x} \quad (x < 0). \tag{428}$$

If the point $(1-\lambda)$ lies to the left of the imaginary axis, both poles will lie on the left, and we get the bounded solution

$$\begin{aligned} \varphi(x) &= \frac{2}{2-\lambda}\left(e^{-x} - e^{(1-\lambda)x}\right) \quad (x > 0) \\ \varphi(x) &= 0 \qquad\qquad\qquad\qquad\quad (x < 0). \end{aligned} \tag{429}$$

We assume here that $\lambda \neq 2$. With $\lambda = 2$ the integrand in (426) has a double pole at $s = -1$, and we obtain

$$\varphi(x) = \begin{cases} -2xe^{-x} & (x > 0) \\ 0 & (x < 0). \end{cases} \tag{430}$$

On returning to equation (425), we obtain:

$$\psi(x) = \int_x^\infty e^{-t}\varphi(t)\,dt,$$

whence

$$\psi'(x) = -e^{-x}\varphi(x). \tag{431}$$

With $x > 0$, equation (425) becomes

$$-\psi'(x) = e^{-2x} + \lambda\psi(x),$$

so that, if $\lambda \neq 2$:

$$\psi(x) = \frac{e^{-2x}}{2-\lambda} C_1 e^{-\lambda x} \quad (x > 0).$$

Similarly, we obtain with $x < 0$:

$$\psi(x) = -\frac{1}{\lambda} + C_2 e^{-\lambda x} \quad (x < 0).$$

The continuity of $\psi(x)$ at $x = 0$ implies that

$$C_2 = C_1 + \frac{2}{2-\lambda} + \frac{1}{\lambda},$$

and finally, on taking (431) into account and putting $C = C_1\lambda$, we obtain

$$\begin{aligned} \varphi(x) &= \frac{1}{2-\lambda} e^{-x} + Ce^{(1-\lambda)x} \quad (x > 0) \\ \varphi(x) &= \left(\frac{2}{2-\lambda} + C\right) e^{(1-\lambda)x} \quad (x < 0), \end{aligned} \tag{432}$$

where C is an arbitrary constant. The case $\lambda = 2$ has to be treated separately, and we shall not dwell on this. Solution (427) and (428) is obtained with $C = 0$, and solution (429) with $C = 2 : (2 - \lambda)$. If the real part of λ is negative or zero, we have to put $C = 0$ in expressions (432), since otherwise the integral appearing in equation (425) loses its meaning. If $(1 - \lambda)$ is pure imaginary or zero, solution (432) will be bounded with any C. In this case:

$$A = \int_{-\infty}^0 e^t\,dt = 1,$$

and condition (418) has the form $|\lambda| < 1$. There is seen to be an infinite set of bounded solutions even with $\lambda = 1$. The presence of an arbitrary constant in expressions (432) shows that the function

$$\omega(x) = e^{(1-\lambda)x} \tag{433}$$

must be a solution of the homogeneous equation:

$$\omega(x) = \lambda \int_x^\infty e^{x-t} \omega(t)\, dt\,, \tag{434}$$

if the real part of λ is positive. In the present case equation (421) becomes

$$\lambda \int_{-\infty}^{0} e^{(1-a)t}\, dt = 1, \quad \text{i.e.} \quad \frac{\lambda}{1-a} = 1 \text{ or } a = 1 - \lambda,$$

whence we in fact obtain solution (433) of equation (434). Every λ with a positive real part is thus seen to be an eigenvalue of equation (434). The corresponding eigenfunction (433) will not be bounded in the interval $(-\infty, +\infty)$, except when $(1 - \lambda)$ is pure imaginary or zero. As in Fourier's integral equation [55], the deviation from the theorems proved in the general theory of integral equations is explained by the unbounded nature of the interval of integration.

2. We take the equation

$$\varphi(x) = f(x) + \lambda \int_{-\infty}^{+\infty} e^{-|x-t|} \varphi(t)\, dt. \tag{435}$$

Here, the kernel is symmetric. We apply a two-sided Laplace transformation to the function $\lambda K(x) = \lambda e^{-|x|}$ (see the previous example):

$$L(s) = \lambda \int_{-\infty}^{+\infty} e^{-sx} e^{-|x|}\, dx = \frac{2\lambda}{1-s^2} \quad (-1 < \sigma < 1).$$

We now form $M(s)$, in accordance with formula (327):

$$M(s) = \frac{2\lambda}{1 - 2\lambda - s^2}$$

and the resolvent from formula (328):

$$R(x) = \frac{2\lambda}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{sx}}{1 - 2\lambda - s^2}\, ds. \tag{436}$$

In the present case:

$$A = \int_{-\infty}^{+\infty} e^{-|x|}\, dx = 2,$$

and condition (418) becomes

$$|\lambda| < \frac{1}{2}\,. \tag{437}$$

We shall assume that this condition is satisfied. The real part of $(1 - 2\lambda)$ is now certainly greater than zero, and on writing $\sqrt{1-2\lambda}$ for the value of the radical, the real part of which is positive, we obtain from (436):

$$R(x) = \frac{\lambda}{\sqrt{1-2\lambda}}\, e^{-|x|\sqrt{1-2\lambda}}\,. \tag{438}$$

Formula (438) follows from (436) by the residue theorem as in the previous example, the cases $x < 0$ and $x > 0$ being considered separately. It can be obtained in another way. We only need to put $x = y\sqrt{1-2\lambda}$ and $s = \sigma : \sqrt{1-2\lambda}$ in the expression

$$\int_{-\infty}^{+\infty} e^{-sx} e^{-|x|} \, dx = \frac{2}{1-s^2},$$

which gives us

$$\int_{-\infty}^{+\infty} e^{-\sigma y} e^{-|y|\sqrt{1-2\lambda}} \, dy = \frac{2\sqrt{1-2\lambda}}{1-2\lambda-\sigma^2}.$$

Inversion of this two-sided Laplace transformation leads us at once to expression (438) for integral (436). Having found the resolvent (438), we obtain by analogy with (325):

$$\varphi(x) = f(x) + \frac{\lambda}{\sqrt{1-2\lambda}} \int_{-\infty}^{+\infty} e^{-|x-t|\sqrt{1-2\lambda}} f(t) \, dt. \tag{439}$$

If $f(x)$ is bounded in the interval $(-\infty, +\infty)$, this formula obviously gives a bounded solution of equation (435). This will hold not only under condition (437), but for any λ except for the case when $(1 - 2\lambda)$ is zero or a real negative number, i.e. except when $\lambda \geqslant 1/2$.

Equation (421) becomes in the present case:

$$\lambda \int_{-\infty}^{+\infty} e^{-|t|} e^{-at} \, dt = 1,$$

i.e.

$$\frac{2\lambda}{1-a^2} = 1 \text{ or } a = \pm\sqrt{1-2\lambda},$$

so that the homogeneous equation

$$\omega(x) = \lambda \int_{-\infty}^{+\infty} e^{-|x-t|} \omega(t) \, dt \tag{440}$$

has the solution

$$\omega(x) = C_1 e^{\sqrt{1-2\lambda}x} + C_2 e^{-\sqrt{1-2\lambda}x}. \tag{441}$$

When $\lambda = 1/2$, the equation in a has a double root $a = 0$, and we have instead of (441):

$$\omega(x) = C_1 + C_2 x.$$

For the integral to have a meaning when expression (441) is substituted in the right-hand side of (440), we have to require that the positive real part of $\sqrt{1-2\lambda}$ be less than unity or that this real part be zero. If $1 - 2\lambda < 0$, i.e. $\lambda > 1/2$, (441) yields a bounded solution of equation (440). With $\lambda = 1/2$, an orthogonal solution of equation (440) will be an arbitrary constant. In this case all the real values $\lambda \geqslant 1/2$ will be eigenvalues of equation (440), to which

there correspond bounded eigenfunctions. When $\lambda > 1/2$, the eigenfunctions can be written, by (441), as: $\sin\sqrt{2\lambda-1}$ and $\cos\sqrt{2\lambda-1}$. If the real part of $\sqrt{1-2\lambda}$ is positive and less than unity, (441) gives an unbounded solution of (440) with any choice of arbitrary constants C_1 and C_2.

3. The homogeneous equation

$$\psi(x) = \lambda \int_0^{\infty} \frac{\psi(t)}{x+t}\,\mathrm{d}t \tag{442}$$

reduces, with the aid of the substitutions

$$x = \mathrm{e}^{\xi}, \quad t = \mathrm{e}^{\tau}; \quad \mathrm{e}^{1/_2\xi}\,\psi(\mathrm{e}^{\xi}) = \varphi(\xi) \tag{443}$$

to the form

$$\varphi(\xi) = \lambda \int_{-\infty}^{+\infty} \frac{\varphi(\tau)}{2\cosh\frac{1}{2}(\xi-\tau)}\,\mathrm{d}\tau. \tag{444}$$

The corresponding equation (421) becomes

$$\lambda \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-at}}{2\cosh\frac{1}{2}t}\,\mathrm{d}t = 1, \tag{445}$$

where the existence of the integral requires the assumption that the real part of a lies inside the interval $(-1/2, 1/2)$. The last integral is evaluated by introducing a new variable of integration $x = \mathrm{e}^{t}$:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-at}}{2\cosh\frac{1}{2}t}\,\mathrm{d}t = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-at}}{\mathrm{e}^{\frac{1}{2}t}+\mathrm{e}^{-\frac{1}{2}t}}\,\mathrm{d}t = \int_0^{\infty} \frac{x^{-(a+1/_2)}}{1+x}\,\mathrm{d}x.$$

On putting $-(a+1/2) = b-1$, i.e. $b = 1/2 - a$, and using the familiar integral [III$_2$, 62]:

$$\int_0^{\infty} \frac{x^{b-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin b\pi} = \frac{\pi}{\cos a\pi},$$

we can rewrite equation (445) as

$$\frac{\pi\lambda}{\cos a\pi} = 1,$$

where we have to take the roots whose real parts lie inside the interval $(-1/2, +1/2)$. If $\lambda = 1:\pi$, the equation has a double root $a = 0$. Equation (444) now has the solution $\varphi(\xi) = C_1 + C_2\,\xi$, and by (443), we get the following solution of equation (442):

$$\psi(x) = \frac{C_1 + C_2\log x}{\sqrt{x}},$$

where C_1 and C_2 are arbitrary constants.

58. The case of a semi-infinite interval. When the basic interval of integration in an integral equation with a kernel depending on a difference is $(0, +\infty)$ instead of $(-\infty, +\infty)$, the problem becomes more complicated. Equations of this type have been investigated by Wiener and Hopf *(Sitz Preuss. Akad.*, 1931), V. A. Fok *(Matem. sb.*, vol. 14, 1944), and I. M. Rapoport *(DAN SSSR.*, vol. 59, no. 8, 1948). The last work establishes the connection between this type of equation and Hilbert's boundary problem. We shall follow this method, confining ourselves to general indications of the way in which it is used.

We consider the equation

$$\varphi(x) = f(x) + \int_0^\infty K(x-t)\,\varphi(t)\,dt. \tag{446}$$

The function $K(x)$ is given in the interval $-\infty < x < +\infty$ and $f(x)$ in the interval $0 \leqslant x < +\infty$. It is required to find $\varphi(x)$ for $0 \leqslant x < +\infty$. We shall assume that the given functions are continuous and that, for some real c, the products

$$K(x)\,e^{-cx}, \quad f(x)\,e^{-cx}$$

are absolutely integrable and have a finite number of intervals of increase and decrease for $-\infty < x < +\infty$ and $0 \leqslant x < +\infty$ respectively.

We complete the definition of $f(x)$ for $x < 0$ by putting $f(x) = 0$ for $x < 0$, and seek the $\varphi(x)$ such that equation (446) is satisfied throughout the interval $-\infty < x < +\infty$. We shall also assume that $\varphi(x)\,e^{-cx}$ is absolutely integrable for $-\infty < x < +\infty$.

In order to apply the convolution theorem for the two-sided Laplace transformation, we must have the limits of integration $-\infty < x < +\infty$. We proceed as follows. We introduce the functions

$$\varphi_+(x) = \begin{cases} 0 & (x > 0) \\ \varphi(x) & (x < 0) \end{cases} \qquad \varphi_-(x) = \begin{cases} -\varphi(x) & (x > 0) \\ 0 & (x < 0) \end{cases} \tag{447}$$

Equation (446) can now be written as

$$\varphi_+(x) - \varphi_-(x) = f(x) - \int_{-\infty}^{+\infty} K(x-t)\,\varphi_-(t)\,dt. \tag{448}$$

We introduce the two-sided Laplace transforms

$$\Phi^+(s) = L_2(\varphi_+); \quad \Phi^-(s) = L_2(\varphi_-); \quad F(s) = L_2(f); \quad L(s) = L_2(K),$$

putting $s = c + \tau i$. In view of our assumptions regarding the absolute convergence of the integrals, these transformations are realizable, and, by (448), we obtain

$$\varphi^+(s) - \Phi^-(s) = F(s) - L(s)\,\Phi^-(s) \quad (s = c + \tau i). \tag{449}$$

Since $f(x) = 0$ for $x < 0$, we have

$$F(s) = \int_0^\infty e^{-sx} f(x)\,dx,$$

and the last integral is absolutely convergent if the real part of $s = \sigma + \tau i$ satisfies the inequality $\mathcal{R}s = \sigma \geqslant c$, whilst the function $F(s)$ is regular to the right of the straight line $\sigma = c$ and is continuous up to this line [44]. Similarly, since we have assumed the absolute convergence of the integral

$$\int_{-\infty}^{+\infty} e^{-cx} \varphi(x)\, dx,$$

the integrals

$$\int_{-\infty}^{+\infty} e^{-sx} \varphi_+(x)\, dx \quad \text{and} \quad \int_{-\infty}^{+\infty} e^{-sx} \varphi_-(x)\, dx$$

must be absolutely convergent if σ satisfies respectively $\sigma \leqslant c$ and $\sigma \geqslant c$, and we can assert that $\varphi^+(s)$ must be regular in the half-plane $\mathcal{R}s = \sigma < c$, and $\varphi^-(s)$ in the half-plane $\mathcal{R}s = \sigma > c$, where both functions must be continuous up to the line $\sigma = c$. It can also be shown that both functions must tend to zero in these domains as $|s| \to +\infty$. Equation (449) relates the boundary values of the functions $\varphi^+(s)$ and $\varphi^-(s)$ on $\sigma = c$. On writing the former as

$$\Phi^+(s) = [1 - L(s)]\, \Phi^-(s) + F(s), \tag{450}$$

we see that a non-homogeneous Hilbert problem is obtained (see [51]). Problem (450) differs from problems (371) only in the closed contour L, containing the origin inside it and an infinitely remote point outside, being replaced by the straight line $s = c + \tau i$. This change leads to certain modifications in the formulae. The factor τ^{-k} of (363) is replaced by

$$\left(\frac{s-\alpha}{s-\beta}\right)^{-k} \qquad (\alpha < c;\ \beta > c),$$

the amplitude of which receives the increment $(-2k\pi)$ on an upward displacement along the straight line $\sigma = c$. Formula (363) now becomes

$$g_0(s) = \left(\frac{s-\alpha}{s-\beta}\right)^{-k} [1 - L(s)],$$

where

$$k = \frac{1}{2\pi} \{\arg [1 - L(s)]\}_{c-i\infty}^{c+i\infty}$$

(here and later we assume that $1 - L(c + \tau i) \neq 0$). Instead of (368) we have

$$\varphi_0^+(s) = \psi_0^+(s); \quad \varphi_0^-(s) = \left(\frac{s-\alpha}{s-\beta}\right)^{-k} \psi_0^-(s),$$

and instead of the polynomial $P(z)$ the fraction

$$\frac{P(s)}{(s-\beta)^k},$$

since we now have to apply the generalized Liouville theorem to a function regular everywhere except at $s = \beta$.

We assume that functions $L(c + \tau i)$ and $F(c + \tau i)$ satisfy the Lipschitz condition and write down the solution of Problem (450) vanishing at infinity [cf. (379)]:

$$\frac{\varphi_0(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(\tau)}{\varphi_0^+(\tau)(\tau-s)}\,d\tau + \frac{P_{k-1}(s)}{(s-\beta)^k}\varphi_0(s) = \begin{cases}\Phi^+(s) & (\mathcal{R}s < c)\\ \Phi^-(s) & (\mathcal{R}s > c)\end{cases}. \tag{451}$$

Here

$$\varphi_0(s) = \begin{cases}\varphi_0^+(s) & (\mathcal{R}s < c)\\ \varphi_0^-(s) & (\mathcal{R}s > c)\end{cases},$$

$$\varphi_0^+(s) = \exp\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\log\left\{\left(\frac{\tau-\alpha}{\tau-\beta}\right)^{-k}[1-L(\tau)]\right\}\frac{d\tau}{\tau-s},$$

$$\varphi_0^-(s) = \left(\frac{s-\alpha}{s-\beta}\right)^{-k}\exp\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\log\left\{\left(\frac{\tau-\alpha}{\tau-\beta}\right)^{-k}[1-L(\tau)]\right\}\frac{d\tau}{\tau-s}$$

$$(\exp x = e^x),$$

$P_{k-1}(s)$ is a polynomial with arbitrary coefficients of degree not higher than $k - 1$; when $k \leqslant 0$ we have to put $P_{k-1}(s) \equiv 0$.

In accordance with the results obtained at the end of [51], we have three cases:

$k > 0$. Problem (450) has a solution depending on k arbitrary constants.

$k = 0$. The problem has a unique solution.

$k < 0$. The integral in (451) must have a zero of order $(-k)$ at $s = \beta$, compensating the pole of the factor $[(s - \alpha)/(s - \beta)]^{-k}$ appearing in $\varphi_0^-(s)$. This leads to the necessary and sufficient conditions for solubility:

$$\int_{c-i\infty}^{c+i\infty}\frac{F(\tau)}{\varphi_0^+(\tau)(\tau-\beta)^m}\,d\tau = 0 \quad (m = 0, 1, \ldots, -k-1). \tag{452}$$

Having found the function $\Phi^-(s)$ in accordance with (450), we find by inversion formula (308) the function $\varphi(x)$ with $x > 0$:

$$\varphi(x) = -\varphi_-(x) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx}\Phi^-(s)\,ds \qquad (\sigma > c, x > 0). \tag{453}$$

It can be shown that the product $\varphi(x)\,e^{-cx}$ is absolutely integrable on the interval $0 \leqslant x < +\infty$. After writing the inversion formula for $\Phi^+(s)$, we can find $\varphi(x)$ for $x < 0$.

Let us state the result of the investigation. If k is a number equal to the increment of the amplitude of the function $1 - L(s)$ when s varies from $c - i_\infty$ to $c + i_\infty$ divided by 2π, in the case $k > 0$ equation (446) has k linearly independent solutions; when $k = 0$ the equation has a unique solution, zero for the homogeneous equation; when $k < 0$ the necessary and sufficient conditions

for the equation to be soluble are given by (452). If (452) are fulfilled, the equation has a unique solution. When $k > 0$ the homogeneous equation (446) $(f(x) \equiv 0)$ has k linearly independent solutions, which are obtained if we put $F(s) \equiv 0$ in (451).

59. Examples. 1. Let us consider the symmetric kernel

$$K(x) = \lambda e^{-|x|} = \begin{cases} \lambda e^{-x} & (x \geqslant 0) \\ \lambda e^{x} & (x \leqslant 0) \end{cases},$$

where λ is a real parameter. We can take as c any number satisfying $-1 < c < 1$. The function $L(s)$ becomes

$$L(s) = \lambda \int_0^\infty e^{-sx} e^{-x}\, dx + \lambda \int_{-\infty}^0 e^{-sx} e^{x}\, dx = \frac{2\lambda}{1 - s^2}$$

and

$$1 - L(s) = \frac{(s-\lambda_1)(s-\lambda_2)}{(s-1)(s+1)}, \tag{454}$$

where

$$\lambda_{1,2} = \pm\sqrt{1-2\lambda} \qquad (\mathcal{R}\lambda_1 \geqslant \mathcal{R}\lambda_2).$$

The number k will depend on λ and on the class of solutions, characterized by the number c. If $\mathcal{R}\lambda_1 < c$ and $\mathcal{R}\lambda_2 < c$, then $k = +1$; if $\mathcal{R}\lambda_1 > c$, and $\mathcal{R}\lambda_2 < c$, then $k = 0$; if $\mathcal{R}\lambda_1 > c$ and $\mathcal{R}\lambda_2 > c$, $k = -1$.

(1) The case $\lambda < 0$. With any c, $|c| < 1$, we have $k = 0$, since $\lambda_1 \geqslant 1$ and $\lambda_2 \leqslant -1$. Problem (450) becomes

$$\Phi^+(s) = \frac{(s-\lambda_1)(s-\lambda_2)}{(s-1)(s+1)} \Phi^-(s) + F(s). \tag{455}$$

Due to the special simplicity of coefficient (454) this problem can be solved without using a Cauchy type integral. It is obvious that

$$\varphi_0^+(s) = \frac{s-\lambda_1}{s-1}; \qquad \varphi_0^-(s) = \frac{s+1}{s-\lambda_2}.$$

We rewrite condition (455) in the form (372):

$$\frac{s-1}{s-\lambda_1}\Phi^+(s) - \frac{s-\lambda_2}{s+1}\Phi^-(s) = \frac{s-1}{s-\lambda_1}F(s). \tag{456}$$

The product

$$\frac{s-1}{s-\lambda_1}F(s)$$

can easily be expressed as a difference of the form (357) on the assumption that $F(s)$ is regular for $\mathcal{R}s > c$:

$$\frac{s-1}{s-\lambda_1}F(s) = \frac{\lambda_1 - 1}{s-\lambda_1}F(\lambda_1) - \frac{(\lambda_1 - 1)F(\lambda_1) - (s-1)F(s)}{s-\lambda_1}.$$

Hence [cf. 51]:

$$\frac{s-\lambda_2}{s-1}\Phi^-(s) = \frac{(\lambda_1-1)F(\lambda_1) - (s-1)F(s)}{s-\lambda_1},$$

so that

$$\Phi^-(s) = \frac{(s+1)\,[(\lambda_1 - 1)\,F(\lambda_1) - (s-1)\,F(s)]}{(s-\lambda_1)\,(s-\lambda_2)}, \tag{457}$$

and the solution of the equation

$$\varphi(x) = f(x) + \lambda \int_0^\infty e^{-|x-t|}\,\varphi(t)\,dt \tag{458}$$

with $\lambda \leqslant 0$ becomes

$$\varphi(x) = -\varphi_-(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{sx} \frac{(s^2-1)\,F(s) - (\lambda_1 - 1)\,(s+1)\,F(\lambda_1)}{(s-\lambda_1)\,(s-\lambda_2)}\,ds.$$

(2) We take the case $0 < \lambda < 1/2$ and suppose that c satisfies the condition $\sqrt{1-2\lambda} < c < 1$. We have $\lambda_2 < \lambda_1 < c$, whence $k = +1$. The function on the right-hand side of (455) can have a simple pole $s = 1$ in the domain $\sigma > c$ and must tend to zero as $|s| \to \infty$. It is easy to see that we must put

$$\Phi^+(s) = \frac{(s-\lambda_1)\,(s-\lambda_2)}{(s-1)\,(s+1)}\,\Phi^-(s) + F(s) = \frac{A}{s-1},$$

where A is an arbitrary constant. We find that

$$\Phi^-(s) = -\frac{s^2-1}{(s-\lambda_1)\,(s-\lambda_2)}\,F(s) + \frac{(s+1)\,A}{(s-\lambda_1)\,(s-\lambda_2)},$$

and the solution of equation (458) with $0 < \lambda < 1/2$ and $\sqrt{1-2\lambda} < c < 1$ will be

$$\varphi(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx} \frac{s^2-1}{(s-\lambda_1)\,(s-\lambda_2)}\,F(s)\,ds - \frac{A}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx} \frac{s+1}{(s-\lambda_1)\,s-\lambda_2)}\,ds.$$

The second term is the solution of the homogeneous equation. On adding a semicircle of large radius to the left of the line of integration, and applying Jordan's lemma and the residue theorem, we obtain the solution of the homogeneous equation in the form

$$\varphi_0(x) = -A\,\frac{\sqrt{1-2\lambda}+1}{2\sqrt{1-2\lambda}}\,e^{\sqrt{1-2\lambda}\,x} - A\,\frac{\sqrt{1-2\lambda}-1}{2\sqrt{1-2\lambda}}\,e^{-\sqrt{1-2\lambda}\,x}.$$

(3) The case $0 < \lambda < 1/2$, $-\sqrt{1-2\lambda} < c < \sqrt{1-2\lambda}$. We have $k = 0$ for this narrower class of solutions. The method of solution is the same as in case (1).

(4) The case $0 < \lambda < 1/2$, $-1 < c < -\sqrt{1-2\lambda}$. The class of solutions is still narrower ($k = -1$). The method of solution differs from that of case (1) only in the fact that the expression in square brackets in (457) must now vanish for $s = \lambda_2$. We find from this the necessary and sufficient condition for equation (458) to be soluble in the present case:

$$(\lambda_1 - 1)\,F(\lambda_1) - (\lambda_2 - 1)\,F(\lambda_2) = 0. \tag{459}$$

(5) The case $\lambda \geqslant 1/2$, $0 < c < 1$ ($k = 1$). The method of solution is the same as in case (2). The solution of the homogeneous equation with $\lambda > 1/2$ can be written as

$$\varphi_0(x) = \cos \nu x + \frac{\sin \nu x}{\nu},$$

where $\nu^2 = 2\lambda - 1$. With $\lambda = 1/2$ the homogeneous equation has the solution

$$\varphi_0(x) = 1 + x.$$

(6) The case $\lambda \geqslant 1/2$, $-1 < c < 0$ ($k = -1$). The method of solution is the same as in case (4). We get a solution vanishing at infinity if condition (459) is satisfied. It can be shown that this condition is equivalent to

$$\int_0^x \left(\cos \nu x + \frac{\sin \nu x}{\nu}\right) f(x)\, dx = 0$$

or

$$\int_0^x (1 + x) f(x)\, dx = 0.$$

2. We take a homogeneous equation, the kernel of which is given by (Milne's equation):

$$K(x) = \frac{1}{2} \int_{|x|}^{\infty} \frac{e^{-t}}{t}\, dt.$$

This function tends to infinity of order $\log x$ at $x = 0$. This fact does not prevent the application of the previous method. We form the function $L(s)$:

$$L(s) = \int_{-\infty}^{+\infty} e^{sx} \left[\frac{1}{2} \int_{|x|}^{\infty} \frac{e^{-t}}{t}\right] dx$$

$$= \frac{1}{2} \int_0^{+\infty} e^{sx} \left[\int_x^{\infty} \frac{e^{-t}}{t}\, dt\right] dx + \frac{1}{2} \int_{-\infty}^{0} e^{sx} \left[\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt\right] dx.$$

The repeated integration in the first term is equivalent to finding the double integral over the part of the first quadrant of the (x, t) plane in which $t \geqslant x$. By changing the order of integration, we can rewrite this first term as

$$\frac{1}{2} \int_0^{\infty} \frac{e^{-t}}{t} \left[\int_0^t e^{sx}\, dx\right] dt = \frac{1}{2s} \int_0^{\infty} \frac{e^{(s-1)t} - e^{-t}}{t}\, dt.$$

The last integral can be evaluated say by differentiation with respect to the parameter s, and we get:

$$\frac{1}{2s} \int_0^{\infty} \frac{e^{(s-1)t} - e^{-t}}{t}\, dt = -\frac{1}{2s} \log(1 - s),$$

where the real part of s is assumed less than unity. The second term in the expression for $L(s)$ is evaluated in the same way, and we obtain

$$L(s) = \frac{1}{2} \log \frac{1+s}{1-s} \qquad (s = \sigma + \tau i; \quad -1 < \sigma < 1),$$

where we have to take the value of the logarithm which vanishes for $s = 0$. On expanding the logarithm as a power series, the equation

$$1 - \frac{1}{2s} \log \frac{1+s}{1-s} = 0$$

is seen to have a double root $s = 0$. It can be shown that it has no other roots, for which the real part lies inside the interval $(-1, 1)$.

If we take $0 < c < 1$, the number

$$k = \frac{1}{2\pi} \left\{ \arg \left[1 - \frac{1}{2s} \log \frac{1+s}{1-s} \right] \right\}_{c-i\infty}^{c+i\infty} = 1$$

and we can find the solutions from (451) and (453) with $F(s) \equiv 0$.

60. More general equations. The more general equations

$$\varphi(x) = f(x) + \int_0^\infty K_1(x-t)\,\varphi(t)\,\mathrm{d}t + \int_{-\infty}^0 K_2(x-t)\,\varphi(t)\,\mathrm{d}t \tag{460}$$

and

$$\left.\begin{aligned} \varphi(x) &= f(x) + \int_{-\infty}^{+\infty} K_1(x-t)\,\varphi(t)\,\mathrm{d}t \ (x > 0), \\ \varphi(x) &= f(x) + \int_{-\infty}^{+\infty} K_2(x-t)\,\varphi(t)\,\mathrm{d}t \ (x < 0), \end{aligned}\right\} \tag{461}$$

are reducible to the Hilbert boundary value problem. If all the functions are absolutely integrable, the methods of solution for these equations are similar to the methods of [58] (see I. M. Rapoport, *Sb. tr. Inst. matem. AN SSSR*, no. 12, 1949; Yu. I. Cherskii, *Uchen. zap. Kaz. in-ta*, vol. 113, kn. 10, 1953). The problem becomes more difficult if exponential growth is permitted in the required function.

Let the kernels in equation (460) be such that the functions $K_1(x)\,\mathrm{e}^{-x}$ and $K_2(x)\,\mathrm{e}^{x}$ are absolutely integrable in the interval $-\infty < x < +\infty$; the function $f(x)\,\mathrm{e}^{-|x|}$ is also assumed absolutely integrable. The solution $\varphi(x)$ can increase to infinity, but in such a way that the product $\varphi(x)\,\mathrm{e}^{-|x|}$ remains absolutely integrable. Using notation (447), we rewrite (460) as:

$$\begin{aligned} &\varphi_+(x) - \int_{-\infty}^{+\infty} K_2(x-t)\,\varphi_+(t)\,\mathrm{d}t - f_+(x) \\ &\qquad = \varphi_-(x) - \int_{-\infty}^{+\infty} K_1(x-t)\,\varphi_-(t)\,\mathrm{d}t - f(x) = \omega(x). \end{aligned} \tag{462}$$

The new unknown function is easily shown to have the property that the integral

$$\int_{-\infty}^{+\infty} \omega(x)\,\mathrm{e}^{-\sigma x}\,\mathrm{d}x$$

is absolutely convergent for $-1 < \sigma < 1$. On writing

$$\Phi^+(s) = L_2(\varphi_+); \quad \Phi^-(s) = L_2(\varphi_-); \quad F^+(s) = L_2(f_+),$$
$$F^-(s) = L_2(f_-); \quad L_1(s) = L_2 K_1); \quad L_2(s) = L_2(K_2),$$
$$\Omega(s) = L_2(\omega),$$

we obtain two conditions from equation (462):

$$\begin{aligned} &\Phi^+(s) - L_2(s)\,\Phi^+(s) - F^+(s) = \Omega(s) \qquad (s = -1 + \tau i), \\ &\Phi^-(s) - L_1(s)\,\Phi^-(s) - F^-(s) = \Omega(s) \qquad (s = +1 + \tau i), \end{aligned} \qquad (463$$

The unknown functions $\Phi^+(s)$, $\Phi^-(s)$ and $\Omega(s)$ are regular in the domains $\mathrm{R}s < -1$, $\mathrm{R}s < +1$ and $-1 < \mathrm{R}s < +1$ respectively, and vanish at infinity.

Problem (463) is a non-homogeneous Hilbert problem [cf. 51], the contour L being made up of the straight lines $\mathrm{R}s = -1$ and $\mathrm{R}s = +1$.

This and other cases for equations (460) and (461) are treated in detail in an article by F. D. Gakhov and Yu. I. Cherskii *(Izv. Akad. Nauk SSSR,* 20, 1956). The latter considered these equations for the general case of growth or decrease of exponential order of kernels $K_1(x)$ and $K_2(x)$; he also showed that, if (460) has a further term consisting of a completely continuous operator, the equation reduces after Laplace transformation to a singular equation, the theory of which may be found in the cited work by Muskhelishvili. Post-graduate students of Rostov university have developed a theory of systems of equations of type (460), a theory of integro-differential equations with kernels depending on a difference, etc.

Krein and Hochberg have shown that the Lipschitz condition that we imposed on functions $L(s)$ and $K(s)$ in [58] is not essential.

The treatment of the last three sections is due to Cherskii.

CHAPTER II

THE CALCULUS OF VARIATIONS

61. Statement of the problem. We shall consider some concrete problems that indicate the aims and ideas of the calculus of variations. Suppose we have a non-homogeneous isotropic medium at every point (x, y, z) of which a velocity $v(x, y, z)$ is defined, independent of direction. We try to find the time required for a point moving with this velocity to describe a given curve l. The elementary path ds will be traversed in time ds/v, and the entire curve l will be traversed in a time given by the integral:

$$T = \int_l \frac{ds}{v(x, y, z)}. \tag{1}$$

We fix the extremities (x_0, y_0, z_0) and (x_1, y_1, z_1) of the curve l, and let the curve itself vary. The magnitude T will now vary in relationship to l. We say that T is a *functional of the curve* l. Given a definite choice of l, the functional T will have a definite numerical value. One of the problems of geometrical optics is the following: given the ends (x_0, y_0, z_0) and (x_1, y_1, z_1), to find l such that the functional T has a minimum value. Suppose we take x as the parameter in the equation of l, so that y and z are functions of x. Integral (1) now becomes:

$$T = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2 + z'^2}}{v(x, y, z)}\, dx, \tag{2}$$

where y' and z' are the derivatives of the above-mentioned functions of x. The problem amounts to seeking functions $y(x)$ and $z(x)$ such that (2) has a minimum value, whilst the required functions must satisfy the following conditions:

$$y(x_0) = y_0; \quad z(x_0) = z_0;$$
$$y(x_1) = y_1; \quad z(x_1) = z_1.$$

In the case of a plane, functional (2) becomes

$$T = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{v(y, x)}\, dx,$$

and the problem amounts to finding the function $y(x)$, satisfying the two boundary conditions:

$$y(x_0) = y_0; \quad y(x_1) = y_1.$$

We now consider the problem for a multiple integral. Given a closed curve l in space, we want to stretch a surface over it which has a minimum area. Let λ be the projection of l on the (x, y) plane and B the domain bounded by λ. The equation of the required surface can be written explicitly as $z = z(x, y)$. The surface area is now given by

$$S = \iint_B \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy, \tag{3}$$

where z_x and z_y are the partial derivatives of $z(x, y)$ with respect to x and y.

Given a definite choice of surface, S will have a definite value in other words, S is a functional of the surface. The problem amounts to choosing the function $z(x, y)$ such that S has a least value. The boundary conditions in the present case consist in specifying the values of the required function on the contour λ. These values must yield the ordinates z of the contour l on which the surface has to be stretched.

The fundamental problem of the calculus of variations is in fact seeking the maximum and minimum values of functionals of curves and surfaces, expressed by certain definite integrals. This problem is analogous to the problem of the differential calculus of finding the maxima and minima of a given function. As we know, this latter problem is directly bound up with the problem of seeking the extrema of a function, i.e. seeking the values of the independent variables for which the function takes its greatest or least values in comparison with all the neighbouring values. We shall treat the problem for functionals in a similar manner. For instance, in the case of functional (2) we shall seek the curve l such that the value of T for this curve is not greater than its values for all curves sufficiently close to l. If, for a certain curve or surface, the functional has values not less than (or not greater than) its values for all neighbouring curves or surfaces, we simply say that the functional has an extremum for this curve or surface.

We shall give below a strict statement of the problem and shall define the concept of closeness for curves and surfaces, which play the part of the independent variables of the ordinary differential calculus. We know that the equation $f'(x) = 0$ has to be solved in order to find the x for which the function $f(x)$ has an extremum. It is shown in the calculus of variations that the curve $y = y(x)$ or surface $z = z(x, y)$, yielding the extremum of a certain functional, must satisfy a certain differential equation. Our first problem is to obtain this differential equation. Satisfaction of the equation represents the necessary condition for an extremum of the functional, just as the equation $f'(x) = 0$ is the necessary condition for a given function $f(x)$ to have an extremum for a certain value of x. In order to deduce the required equation we need two lemmas, which are given in the next section.

62. Fundamental lemmas. LEMMA 1. *If the integral*

$$\int_{x_0}^{x_1} f(x)\, \eta(x)\, dx, \tag{4}$$

where $f(x)$ is a fixed function continuous in the interval $[x_0, x_1]$, vanishes for every function $\eta(x)$ which is continuous together with its derivative and vanishes at the ends: $\eta(x_0) = \eta(x_1) = 0$, then $f(x)$ is identically zero in the interval.

Let us suppose the contrary. Let $f(x)$ be non-zero at some point $x = \xi$ inside the interval, e.g. $f(\xi) > 0$. Since $f(x)$ is continuous it will be positive in some interval $[\xi_0, \xi_1]$ containing the point ξ and lying inside $[x_0, x_1]$. We now define the function $\eta(x)$ as follows:

$$\eta(x) = \begin{cases} 0 & \text{for } x_0 \leqslant x \leqslant \xi_1 \\ (x - \xi_1)^2 (x - \xi_2)^2 & \text{for } \xi_1 \leqslant x \leqslant \xi_2 \\ 0 & \text{for } \xi_2 \leqslant x \leqslant x_1. \end{cases} \tag{5}$$

The function $\eta(x)$ thus constructed satisfies all the conditions of the lemma. For, by construction, $\eta(x_0) = \eta(x_1) = 0$. The product $(x - \xi_1)^2 (x - \xi_2)^2$ and its derivative with respect to x vanish for $x = \xi_1$ and $x = \xi_2$. Outside the interval $[\xi_1, \xi_2]$, $\eta(x)$ is identically zero. Hence it follows that the function and its derivative are continuous throughout $[x_0, x_1]$. On recalling that $\eta(x)$ is identically zero outside $[\xi_1, \xi_2]$, we can write integral (4) as

$$\int_{\xi_1}^{\xi_2} f(x)\, (x - \xi_1)^2 (x - \xi_2)^2\, dx,$$

whence it follows that it has a positive value, since the integrand is continuous and positive inside the interval of integration. But the integral must vanish by hypothesis. This contradiction proves the lemma.

There is a similar lemma for a double integral.

LEMMA 2. *If the integral*

$$\int\int_B f(x, y)\, \eta(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{6}$$

where $f(x, y)$ is a fixed continuous function in the domain B, vanishes for every function $\eta(x, y)$ which is continuous together with its first order partial derivatives in B and vanishes on the contour l of B, then $f(x, y)$ is identically zero in B.

Suppose that $f(x, y)$ is positive at a point (ξ, η) inside the domain B. It will now be positive in some circle with centre (ξ, η) and radius ϱ lying inside B. We define $\eta(x, y)$ as follows:

$$\eta(x, y) = \begin{cases} 0 & \text{for } (x-\xi)^2 + (y-\eta)^2 \geqslant \varrho^2 \\ \left[(x-\xi)^2 + (y-\eta)^2 - \varrho^2\right]^2 & \text{for } (x-\xi)^2 + (y-\eta)^2 < \varrho^2. \end{cases}$$

It is easily shown that $\eta(x, y)$ satisfies all the conditions of the lemma, whilst integral (6) reduces to the integral over the circle of a continuous positive function, i.e. is positive, which contradicts the hypothesis of the lemma.

We remark that both lemmas still hold if we impose stronger restrictions on the function η, e.g. require that it has continuous derivatives up to some order n. The proof remains as before and it is sufficient merely to replace the power 2 say in formula (5) by $(n+1)$ or $(n+2)$. We remark also that the lemma can easily be proved for triple integrals and in general for integrals of any multiplicity.

63. Euler's equation in the elementary case. We take the elementary functional:

$$J = \int_{x_0}^{x_1} F(x, y, y')\, \mathrm{d}x. \tag{7}$$

where F is a given function of all three arguments. We shall assume it to be continuous together with its derivatives up to the second order in a domain B of the (x, y) plane and for any values of y'.

The functional J has a definite numerical value if we fix the function $y = y(x)$, or what amounts to the same thing, the curve $y = y(x)$, which we shall always assume to belong to the domain B.

Let the values of the function $y(x)$ be given at the ends of the interval of integration:

$$y(x_0) = y_0; \quad y(x_1) = y_1. \tag{8}$$

We shall assume that the required function has a continuous derivative. The class of functions having a continuous derivative in the interval $[x_0, x_1]$ will be referred to as class C_1 (and similarly, the class of functions having n continuous derivatives will be called C_n), and we shall assume that all the functions mentioned in future belong to this class. We define an ε-neighbourhood of the curve $y = y(x)$ as all the possible curves $y_1(x)$ which satisfy the inequality $| y_1(x) - y(x) | \leqslant \varepsilon$ throughout $[x_0, x_1]$. Sometimes we shall add the further inequality $| y_1'(x) - y'(x) | \leqslant \varepsilon$, i.e. we require an ε-closeness of the slopes of the tangents as well as of the ordinates. We shall occasionally speak of the ε-closeness of zero order in the first case, and of an ε-closeness of the first order in the second, when both inequalities are satisfied.

DEFINITION. *A functional J is said to attain a relative extremum for the curve $y(x)$, lying inside the domain B, belonging to class C_1 and satisfying condition* (8), *if the magnitude of the functional for $y(x)$ is not less than (or not greater than) its magnitude for any other curves of class C_1 belonging to an ε-neighbourhood of $y(x)$ and satisfying condition* (8).

This concept of relative extremum is completely analogous to the concept of maximum and minimum of a function [I, 58]. The concept of *absolute extremum* can be brought in, in addition to that of relative extremum. Suppose we have a class D of functions $y(x)$ for which integral (7) has a meaning. The functional J is said to attain an absolutely extremum in class D for the curve $y(x)$ if the value of the functional for $y(x)$ is not less than (or not greater than) its values for all the other curves of class D.

We shall only be concerned for the present with relative extrema and leave a brief discussion of absolute extrema to the end of the chapter. For brevity, we shall simply speak of extrema instead of relative extrema. In the following sections we shall discuss functionals differing from functional (7). The problem of both relative and absolute extrema may be stated for these latter functionals. This will not be mentioned explicitly each time and we shall be concerned at the start only with relative extrema.

Let us deduce the necessary conditions to be satisfied by $y(x)$ in order for functional J to have an extremum. We take any function $\eta(x)$, vanishing at the ends of the interval of integration, and in addition to the $y(x)$ which has to yield the extremum of the functional J, we form the new function $y(x) + \alpha\eta(x)$, where α is a small numerical parameter. This new function satisfies the same boundary conditions as $y(x)$. On substituting it in functional J, we obtain as a result of integration a function of the parameter α:

$$J(\alpha) = \int_{x_0}^{x_1} F(x,\ y(x) + \alpha\eta(x),\ y'(x) + \alpha\eta'(x))\,\mathrm{d}x. \tag{9}$$

Given any positive ε, the function $y(x) + \alpha\eta(x)$ lies in the ε-neighbourhood (even of the first order) of the curve $y(x)$ for all values of the parameter α sufficiently close to zero. Consequently, since $y(x)$ gives an extremum of functional J, function (9) must have an extremum for the value $\alpha = 0$, so that its derivative must vanish for $\alpha = 0$. On differentiating under the integral sign and denoting the derivatives by subscripts, we have:

$$J'(0) = \int_{x_0}^{x_1} \left[F_y(x, y, y')\,\eta(x) + F_{y'}(x, y, y')\,\eta'(x)\right]\mathrm{d}x.$$

On integrating by parts, we can write:

$$J'(0) = \left[F_{y'}\,\eta(x)\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta(x)\left[F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right]\mathrm{d}x. \tag{10}$$

The term outside the integral vanishes, since by hypothesis $\eta(x)$ vanishes at the ends of the interval, so that

$$J'(0) = \int_{x_0}^{x_1} \eta(x)\left[F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right]\mathrm{d}x = 0.$$

By applying the fundamental lemma, we can assert that the curve $y(x)$ yielding an extremum of integral (7) must satisfy the following differential equation:

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'} = 0. \tag{11}$$

On expanding the total derivative with respect to x, we can write this equation as

$$F_{y'y'}\,y'' + F_{yy'}\,y' + F_{xy'} - F_y = 0, \tag{12}$$

where, for instance, $F_{xy'}$ is the second order partial derivative with respect to x and y'. This equation is due to Euler and is generally known as *Euler's equation*. It is a second order differential equation and its general solution contains two arbitrary constants which must be determined from the two boundary conditions (8).

The product $J'(0)\,a$, which is the differential of the function $J(a)$ for $a = 0$, is usually called the first variation of functional (7) and is written as δJ. On taking (10) into account, we can write

$$\delta J = J'(0)\,a = [F_{y'}\,\delta y]_{x_0}^{x_1} + \int\limits_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \delta y \,\mathrm{d}x \tag{13}$$

$$\big(\delta y = a\eta(x)\big).$$

It may be remarked that we have also used the second derivative $y''(x)$ in deducing equation (11), so that, strictly speaking, we assume when establishing the necessary condition for an extremum that $y(x)$ belongs to the class C_2 of functions having continuous derivatives up to the second order.

It can be shown by suitable modification to the lemma that if $y(x)$ of C_1 satisfies the condition

$$J'(0) = \int\limits_{x_0}^{x_1} \big[F_y\,\eta(x) + F_{y'}\,\eta'(x)\big]\,\mathrm{d}x = 0,$$

for any choice of $\eta(x)$ having a continuous derivative and vanishing at $x = x_0$ and $x = x_1$, where $F_{y'y'} \neq 0$ on $y = y(x)$, then $y(x)$ belongs to C_2 and must therefore satisfy equation (11).

64. The case of several functions and higher order derivatives. Euler's equation may readily be written for the case when the functional depends on several functions, as was the case, for instance, in functional (2).

We shall confine ourselves to the case of two functions:

$$J = \int\limits_{x_0}^{x_1} F(x, y, y', z, z')\,\mathrm{d}x. \tag{14}$$

We construct two functions closed to $y(x)$ and $z(x)$:

$$y(x) + a\eta(x); \quad z(x) + a_1\eta_1(x),$$

where $\eta(x)$ and $\eta_1(x)$ are arbitrary functions which vanish at the ends of the interval. On substituting these in integral (14) we obtain

a function $J(a, a_1)$ of a and a_1, and the necessary conditions for $y(x)$ and $z(x)$ to yield an extremum of functional (14) are that the partial derivatives of $J(a, a_1)$ with respect to a and a_1 vanish at $a = a_1 = 0$. We use working precisely similar to the above and find the following expressions for these partial derivatives:

$$\left.\begin{aligned} J_a(0,0) &= [F_{y'}\eta]_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta(x)\left(F_y - \frac{d}{dx}F_{y'}\right)dx, \\ J_{a_1}(0,0) &= [F_{z'}\eta_1]_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta_1(x)\left(F_z - \frac{d}{dx}F_{z'}\right)dx, \end{aligned}\right\} \tag{15}$$

and since the terms outside the integrals vanish, it is seen as above that the necessary conditions for $y(x)$ and $z(x)$ to yield an extremum of (14) are that they satisfy the following system of two second order equations:

$$F_y - \frac{d}{dx}F_{y'} = 0; \quad F_z - \frac{d}{dx}F_{z'} = 0. \tag{16}$$

Apart from these equations we also have the boundary conditions:

$$y(x_0) = y_0; \quad y(x_1) = y_1; \quad z(x_0) = z_0; \quad z(x_1) = z_1,$$

containing the fixed ends of the required spatial curve.

By (15), the variation of integral (14) is given by

$$\begin{aligned} \delta J &= J_a(0,0)\,a + J_{a_1}(0,0)\,a_1 = \\ &= [F_{y'}\delta y + F_{z'}\delta z]_{x_0}^{x_1} + \int_{x_0}^{x_1}\left[\left(F_y - \frac{d}{dx}F_{y'}\right)\delta y + \left(F_z - \frac{d}{dx}F_{z'}\right)\delta z\right]dx \\ &\qquad (\delta y = a\eta(x);\ \delta z = a_1\eta_1(x)). \end{aligned} \tag{17}$$

In the case of a functional dependent on n functions: $y_1(x), \ldots, y_n(x)$:

$$J = \int_{x_0}^{x_1} F(x, y_1, y_1', y_2, y_2', \ldots, y_n, y_n')\,dx, \tag{18}$$

the necessary conditions for an extremum are given by a system of n equations of the second order:

$$F_{y_k} - \frac{d}{dx}F_{y_k'} = 0 \qquad (k = 1, 2, \ldots, n), \tag{19}$$

whilst the boundary conditions for the fixed ends take the form:

$$y_k(x_0) = y_k^{(0)}; \quad y_k(x_1) = y_k^{(1)} \qquad (k = 1, 2, \ldots, n).$$

The first variation of functional (18) has the form:

$$\delta J = \sum_{k=1}^{n} J_{a_k}(0, 0, \ldots, 0)\, a_k =$$

$$= \left[\sum_{k=1}^{n} F_{y'_k} \delta y_k\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \sum_{k=1}^{n} \left(F_{y_k} - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'_k}\right) \delta y_k \,\mathrm{d}x \tag{20}$$

$$\left(\delta y_k = a_k \eta_k(x)\right).$$

We now take the case when the integral contains derivatives of the required function of order higher than the first:

$$J = \int_{x_0}^{x_1} F(x, y, y', \ldots, y^{(n)})\,\mathrm{d}x. \tag{21}$$

As above, we construct a neighbouring curve $y(x) + a\eta(x)$, substitute in integral (21), differentiate with respect to a and put $a = 0$. We thus obtain:

$$J'(0) = \int_{x_0}^{x_1} \left[F_y \eta(x) + F_{y'} \eta'(x) + \ldots + F_{y^{(n)}} \eta^{(n)}(x)\right] \mathrm{d}x. \tag{22}$$

We transform all the terms on the right-hand side except the first by integrating by parts several times:

$$\int_{x_0}^{x} F_{y^{(k)}} \eta^{(k)}(x)\,\mathrm{d}x = \Big[F_{y^{(k)}} \eta^{(k-1)}(x) - \frac{\mathrm{d}}{\mathrm{d}x} F_{y^{(k)}} \eta^{(k-2)}(x) + \ldots +$$

$$+ (-1)^{k-1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} F_{y^{(k)}} \eta(x)\Big]_{x_0}^{x_1} +$$

$$+ (-1)^k \int_{x_0}^{x_1} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F_{y^{(k)}} \eta(x)\,\mathrm{d}x. \tag{23}$$

We assume that $\eta(x)$ and its derivatives up to order $(n-1)$ vanish at the ends. In view of this, the terms outside the integrals fall out; on equating $J'(0)$ to zero, we get the condition:

$$J'(0) = \int_{x_0}^{x_1} \eta(x)\left[F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} + \ldots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} F_{y^{(n)}}\right] \mathrm{d}x = 0,$$

which, by the fundamental lemma, leads to the following Euler equation:

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} + \ldots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} F_{y^{(n)}} = 0. \tag{24}$$

This is a differential equation of order $2n$. Its general solution contains $2n$ arbitrary constants, and we must also have $2n$ boundary conditions. In the simplest case these conditions reduce to specifying the function and its derivatives up to order $(n - 1)$ at the ends of the interval. It follows from these boundary conditions that the corresponding magnitudes for $\eta(x)$ must vanish. We remark further that all the functions appearing in the above formulae are assumed continuous, so that, for instance, the required function $y(x)$ is assumed to belong to class C_{2n} of functions which are continuous with their derivatives up to order $2n$.

65. The case of multiple integrals. *Ostrogradskii's equation.* We now deduce the necessary condition for an extremum in the case of a multiple integral. Such conditions were first indicated by M. V. Ostrogradskii in his work *On the Differential Equations of the Isoperimetric Problem* ("O differentsial'nykh uravneniyakh zadach izoperimetrov"), published in the *Memoirs of the Petersburg Academy of Sciences*, vol. IV, ser. 5, 1850.

We consider the double integral:

$$J = \int\limits_B\int F(x, y, u, u_x, u_y)\,\mathrm{d}x\,\mathrm{d}y, \tag{25}$$

where u_x and u_y denote the partial derivatives of the function $u(x, y)$. We seek the function $u(x, y)$ which is continuous together with its derivatives up to the second order in the domain B, has given values on the contour l of the domain and yields an extremum of functional (25). We form the neighbouring functions $u(x, y) + a\eta(x, y)$, where $\eta(x, y)$ is an arbitrary function which vanishes on l. On substituting this function in integral (25), differentiating with respect to a and putting $a = 0$, we get the following expression for the first variation of the functional:

$$\delta J = J'_a(0)\,a = a\int\limits_B\int (F_u\eta + F_{u_x}\eta_x + F_{u_y}\eta_y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

We transform the last two terms by using the familiar Riemann formula:

$$\int\limits_B\int\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int\limits_l P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

which gives us:

$$\iint\limits_B (F_{u_x}\eta_x + F_{u_y}\eta_y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_B \left[\frac{\partial}{\partial x}(\eta F_{u_x}) + \frac{\partial}{\partial y}(\eta F_{u_y})\right]\mathrm{d}x\,\mathrm{d}y - $$

$$- \iint\limits_B \eta\left(\frac{\partial}{\partial x}F_{u_x} + \frac{\partial}{\partial y}F_{u_y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \int\limits_l \eta F_{u_x}\mathrm{d}y - \eta F_{u_y}\mathrm{d}x - \iint\limits_B \eta\left(\frac{\partial}{\partial x}F_{u_x} + \frac{\partial}{\partial y}F_{u_y}\right)\mathrm{d}x\,\mathrm{d}y.$$

We thus find the following expression for the first variation:

$$\delta J = \int\limits_l \delta u\,(F_{u_x}\mathrm{d}y - F_{u_y}\mathrm{d}x) + \iint\limits_B \left(F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y}\right)\delta u\,\mathrm{d}x\,\mathrm{d}y$$
$$\big(\delta u = a\eta(x, y)\big). \tag{26}$$

This first variation must vanish in the case of an extremum, or on using the fact that $\eta(x, y)$ vanishes on l, we can assert that the double integral on the right-hand side of (26) must vanish. Hence, by the fundamental lemma, the following equation (of Ostrogradskii) is obtained for the required function $u(x, y)$ that yields an extremum of functional (25):

$$F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y} = 0, \tag{27}$$

or in the expanded form:

$$F_{u_xu_x}u_{xx} + 2F_{u_xu_y}u_{xy} + F_{u_yu_y}u_{yy} + F_{u_xu}u_x +$$
$$+ F_{u_yu}u_y + F_{xu_x} + F_{yu_y} - F_u = 0. \tag{28}$$

We have obtained a second order partial differential equation which has to be satisfied inside the domain. As already remarked, the boundary condition consists in specifying u on the contour l.

In the case of a multiple integral dependent on several functions, we get a system of such equations. In the case of a triple integral and a function $u(x, y, z)$ dependent on three independent variables, an equation of the following form is obtained:

$$F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y} - \frac{\partial}{\partial z}F_{u_z} = 0. \tag{29}$$

If the derivatives of $u(x, y)$ up to order n appear under the integral sign, Ostrogradskii's equation becomes:

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} + \frac{\partial^2}{\partial x^2} F_{u_{xx}} + \frac{\partial^2}{\partial x \partial y} F_{u_{xy}} + \\ + \frac{\partial^2}{\partial y^2} F_{u_{yy}} - \ldots + (-1)^n \frac{\partial^n}{\partial y^n} F_{u_{yy\ldots y}} = 0. \qquad (30)$$

Throughout the above discussion we have assumed as usual that all the functions present are continuous. In addition, we have assumed that the transformation of a double to a line integral is permissible when deducing (26); this is bound up with the behaviour of the partial derivatives u_x and u_y in the neighbourhood of the contour l of domain B. We shall return to this question later when considering the problem of absolute extrema.

It may be remarked further that the functions satisfying equation (24) or (27) or, to be more precise, the geometrical shapes corresponding to them, are usually called *extremals* of the problem. These are curves in the case of a single integral, and surfaces in the case of a double integral. Since the Euler and Ostrogradskii equations are merely necessary conditions for the respective functionals to have an extremum, we naturally cannot assert that every extremal gives an extremum of the functional in comparison with all the sufficiently neighbouring curves or surfaces.

66. Remarks on the Euler and Ostrogradskii equations. Let us first take Euler's equation (11) in the simplest case. Suppose that function F does not contain y. The equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0$$

and has the obvious first integral $F_{y'} = C$. If F does not contain x, it is easily shown that the first integral exists:

$$F - y' F_{y'} = C. \qquad (31)$$

For:

$$\frac{\mathrm{d}}{\mathrm{d}x}(F - y' F_{y'}) = F_y y' + F_{y'} y'' - F_{y'} y'' - F_{y'y} y'^2 - F_{y'y'} y' y'' = \\ = -y'(F_{y'y'} y'' + F_{y'y} - F_y).$$

Since F does not contain x, the factor in $-y'$ is the left-hand side of Euler's equation, and consequently, in view of this equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(F - y' F_{y'}) = 0,$$

i.e. we in fact obtain solution (31).

If F does not contain y', Euler's equation (11) becomes:

$$F_y(x, y) = 0,$$

i.e. we have a finite, and not a differential equation. It gives us one or several curves, and not a family depending on two parameters, as was the case for the differential equation, and we cannot in general satisfy the boundary conditions.

We now consider the cases when Euler's equation becomes an identity. Suppose that $F = A(x, y) + B(x, y)\, y'$, where we have the identity:

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = 0. \tag{32}$$

It is easily shown that the left-hand side of equation (11) now becomes identically zero, whilst integral (7) can be written as:

$$J = \int\limits_l [A\,\mathrm{d}x + B\,\mathrm{d}y]; \tag{33}$$

here, by (32), J does not depend on the path, i.e. has the same value whatever the choice of curve l joining the points (x_0, y_0) and (x_1, y_1), this being explained by the fact that Euler's equation is an identity. It will be readily seen that we can write in this case:

$$F(x, y, y') = \frac{\mathrm{d}}{\mathrm{d}x} G(x, y),$$

where $G(x, y)$ is defined as integral (33) with variable upper limit.

In the same way, if the integrand in integral (21) is the total derivative with respect to x of a certain function that depends on $(x, y, y', \ldots, y^{(n-1)})$:

$$F(x, y, y', \ldots, y^{(n)}) = \frac{\mathrm{d}}{\mathrm{d}x} G(x, y, y', \ldots, y^{(n-1)}),$$

Euler's equation (24) reduces to an identity.

We now take functional (25) and suppose that the integrand has the form

$$F(x, y, u, u_x, u_y) = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}, \tag{34}$$

where A and B are certain functions of (x, y, u). It can be shown by direct substitution that Ostrogradskii's equation now becomes an identity. This is due in essence to the fact that, by Riemann's formula, the double integral of (34) is equal to the integral over the contour:

$$\int_l (A\,\mathrm{d}y - B\,\mathrm{d}x)$$

so that the value of the double integral is completely determined by the values taken by function u on the contour l of domain B. If we fix the value of u on the contour l, the double integral over B will have the same value whatever the choice of function u.

Expressions of the form (34) may be termed expressions of a divergence type. It will be observed that, if we add to the integrand of any functional (25) an expression of a divergence type, this will have no effect whatever on Ostrogradskii's equation, i.e. the new functional will have the same Ostrogradskii equation as the previous one. This follows at once from the fact that the left-hand side of equation (27) is a linear homogeneous form in F and its partial derivatives.

We have seen that, if the integrand in integral (25) is an expression of a divergence type, Ostrogradskii's equation becomes an identity. The converse can also be proved.

If the integrand F contains partial derivatives of order higher than the first, condition (34) is, as above, necessary and sufficient for Ostrogradskii's equation (30) to reduce to an identity. But A and B may now contain partial derivatives of the same order as F. For instance:

$$F = u_{xx}\,u_{yy} - u_{xy}^2 = (u_x\,u_{yy})_x - (u_x\,u_{xy})_y,$$

and it is easily shown that equation (30) now reduces to an identity.

67. Examples. 1. We take functional (21), putting $v(x, y) = \sqrt{y}$:

$$J = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\,\mathrm{d}x. \tag{35}$$

The so-called brachistochrone problem leads to such a functional; the problem is to find the curve among all those joining two given points (x_0, y_0) and (x_1, y_1) such that a freely falling material particle traverses the curve in the shortest time. It is assumed that the y axis is directed vertically downwards,

i.e. in the direction of the force of gravity. The integrand in functional (35) does not contain x, and we can write down at once the first integral of Euler's equation:

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} = \frac{1}{\sqrt{C_1}},$$

or

$$y'^2 = \frac{C_1 - y}{y}. \tag{36}$$

On putting

$$y = \frac{C_1}{2}(1 - \cos u),$$

whence

$$y' = \frac{C_1}{2} u' \sin u,$$

we find after substituting in (36) and simplifying:

$$\frac{C_1}{2}(1 - \cos u)\,\mathrm{d}u = \pm\,\mathrm{d}x,$$

so that:

$$x = \pm\frac{C_1}{2}(u - \sin u) + C_2; \quad y = \frac{C_1}{2}(1 - \cos u).$$

It is clear from this that the extremals of functional (35) are cycloids. The constants C_1 and C_2 are found from the given initial and final points. If one of these points is the origin we must put $C_2 = 0$, and the origin is obtained when the parameter u is zero. It may be observed that the problem now has a special feature in that $y' = \mathrm{d}y/\mathrm{d}x$ becomes infinite at $u = 0$, as is easily shown, whilst the denominator in the integrand of (35) vanishes. If we return to the variable u in the integral, the singularity at $u = 0$ disappears.

2. Let (u, v) be coordinate parameters defining the position of a point on a surface, and let

$$\mathrm{d}s^2 = E(u, v)\,\mathrm{d}u^2 + 2F(u, v)\,\mathrm{d}u\,\mathrm{d}v + G(u, v)\,\mathrm{d}v^2$$

be the square of an elementary arc on the surface [II; 130].

The geodesics on the surface are defined as the curves given by the necessary condition for a minimum of the integral

$$\int_{u_0}^{u_1} \sqrt{E + 2Fv' + Gv'^2}\,\mathrm{d}u, \tag{37}$$

which expresses the length of the curve, v being assumed a function of u along the curve. Euler's equation becomes

$$\frac{1}{2}\frac{E_v + 2F_v v' + G_v v'^2}{\sqrt{E + 2Fv' + Gv'^2}} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} = 0.$$

We consider a sphere with centre at the origin and unit radius:

$$x = \sin\theta\cos\varphi; \quad y = \sin\theta\sin\varphi; \quad z = \cos\theta.$$

Now,

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2,$$

and integral (37) becomes

$$\int_{\theta_0}^{\theta_1} \sqrt{1 + \sin^2\theta\varphi'^2}\, d\theta,$$

where φ' is the derivative of φ with respect to θ. The integrand does not contain φ and we therefore have the solution:

$$\frac{\sin^2\theta\varphi'}{\sqrt{1+\sin^2\theta\varphi'^2}} = C_1.$$

Putting $C_1 = 0$, we get the obvious solution $\varphi = \text{const}$. In other words, the geodesics of a sphere are the meridians, i.e. the great circles passing through the poles of the sphere, at which $\theta = 0$ and π. In view of the arbitrary choice of pole, all the great circles of a sphere are obviously geodesics.

3. We consider the problem of geometrical optics in space:

$$J = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2+z'^2}}{v(y,z)}\, dx = \int_{x_0}^{x_1} n(y,z)\sqrt{1+y'^2+z'^2}\, dx \tag{38}$$

in the case when the velocity v or refractive index $n = 1:v$ do not depend on x.

If we write Euler's equations for integral (38) and solve them for y'' and z'', we get:

$$ny'' = n_y(1+y'^2+z'^2); \quad nz'' = n_z(1+y'^2+z'^2), \tag{39}$$

and it can easily be seen that the first integral is

$$n = C\sqrt{1+y'^2+z'^2}. \tag{40}$$

If n does not contain the variable y, the first of equations (39) gives $y'' = 0$, i.e. $y = C_1x + C_2$, so that every extremal is a plane curve lying in a plane parallel to the z axis. If we put $v = \sqrt{z}$, we get the brachistochrone problem in space, with gravity acting along z axis.

We now put $n = 1:z$, and consider the half-space at points of which z has a positive value. On substituting $a = 1:z$ and $y = C_1x + C_2$ in (40), we obtain a first order equation with separable variables for z:

$$\frac{Cz\, dz}{\sqrt{1-(1+C_1^2)C^2z^2}} = dx,$$

whence

$$(x-C_3)^2 + C_1^2(x-C_3)^2 + z^2 = \frac{1}{(1+C_1^2)C^2}.$$

On introducing the new arbitrary constant $C_4^2 = 1/[C^2(1 + C_1^2)]$ in place of C, and the constant $C_2' = C_2 + C_1 C_3$ in place of C_2, and observing that, since $y = C_1 x + C_2$:

$$C_1^2 = \frac{(y - C_2')^2}{(x - C_3)^2},$$

the previous expression can be rewritten as

$$(x - C_3)^2 + (y - C_2')^2 + z^2 = C_4^2. \tag{41}$$

The extremals of the integral

$$J = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2 + z'^2}}{z}\, \mathrm{d}x$$

are therefore semicircles, formed by the intersection of spheres (41) with centres in the $z = 0$ plane with the planes $y = C_1 x + C_2$ perpendicular to the $z = 0$ plane.

The result obtained can be given an interesting geometrical interpretation. If we define an element of length, i.e. a metric, in the half-space $z > 0$ by the expression

$$\mathrm{d}s = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}}{z},$$

integral (38) will express the length of a curve in this metric. In view of the presence of z in the denominator of the integrand, the length of a curve will increase indefinitely as it approaches the $z = 0$ plane, i.e. this plane will be the plane at infinity for a geometry in this metric.

The above-mentioned semicircles will play the role of straight lines in this geometry. In addition to these semicircles, we shall refer to the half-straight lines perpendicular to the $z = 0$ plane as straight lines in this geometry. These half-straight lines are degenerate cases of the semicircles. We shall call planes the hemispheres with centres in the $z = 0$ plane or the half-planes perpendicular to the $z = 0$ plane. Given these definitions of points, straight lines and planes, this new geometry is easily seen to satisfy the axioms of ordinary Euclidean geometry, apart from the axioms about parallel straight lines, i.e. we have a simple realization of a Lobachevskii geometry in the half-plane $z = 0$. We remark that x cannot be taken as the independent variable in the case of a straight line perpendicular to the $z = 0$ plane. In order not to restrict the choice of independent variable, we must seek the equations of extremals in the parametric form: $x(t)$, $y(t)$, $z(t)$. The integral given above now takes the form:

$$J = \int_{t_0}^{t_1} \frac{\sqrt{x_t^2 + y_t^2 + z_t^2}}{z}\, \mathrm{d}t.$$

We shall consider later the fundamental problem of the calculus of variations in the case when curves are specified parametrically. Given such a parametric specification, the half-lines mentioned above will also be extremals of the integral J.

In the plane case, we have an integral of the form

$$J = \int_{M_0}^{M_1} \frac{\sqrt{dx^2 + dy^2}}{y} = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{y}\, dx,$$

and the extremals becomes circles with centres on the x axis, or straight lines perpendicular to this axis. In the half-plane $y > 0$ the semicircles and half-lines mentioned will play the role of straight lines, and we have the realization of a Lobachevskii plane geometry in the semicircle mentioned. In particular, one and only one extremal will pass through any two points M_0 and M_1 of the semicircle.

4. Suppose we want to find the curve joining points M_0 and M_1 of the XY plane such that it forms the surface with least area when revolved about OX. The area of the surface of revolution is given by the integral [I, 106]:

$$S = 2\pi \int_{M_0}^{M_1} y\sqrt{dx^2 + dy^2} = 2\pi \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx,$$

and, on throwing away the factor 2π, we arrive at the extremal problem for the integral:

$$J = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx.$$

The integrand here does not contain x, and we can write solution (31) of Euler's equation:

$$y\sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} = C_1, \quad \text{whence} \quad \frac{C_1\, dy}{\sqrt{y^2 - C_1^2}} = dx.$$

Integration gives us

$$x - C_2 = C_1 \log\left(y + \sqrt{y^2 - C_1^2}\right) - C_1 \log C_1$$

or

$$y + \sqrt{y^2 - C_1^2} = C_1 e^{\frac{x - C_2}{C_1}},$$

and finally:

$$y = \frac{C_1}{2}\left(e^{\frac{x - C_2}{C_1}} + e^{-\frac{x - C_2}{C_1}}\right) = C_1 \cosh \frac{x - C_2}{C_1}.$$

The extremals are therefore catenaries, with axes of symmetry parallel to OY [I, 178]. It can be shown that, in the present problem, there is not always one and only one extremal passing through two given points M_0 and M_1. There may be two, one or no extremals, depending on the positions of the points.

As we saw above, the geodesics on a sphere are its great circles. If points M_0 and M_1 of the sphere are not the ends of the same diameter of the sphere, they can be joined by two arcs of one and only one great circle. Whereas if M_0 and M_1 are at the ends of the same diameter, they can be joined by an infinite set of semicircles of great circles.

Euler's equation is merely a necessary condition for an extremal of a functional, so that we cannot assert that any extremal obtained in fact provides an extremum of the functional. We shall indicate some sufficient conditions later. In the case of geodesics on a sphere, the minimum distance will be given by the lesser of the two arcs of a great circle joining points M_0 and M_1. There is no curve on the surface, joining points M_0 and M_1, which gives a maximum distance between the points. We can obviously always draw a curve as close as desired to a given curve joining M_0 and M_1 such that its length is greater.

5. We consider the problem of the extremum of the integral

$$J = \iint\limits_B \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy.$$

We have seen in [61] that the problem of finding the surface of least area stretched over a given contour leads to this problem. If we stretch a surface over the contour it is quite obvious that we can stretch another surface as close as desired to it over the same contour such that the second has a greater area. Thus in the present case the extremum of the integral can only be a minimum. On substituting the integrand in (27), we obtain the following second order differential equation for the required minimal surface:

$$r(1 + q^2) - 2spq + t(1 + p^2) = 0 \tag{42}$$

$$(p = u_x; \quad q = u_y; \quad r = u_{xx}, \quad s = u_{xy}, \quad t = u_{yy}).$$

We recall that the mean curvature of the surface is given by [II, 134]:

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{EN - 2FM + GL}{2(EG - F^2)}, \tag{43}$$

where $E, F, \ldots, M, N$ are the coefficients of the first and second Gaussian forms. When the equation of the surface is given explicitly, we have [II, 131]:

$$E = 1 + p^2; \quad F = pq; \quad G = 1 + q^2;$$

$$L = \frac{r}{\sqrt{1 + p^2 + q^2}}; \quad M = \frac{s}{\sqrt{1 + p^2 + q^2}}; \quad N = \frac{t}{\sqrt{1 + p^2 + q^2}},$$

and equation (42) expresses the fact that the mean curvature must be zero at all points of the minimal surface. This result was obtained earlier with the aid of variation of the elementary area of the surface [II, 139].

Equation (42) is a second order partial differential equation with two independent variables, analogous to Laplace's equation. We show that solutions of (42) can be obtained with the aid of analytic functions of a complex variable by a similar method to that used earlier to find solutions of Laplace's equation [III_2, 2]. It follows at once from (43) that we have $H = 0$, if the condition $E = G = N = 0$ is fulfilled for the surface. Let $\mathbf{r}$ be the radius vector of the surface, with components (x, y, z). The above condition can be written as [II, 130]:

$$\mathbf{r}_u'^2 = \mathbf{r}_v'^2 = \mathbf{r}_{uv}'' \cdot \mathbf{m} = 0,$$

where $\mathbf{m}$ is the unit vector normal to the surface. This condition is certainly satisfied if we subject $\mathbf{r}$ to the conditions:

$$\mathbf{r}_u'^2 = 0; \quad \mathbf{r}_v'^2 = 0; \quad \mathbf{r}_{uv}'' = 0.$$

The first two are scalar equations and the third a vector equation. They can be written in the developed form as:

$$\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2=0, \qquad \left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=0; \tag{44}$$

$$\frac{\partial^2 x}{\partial u\,\partial v}=\frac{\partial^2 y}{\partial u\,\partial v}=\frac{\partial^2 z}{\partial u\,\partial v}=0. \tag{45}$$

Equations (44) are obviously not fulfilled if the derivatives of the coordinates with respect to u and v have real values. Suppose that the coordinates are analytic functions of the complex variables u and v.

Equations (45) show that (x, y, z) must be expressible as sums of functions of u only and v only [II, 164]:

$$x=\varphi_1(u)+\psi_1(v);\quad y=\varphi_2(u)+\psi_2(v);\quad z=\varphi_3(u)+\psi_3(v), \tag{46}$$

where, by (44), we must have

$$\sum_{s=1}^{3}\varphi_s'^2(u)=0;\qquad \sum_{s=1}^{3}\psi_s'^2(v)=0.$$

We put $u=\varrho+\sigma i$. In order to have a real surface, we suppose that $\psi_s(v)$ has values which are the complex conjugates of $\varphi_s(u)$. To be more precise, we shall assume that $v=\varrho-\sigma i$ and that functions $\psi_s(v)$ have complex conjugate values with respect to $\varphi_s(u)$ at points symmetric with respect to the real axis. Expressions (46) now become

$$x=2\mathcal{R}\varphi_1(u);\qquad y=2\mathcal{R}\varphi_2(u);\qquad z=2\mathcal{R}\varphi_3(u),$$

where R indicates the real part. By including the factor 2 under the sign of the real part and of the functional relationship, we can rewrite our expressions as

$$x=\mathcal{R}\varphi_1(u);\qquad y=\mathcal{R}\varphi_2(u);\qquad z=\mathcal{R}\varphi_3(u), \tag{47}$$

where the analytic functions $\varphi_s(u)$ must satisfy

$$\sum_{s=1}^{3}\varphi_s'^2(u)=0. \tag{48}$$

In the parametric representation (47) the roles of the real parameters are played by ϱ and σ, i.e. the real and imaginary parts of the complex variable u. We can take one of the functions $\varphi_s(u)$ as the independent variable. For example, we can put $t=\varphi_3(u)$ and assume that the first two functions are functions of the complex variable t. They must be connected by the relationship:

$$\varphi_1'^2(t)+\varphi_2'^2(t)+1=0.$$

The set of minimal surfaces obtained by us is thus seen to depend on a single analytic function. For instance, we can write:

$$x=\mathcal{R}\varphi_1(t);\qquad y=\mathcal{R}i\int\sqrt{1+\varphi_1'^2(t)}\,dt;\qquad z=\mathcal{R}t,$$

where $\varphi_1(t)$ is an arbitrary analytic function of the complex variable t.

We can write the expressions more symmetrically as

$$\left.\begin{aligned} x &= \mathcal{R}i\left[f(u) - uf'(u) - \frac{1-u^2}{2}f''(u)\right], \\ y &= \mathcal{R}\left[f(u) - uf'(u) + \frac{1+u^2}{2}f''(u)\right], \\ z &= \mathcal{R}i\,[f'(u) - uf''(u)], \end{aligned}\right\} \tag{49}$$

where $f(u)$ is an arbitrary analytic function. It is easily shown that the function under the sign of the real part in fact satisfies relationship (48).

6. Let us consider the functional

$$D(u) = \int\int_B (u_x^2 + u_y^2)\,dx\,dy, \tag{50}$$

where B is a bounded domain of the (x, y) plane. By (27), Ostrogradskii's equation has the form for this functional:

$$u_{xx} + u_{yy} = 0,$$

i.e. we obtain Laplace's equation. We can justifiably expect that functional (50) is minimized by the harmonic function, having the given boundary values on the contour l of domain B, rather than by any other function continuous in the closed domain B, having continuous first order partial derivatives inside B, and taking on l the same boundary values as the harmonic function. We have no strict proof of this assertion, however, since Ostrogradskii's equation only gives the necessary condition for an extremum, apart from which we must remember that the existence of continuous second order derivatives of the required function was assumed when deducing this equation. We shall assume that B is a circle with centre at the origin and unit radius.

We know that, given any continuous values on the contour, there exists a unique harmonic function v solving the Dirichlet problem for the given contour values. But we can by no means assert how the first derivatives of this function behave on approaching the contour, so that we cannot assert that functional (50) has a finite value for the harmonic function obtained. It turns out, in fact, that continuous boundary conditions can be specified on the contour such that functional (50) is equal to $(+\infty)$ for the harmonic function in question, i.e. to be more precise, if we take integral (50) over a concentric circle C_r of radius r, less than unity, as $r \to 1$ this integral increases indefinitely. It can be shown that in this case (50) becomes $(+\infty)$ for any function with continuous first order derivatives and the same boundary values.

In general, the following theorem holds: *if, given the boundary values on the contour l, functional* (50) *has a finite value for some function u, it also has a finite value for the harmonic function v with the same boundary values, where $D(v) < D(u)$ and the sign of equality only holds when u coincides with v.*

The proof of this proposition will be given below; we shall prove it for the time being with the supplementary assumption that the harmonic function v has bounded first order partial derivatives inside the circle. Integral (50) obviously has a finite value for this function. We can write the function u as $u = v + \varphi$, where φ vanishes on the boundary and has continuous first order

partial derivatives inside the domain. Functional (50) can be written for this function as

$$D(v+\varphi) = D(v) + D(\varphi) + 2\int\int_B (v_x\varphi_x + v_y\varphi_y)\,\mathrm{d}x\,\mathrm{d}y. \tag{51}$$

We apply Green's formula to the circle B_r with radius $r < 1$:

$$\int\int_{B_r} (v_x\varphi_x + v_y\varphi_y)\,\mathrm{d}x\,\mathrm{d}y = -\int\int_{B_r} \varphi\,\Delta v\,\mathrm{d}x\,\mathrm{d}y + \int_{C_r} \varphi\frac{\partial v}{\partial n}\,\mathrm{d}s.$$

Since v is a harmonic function, the double integral on the right-hand side vanishes, whilst in the line integral over the circle C_r of radius $r < 1$, when r tends to unity φ tends to zero uniformly with respect to the polar angle and $\partial v/\partial n$ remains bounded; the latter integral thus obviously vanishes in the limit. The integral on the left must therefore also vanish, and (51) can be written as:

$$D(v+\varphi) = D(v) + D(\varphi).$$

But we obviously have $D(\varphi) \geqslant 0$, where the sign of equality can only hold when φ vanishes identically in the circle B. Hence we in fact have $D(v) \leqslant D(u)$, where the sign of equality only holds when u coincides with v.

68. Isoperimetric problems. Let us recall the problem of the relative extrema in the case of functions of several variables [I, 167]. A precisely similar statement can be given in the calculus of variations of the problem of the extrema of a functional on the assumption that the required function must satisfy certain supplementary relationships. In particular, let us pose the problem: among all the curves $y(x)$ for which the integral

$$J_1 = \int_{x_0}^{x_1} G(x, y, y')\,\mathrm{d}x = a \tag{52}$$

has a given value a, find the one that gives the extremum of the integral

$$J = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x. \tag{53}$$

This is usually known as the *isoperimetric problem*. The name owes its origin to the particular problem of this type in which it is required to find the closed curve of given length a which includes the greatest area (the circle). The problem may be reduced to an ordinary problem (without a constraint) of the calculus of variations with the aid of the following theorem:

EULER'S THEOREM. *If the curve $y(x)$ yields an extremum of integral* (53) *under the constraint* (52) *and the usual boundary conditions* (8), *and if $y(x)$ is not an extremal of integral* (52), *there exists a constant λ such that $y(x)$ is an extremal of the integral*

$$\int_{x_0}^{x_1} H(x, y, y')\,\mathrm{d}x, \quad \text{where} \quad H = F + \lambda G. \tag{54}$$

We bring into the discussion a function close to $y(x)$:

$$y(x) + \alpha_1 \eta_1(x) + \alpha_2 \eta_2(x), \tag{55}$$

where α_1 and α_2 are small parameters, whilst $\eta_1(x)$ and $\eta_2(x)$ are functions with the usual properties and vanishing at the ends of the interval of integration. We substitute this function in integral (52):

$$J_1(\alpha_1, \alpha_2) = \int_{x_0}^{x_1} G(x,\ y + \alpha_1\eta_1 + \alpha_2\eta_2,\ y' + \alpha_1\eta_1' + \alpha_2\eta_2')\,\mathrm{d}x.$$

We can write with the aid of the usual working:

$$\left.\frac{\partial J_1}{\partial \alpha_i}\right|_{\alpha_1=\alpha_2=0} = \int_{x_0}^{x_1} \left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right)\eta_i\,\mathrm{d}x \qquad (i = 1, 2).$$

Since $y(x)$ is not an extremal of integral (52), the difference $G_y - (\mathrm{d}/\mathrm{d}x)\, G_{y'}$, is not identically zero in the interval (x_0, x_1), and a function $\eta_2(x)$ can obviously be chosen so that the integral

$$\int_{x_0}^{x_1} \left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right)\eta_2\,\mathrm{d}x$$

differs from zero.

We return to the equation $J_1(\alpha_1, \alpha_2) = a$. It is satisfied for $\alpha_1 = \alpha_2 = 0$, since $y(x)$ is a solution of ourproblem by hypothesis, and, in view of our choice of η_2, the partial derivative of $J_1(\alpha_1, \alpha_2)$ with respect to α_2 differs from zero for $\alpha_1 = \alpha_2 = 0$. Hence, by the implicit function theorem [I, 159], the equation $J_1(\alpha_1, \alpha_2) = a$ defines α_2 as a function of α_1 for all α_1 sufficiently close to zero, the derivative of α_2 with respect to α_1 being evidently given at $\alpha_1 = 0$ by

$$\left.\frac{\mathrm{d}\alpha_2}{\mathrm{d}\alpha_1}\right|_{\alpha_1=0} = -\int_{x_0}^{x_1} \left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right)\eta_1\,\mathrm{d}x : \int_{x_0}^{x_1} \left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right)\eta_2\,\mathrm{d}x = k. \tag{56}$$

We substitute function (55) in integral (53) and differentiate the integral obtained with respect to α_1, remembering that α_2 is a function of α_1:

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha_1}\bigg|_{\alpha_1=0} = \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \eta_1 \, \mathrm{d}x + k \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \eta_2 \, \mathrm{d}x.$$

By using expression (56) for the constant k, we can write

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha_1}\bigg|_{\alpha_1=0} = \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \eta_1 \, \mathrm{d}x + \lambda \int_{x_0}^{x_1} \left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right) \eta_1 \, \mathrm{d}x,$$

where

$$\lambda = - \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \eta_2 \, \mathrm{d}x : \int_{x_0}^{x_1} \left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right) \eta_2 \, \mathrm{d}x,$$

or

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha_1}\bigg|_{\alpha_1=0} = \int_{x_0}^{x_1} \left[\left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) + \lambda\left(G_y - \frac{\mathrm{d}}{\mathrm{d}x} G_{y'}\right)\right] \eta_1 \, \mathrm{d}x.$$

Since $y(x)$ gives an extremum of integral (53) subject to the constraint (52), we must have $\mathrm{d}J/\mathrm{d}\alpha_1 |_{\alpha_1=0} = 0$, whence, on recalling that $\eta_1(x)$ is arbitrary, using the fundamental lemma, and setting $F + \lambda G = H$, we obtain the equation

$$H_y - \frac{\mathrm{d}}{\mathrm{d}x} H_{y'} = 0,$$

which is Euler's equation for integral (54). The general solution of this equation will contain three arbitrary constants, namely: two constants of integration and the constant λ. These constants must be determined from the two boundary conditions and condition (52).

A point should be noted regarding the result obtained. On multiplying the integrand of (53) by an arbitrary constant the extremals of the integral obviously remain as before, and in view of this we can write the function H in the symmetric form $H = \lambda_1 F + \lambda_2 G$, where λ_1 and λ_2 are constants. Since F and G appear symmetrically in the expression for H, we can say that the same extremals are obtained if we seek the extremum of integral (53) subject to the condition that integral (52) retains a constant value, as are obtained if we seek the extremum of integral (52) subject to the condition that integral (53) retains a constant value. This is the so-called *reciprocity principle* in its simplest form. We are assuming here that constants λ_1 and λ_2 differ from zero, i.e. we exclude the curves which are extremals of integral (52) or (53).

We shall explain by means of examples the meaning of the requirement in Euler's theorem that $y(x)$ must not be an extremal of integral (52). The isoperimetric problem has the following form in the more general case: to find the functions $y_i(x)$ $(i = 1, 2, \ldots, n)$ yielding the extremum of the integral

$$J = \int_{x_0}^{x_1} F(x, y_1, y_1', \ldots, y_n, y_n')\,dx,$$

subject to the constraints:

$$\int_{x_0}^{x_1} G_s(x, y_1, y_1', \ldots, y_n, y_n')\,dx = a_s \quad (s = 1, 2, \ldots, p)$$

and the boundary conditions:

$$y_i(x_0) = y_i^{(0)}; \quad y_i(x_1) = y_i^{(1)} \quad (i = 1, 2, \ldots, n).$$

When there is also a supplementary condition that guarantees as above the applicability of the implicit function theorem, we can say that the functions $y_i(x)$ yielding the solution of the problem must be extremals of the integral

$$\int_{x_0}^{x_1} H(x, y_1, y_1', \ldots, y_n, y_n')\,dx, \text{ where } H = F + \sum_{\Omega=1}^{p} \lambda_s G_s$$

and the λ_s are constants. The proof of this assertion is similar to the above. We remark that the number of constraints p may in fact exceed the number of required functions n.

69. Conditional extremum. We shall now discuss problems in which the supplementary constraints have a different form from (52). We start with the simplest case: to find functions $y(x)$ and $z(x)$ yielding an extremum of the integral

$$J = \int_{x_0}^{x_1} F(x, y, y', z, z')\,dx \tag{57}$$

and satisfying the equation

$$G(x, y, z) = 0 \tag{58}$$

together with the boundary conditions for the fixed ends:

$$y(x_0) = y_0; \qquad z(x_0) = z_0,$$
$$y(x_1) = y_1; \qquad z(x_1) = z_1,$$

where the coordinates (x_0, y_0, z_0) and (x_1, y_1, z_1) must evidently satisfy equation (58).

This amounts geometrically to finding the curves on the surface (58) giving an extremum of integral (57). It would be possible to find z from (58) as a function of x and y and sustitute this function in integral (57); we should then arrive at an ordinary problem of the calculus of variations with one required function $y(x)$ and no constraints. Let us use this idea to deduce the equation which must be satisfied by functions $y(x)$ and $z(x)$ that provide the solution of the problem. We shall assume that the partial derivative G_z is non-zero along this solution. Equation (58) will now be soluble with respect to z, and we obtain $z = \varphi(x, y)$. After substituting this expression in integral (57), this latter becomes

$$J = \int_{x_0}^{x_1} F(x, y, y', \varphi, \varphi_x + \varphi_y y')\, \mathrm{d}x. \tag{59}$$

The plane curve l, formed by the projection of our spatial curve on the (x, y) plane, must yield an extremum of integral (59) with fixed ends, and must therefore satisfy Euler's equation for this integral. Let us carry out the preliminary working for forming this equation. We shall write $[F]$ for the integrand in (59). This function depends on (x, y, y'). We shall write F without the square brackets for the previous function $F(x, y, y', z, z')$, so that $[F]$ is obtained from F as a result of the substitutions $z = \varphi(x, y)$ and $z' = \varphi_x + \varphi_x y'$. We have:

$$\frac{\partial [F]}{\partial y} = F_y + F_z \varphi_y + F_{z'} (\varphi_{xy} + \varphi_{yy} y'); \frac{\partial [F]}{\partial y'} = F_{y'} + F_{z'} \varphi_y;$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial [F]}{\partial y'} = \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} + \varphi_y \frac{\mathrm{d}}{\mathrm{d}x} F_{z'} + F_{z'} (\varphi_{xy} + \varphi_{yy'}).$$

Euler's equation for integral (59):

$$\frac{\partial [F]}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial [F]}{\partial y'} = 0$$

becomes by virtue of the above expressions:

$$F_y + \varphi_y \left(F_z - \frac{\mathrm{d}}{\mathrm{d}x} F_{z'} \right) - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0.$$

On the other hand, differentiation of equation (58) with respect to y gives:

$$G_y + G_z \varphi_y = 0,$$

and elimination of φ_y from the last two equations leads us to

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} F_{y'} - F_y \right) : G_y = \left(\frac{\mathrm{d}}{\mathrm{d}x} F_{z'} - F_z \right) : G_z.$$

Along the extremals, both sides of this last equation represent the same function of x, which we shall write as $\lambda(x)$, so that we now have:

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{y'} - [F_y + \lambda(x) G_y] = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{z'} - [F_z + \lambda(x) G_z] = 0.$$

These are the necessary conditions for an extremum. It is easily seen that they can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x} F^*_{y'} - F^*_y = 0; \quad \frac{\mathrm{d}}{\mathrm{d}x} F^*_{z'} - F^*_z = 0, \tag{60}$$

where

$$F^* = F + \lambda(x) G, \tag{61}$$

i.e. the extremals of the problem must be unconditional extremals of the functional with integrand F^* given by (61). We remark that, in the present case, we have the factor $\lambda(x)$, i.e. a function of x, instead of the constant factor λ that figured in the isoperimetric problem. After eliminating the function $\lambda(x)$ and one of the required functions, say z, from (58) and (60), we obtain a second order differential equation in the single function $y(x)$. The two arbitrary constants obtained when this is solved must be determined from the two boundary conditions.

The above discussion can be carried over to problems of a more general type with any number of required functions and constraints, the number of constraints being less than the number of required functions in any given case. The problem of finding the extrema of the integral

$$\int_{x_0}^{x_1} F(x, y_1, y_1', \ldots, y_n, y_n')\,\mathrm{d}x \tag{62}$$

subject to the constraints:

$$G_s(x, y_1, \ldots, y_n) = 0 \quad (s = 1, 2, \ldots, p) \tag{63}$$

and the boundary conditions:

$$y_i(x_0) = y_i^{(0)}; \quad y_i(x_1) = y_i^{(1)} \quad (i = 1, 2, \ldots, n) \tag{64}$$

leads to the equations

$$\frac{\mathrm{d}}{\mathrm{d}x} F^*_{y_i'} - F^*_{y_i} = 0 \quad (i = 1, 2, \ldots, n), \tag{65}$$

where

$$F^* = F + \sum_{s=1}^{k} \lambda_s(x) G_s \tag{66}$$

and $\lambda_s(x)$ are functions of x.

It is assumed here that at least one of the functional determinants of order p, formed from the partial derivatives $\partial G_s/\partial y_i$, differs from zero when the functions yielding an extremum of integral (62) are substituted for the y_i.

Constraints (63), which contain no derivatives of the required functions, are usually called *holonomic constraints*. The assertion made above is still valid for non-holonomic constraints of the form

$$G_s(x, y_1, y_1', \ldots, y_n, y_n') = 0 \quad (s = 1, 2, \ldots, p) \tag{67}$$

i.e. given certain supplementary conditions, the functions y_i yielding an extremum of integral (62) subject to constraints (67) must satisfy the equations

$$\frac{\mathrm{d}}{\mathrm{d}x} F^*_{y_i'} - F^*_{y_i} = 0 \quad (i = 1, 2, \ldots, n), \tag{68}$$

where

$$F^* = F + \sum_{s=1}^{k} \lambda_s(x) G_s(x, y_1, y_1', \ldots, y_n, y_n'). \tag{69}$$

System (68) differs in one essential from the analogous system for the case of holonomic constraints. Since functions (67) contain in the present case the derivatives y_i', the functions $F^*_{y_i'}$ will contain $\lambda_s(x)$, and equations (68) will contain the derivatives of the $\lambda_s(x)$ with respect to x. Finally, equations (67) and (68) give a system of $(n + p)$ differential equations with $(n + p)$ unknown functions y_i and $\lambda_s(x)$, of the second order in y_i and of the first order in $\lambda_s(x)$.

We bring in functions $z_i(x)$ given by

$$z_i(x) = y_i'(x) \quad (i = 1, 2, \ldots, n). \tag{70}$$

After substitution, equations (67) give p holonomic constraints for the functions y_i and z_i, whilst (68) and (70) become a system of $2n$ equations of the first order with $(2n + p)$ functions y_i, z_i and $\lambda_s(x)$. After solving (67) for p of the y_i and z_i and substituting these expressions in (68) and (70), we obtain $2n$ first order equations for $2n$ of the y_i, z_i, $\lambda_s(x)$. The general solution of this system will contain $2n$ arbitrary constants, which must be determined from the $2n$ boundary conditions.

70. Examples. 1. Among all the curves of length l joining two given points A and B, find the curve which, in conjunction with the straight line AB, contains the greatest area. We take the x axis through A and B and let x_0, x_1 be the abscissae of these points. We assume that, for the required curve, y is a single-valued function of x in the interval $[x_0, x_1]$. The problem amounts to finding the greatest value of the integral

$$\int_{x_0}^{x_1} y\,dx \tag{71}$$

subject to the constraint:

$$\int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx = l. \tag{72}$$

The latter integral gives the length of the curve $y(x)$ between the points $x = x_0$ and $x = x_1$. Its extremals will obviously be straight lines. This can be verified at once by forming the Euler equation for the integral. If $l < x_1 - x_0$, there is no curve satisfying (72). If $l = x_1 - x_0$, condition (72) is satisfied only by the straight line AB. The problem is meaningless in both cases and we shall assume in future that $l > x_1 - x_0$. Here:

$$F^* = y + \lambda\sqrt{1+y'^2},$$

and F^* does not contain x, so that the first integral of the corresponding Euler equation will be:

$$F^* - y'F^*_{y'} = y + \lambda\sqrt{1+y'^2} - \frac{\lambda y'^2}{\sqrt{1+y'^2}} = b,$$

whence

$$y' = \frac{\sqrt{\lambda^2-(y-b)^2}}{y-b}$$

or

$$\frac{(y-b)\,dy}{\sqrt{\lambda^2-(y-b)^2}} = dx;$$

integration gives

$$(x-a)^2 + (y-b)^2 = \lambda^2,$$

i.e. the extremals are circles of radius $|\lambda|$.

Let ω be the angle subtended at the centre of a circle by AB, i.e.

$$x_1 - x_0 = 2\lambda \sin\frac{\omega}{2} \quad\text{and}\quad l = \lambda\omega.$$

We have the equation for ω:

$$\sin\frac{\omega}{2} : \frac{\omega}{2} = (x_1 - x_0) : l,$$

which always has a solution, given the above condition. By using the reciprocity principle, we can say that, among the curves bounding an area of given size,

the arc of a circle has an extremal (obviously maximum) length. We remark further that, if $l > \pi(x_1 - x_0)/2$, y will not be a single-valued function of x.

It can be shown by using the result obtained that, if a closed curve of given length bounds a maximum area, the curve must be a circle.

2. It is required to find the position of equilibrium of a heavy homogeneous cord of length l with fixed ends under the action of gravity. Let gravity be directed along the negative y axis. The equilibrium position is defined by the requirement that the centre of gravity of the cord be as low as possible. We shall take it as obvious that any straight line parallel to the y axis meets the cord in not more than one point. The problem amounts to finding the extremum of the integral [cf. example 4 of sec. 67]:

$$\int_{x_0}^{x_1} y\,\mathrm{d}s = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,\mathrm{d}x$$

subject to the constraint:

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\,\mathrm{d}x = l \tag{73}$$

and the boundary conditions: $y(x_0) = y_0$, $y(x_1) = y_1$. Here,

$$F^* = y\sqrt{1 + y'^2} + \lambda\sqrt{1 + y'^2},$$

and the first integral of Euler's equation becomes

$$\frac{y + \lambda}{\sqrt{1 + y'^2}} = a \quad \text{or} \quad \frac{\mathrm{d}y}{\sqrt{(y + \lambda)^2 - a^2}} = \frac{\mathrm{d}x}{a}.$$

If we put

$$y + \lambda = a\cosh z = a\,\frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2},$$

the equation is easily solved and gives:

$$y + \lambda = a\cosh\left(\frac{x}{a} + b\right) = a\,\frac{\mathrm{e}^{\frac{x}{a} + b} + \mathrm{e}^{-\left(\frac{x}{a} + b\right)}}{2} \quad (a > 0),$$

i.e. the extremals are catenaries. The constants a, b and λ must be determined from the boundary conditions:

$$y_0 + \lambda = a\cosh\left(\frac{x_0}{a} + b\right); \quad y_1 + \lambda = a\cosh\left(\frac{x_1}{a} + b\right)$$

and condition (73). On subtracting one boundary condition from the other and transforming the difference between hyperbolic cosines into a product, we get:

$$y_1 - y_0 = 2a\sinh\mu\sinh\nu, \tag{74}$$

where

$$\mu = \frac{x_1 + x_0}{2a} + b; \quad \nu = \frac{x_1 - x_0}{2a}.$$

After substituting the value found for y in condition (73), this latter becomes

$$a\left[\sinh\left(\frac{x_1}{a}+b\right)-\sinh\left(\frac{x_0}{a}+b\right)\right]=l,$$

or

$$2a\cosh\mu\sinh\nu = l. \tag{75}$$

We obtain, by (74):

$$\tanh\mu = \frac{y_1-y_0}{l}. \tag{76}$$

The number l must obviously satisfy the inequality

$$l > \sqrt{(x_1-x_0)^2+(y_1-y_0)^2} > |y_1-y_0|,$$

and equation (76) has a unique root. We obtain from (74) and (75):

$$\sqrt{l^2-(y_1-y_0)^2} = 2a\sinh\nu,$$

or

$$\frac{\sinh\nu}{\nu} = \frac{\sqrt{l^2-(y_1-y_0)^2}}{x_1-x_0}. \tag{77}$$

But

$$\frac{\sinh x}{x} = 1+\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots$$

is a monotonic function increasing from l to $(+\infty)$ for $0 \leqslant x < \infty$, taking every value greater than unity once and only once, so that equation (77) has a unique positive root. After ν and μ have been found, there is no difficulty in obtaining a, b and λ.

3. We consider an elastic homogeneous rod which is straight in the non-deformed state. We know from the theory of elasticity that its potential energy in the deformed state is proportional to the integral over the rod of the square of its curvature. Let the rod, of length l, be constrained at the points (x_0, y_0) and (x_1, y_1). We take as the independent variable the length of rod s measured from (x_0, y_0), and write $\theta(s)$ for the angle formed by the tangent to the rod with the x axis. The curvature is given by the derivative $\theta'(s)$, and the integral whose extremum we require will be

$$\int_0^l \theta'^2\,\mathrm{d}s. \tag{78}$$

As we know,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta,$$

and the two connecting equations are obtained:

$$\int_0^l \cos\theta\,\mathrm{d}s = x_1-x_0; \quad \int_0^l \sin\theta\,\mathrm{d}s = y_1-y_0. \tag{79}$$

In addition, fixing the rod at its ends is equivalent to specifying $\theta(s)$ for $s = 0$ and $s = l$:

$$\theta(0) = a; \quad \theta(l) = b. \tag{80}$$

In the present case:

$$F^* = \theta'^2 + \lambda_1 \cos \theta + \lambda_2 \sin \theta,$$

and this function does not contain the independent variable s, so that the first integral of Euler's equation is obtained directly:

$$\theta'^2 = C + \lambda_1 \cos \theta + \lambda_2 \sin \theta.$$

We bring in the two new constants:

$$h = C + \sqrt{\lambda_1^2 + \lambda_2^2}; \quad k^2 = \frac{2\sqrt{\lambda_1^2 + \lambda_2^2}}{C + \sqrt{\lambda_1^2 + \lambda_2^2}},$$

and the new variable φ in place of θ:

$$\varphi = \frac{\theta - \theta_0}{2},$$

where we have put $\theta_0 = \arctan \lambda_2/\lambda_1$. The above first integral of Euler's equation becomes in the new notation:

$$\frac{d\varphi}{ds} = \frac{\sqrt{h}}{2}\sqrt{1 - k^2 \sin^2 \varphi},$$

whence we obtain s in terms of φ as an elliptic integral:

$$s = \frac{2}{\sqrt{h}} \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} + s_0.$$

The constants k, h, θ_0 and s_0 have to be determined from conditions (79) and (80). We obtain the Cartesian coordinates of the points of the rod from the relationships

$$\frac{dx}{ds} = \cos \theta = \cos (2\varphi + \theta_0); \quad \frac{dy}{ds} = \sin \theta = \sin (2\varphi + \theta_0)$$

simply by using the expression for ds, which gives:

$$dx = \frac{2 \cos (2\varphi + \theta_0)}{\sqrt{h}\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi; \quad dy = \frac{2 \sin (2\varphi + \theta_0)}{\sqrt{h}\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi,$$

whence x and y can now be found with the aid of quadratures.

4. We consider the problem of finding the geodesics on a given surface:

$$G(x, y, z) = 0. \tag{81}$$

This amounts to finding the extrema of the integral

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx$$

subject to the constraint (81). Equations (60) become in the present case:

$$\frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2 + z'^2}} - \lambda G_y = 0; \quad \frac{d}{dx} \frac{z'}{\sqrt{1 + y'^2 + z'^2}} - \lambda G_z = 0. \tag{82}$$

In order to see the fundamental geometrical property of geodesics, we totally differentiate equation (81) with respect to x:

$$G_x + G_y y' + G_z z' = 0.$$

On multiplying both sides by λ and replacing λG_y and λG_z by their expressions from (82), we arrive after some rearrangement at:

$$\frac{d}{dx}\frac{1}{\sqrt{1+y'^2+z'^2}} - \lambda G_x = 0,$$

which is analogous to equations (82), the fractions under the sign of differentiation with respect to x being equal to the direction cosines of the required geodesic, so that we can rewrite these equations as

$$\frac{d\cos\alpha}{dx} = \lambda G_x;\quad \frac{d\cos\beta}{dx} = \lambda G_y;\quad \frac{d\cos\gamma}{dx} = \lambda G_z.$$

By using the formula $dx/ds = \cos\alpha$, we can replace differentiation with respect to x by differentiation with respect to s and hence obtain:

$$\frac{d\cos\alpha}{ds} = \mu G_x;\quad \frac{d\cos\beta}{ds} = \mu G_y;\quad \frac{d\cos\gamma}{ds} = \mu G_z,$$

where $\mu = \lambda\cos\alpha$. But, as we know from [II, 125], the left-hand sides of these last equations are proportional to the direction cosines of the principal normal to the curve, and the right-hand sides to the direction cosines of the normal to the surface, whence it follows at once that *the principal normal along a geodesic is at the same time the normal to the surface.*

5. Let us take the brachistochrone problem in a resistive medium: among the curves joining two given points A, B, find the curve such that a material particle descending it with given initial velocity takes the least time; the medium has a resistance expressed by a given function $R(v)$ of the velocity.

It follows at once from mechanical considerations that the required curve must lie in the plane through the straight line AB and the vertical through A. We take this plane as the (x, y) plane and the y axis vertically downwards. Let (x_0, y_0) and (x_1, y_1) be the coordinates of A and B. The increment in the kinetic energy during motion along the curve will be due to the positive gravity force and the negative work of the resistance, i.e.

$$d\frac{v^2}{2} = g\,dy - R(v)\,ds,$$

where g is the acceleration due to gravity and $ds = \sqrt{dx^2 + dy^2}$. Taking x as the independent variable for the functions v and y, we have:

$$vv' - gy' + R(v)\sqrt{1+y'^2} = 0. \tag{83}$$

and the problem amounts to finding the extrema of the integral

$$\int\limits_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v}\,dx$$

subject to the non-holonomic constraint (83), v and y being the required functions.

The usual boundary conditions amount to specifying the functions at the ends of the interval:

$$y(x_0) = y_0; \qquad y(x_1) = y_1, \tag{84}$$

$$v(x_0) = v_0; \qquad v(x_1) = v_1. \tag{85}$$

The first of conditions (85) is equivalent to specifying the velocity of departure of the particle from the point A. The second amounts to specifying the velocity at the final point of the curve and is not unique from the mechanical point of view. We shall return to this question later. Following the usual method, we must write the Euler equation for the functions:

$$F^* = \sqrt{1 + y'^2}\, H + \lambda(x)\, vv' - \lambda(x)\, gy', \tag{86}$$

where

$$H = \frac{1}{v} + \lambda(x)\, R(v).$$

The function F^* does not contain y and its Euler equation with respect to y has the obvious first integral $F^*_{y'} = C$, or

$$\frac{Hy}{\sqrt{1 + y'^2}} = C + \lambda(x)\, g, \tag{87}$$

whilst the Euler equation for F^* with respect to the function v will be

$$\sqrt{1 + y'^2}\, H_v + \lambda(x)\, v' - \frac{\mathrm{d}}{\mathrm{d}x}[\lambda(x)\, v] = 0,$$

or

$$\frac{v\lambda'(x)}{\sqrt{1 + y'^2}} = H_v. \tag{88}$$

We thus have a system of three equations (83), (87) and (88) for the functions y, v and λ. By differentiating directly the difference $H^2 - (C + g\lambda)^2$ with respect to x and using the above three equations, the existence may be seen of the following solution:

$$H^2 - (C + g\lambda)^2 = a^2, \tag{89}$$

where a is a new arbitrary constant. We can find λ as a function of v from the last equation: $\lambda = \lambda(v)$. On dividing (87) by (88), we get

$$\mathrm{d}y = \frac{(C + g\lambda)\, v\, \mathrm{d}\lambda}{HH_v}. \tag{90}$$

By (87) and (89):

$$y' = \frac{C + g\lambda}{a},$$

whence

$$\mathrm{d}x = \frac{av\, \mathrm{d}\lambda}{HH_v}. \tag{91}$$

On substituting $\lambda = \lambda(v)$ in the right-hand sides of (90) and (91) and carrying out the quadratures, we have:

$$x = d + \varphi(v, a, C); \quad y = e + \psi(v, a, C),$$

where d and e are arbitrary constants. The last two equations give the parametric form of the required brachistochrone, with v playing the part of parameter. The arbitrary constants have to be determined from boundary conditions (84) and (85). As we shall see later, the last of conditions (85) must be replaced by the condition:

$$F^*_{v'}\big|_{x=x_1} = 0,$$

which expresses the fact that the velocity v can have arbitrary values at $x = x_1$. By (86), the above condition has the form $\lambda v\,|_{x=x_1} = 0$. On assuming the velocity different from zero, we obtain $\lambda\,|_{x=x_1} = 0$.

71. Invariance of the Euler and Ostrogradskii equations. When seeking the extrema of a function of a single variable $y = f(x)$, we can change the independent variable from x to ξ, where $x = \varphi(\xi)$ and $\varphi(\xi)$ is assumed to be monotonic and to have a non-zero derivative. The differentiation rule for a function of a function gives:

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = f'(x)\,\varphi'(\xi). \tag{92}$$

The necessary condition for an extremum in the new independent variable will be $f'(x)\,\varphi'(\xi) = 0$, and, since $\varphi'(\xi) \neq 0$, this new condition is equivalent to the earlier $f'(x) = 0$. A formula analogous to (92) can be obtained for the left-hand side of Euler's equation in various cases. We start by considering the elementary functional

$$J = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x \tag{93}$$

and introduce for brevity a special notation for the left-hand side of Euler's equation:

$$[F]_y = F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}.$$

On introducing the new independent variable ξ, we can write:

$$F(x, y, y') = F\left[x(\xi),\ y,\ \frac{\mathrm{d}y/\mathrm{d}\xi}{\mathrm{d}x/\mathrm{d}\xi}\right] = \Phi\left(\xi, y, \frac{\mathrm{d}y}{\mathrm{d}\xi}\right),$$

and the integral J becomes in the new independent variable:

$$\int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x = \int_{\xi_0}^{\xi_1} \Phi\left(\xi, y, \frac{\mathrm{d}y}{\mathrm{d}\xi}\right)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi.$$

We introduce the neighbouring function $y + a\eta$ and carry out the usual working to obtain

$$\frac{\partial}{\partial a}\int\limits_{x_0}^{x_1} F(x, y + a\eta, y' + a\eta')\,\mathrm{d}x\,\big|_{a=0} = \int\limits_{x_0}^{x_1} [F]_y\,\eta\,\mathrm{d}x.$$

This expression in the new independent variable can also be written as

$$\frac{\partial}{\partial a}\int\limits_{\xi_0}^{\xi_1} \Phi\left(\xi, y + a\eta, \frac{\mathrm{d}y}{\mathrm{d}\xi} + a\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi\,\Big|_{a=0} = \int\limits_{\xi_0}^{\xi_1}\left[\Phi\frac{\mathrm{d}x}{\mathrm{d}\xi}\right]_y\eta\,\mathrm{d}\xi;$$

on adding the two results obtained, we can write:

$$\int\limits_{x_0}^{x_1}\left\{[F]_y - \left[\Phi\frac{\mathrm{d}x}{\mathrm{d}\xi}\right]_y\frac{\mathrm{d}\xi}{\mathrm{d}x}\right\}\eta\,\mathrm{d}x,$$

whence, in view of the fact that η is arbitrary, we have by the fundamental lemma:

$$[F]_y = \left[\Phi\frac{\mathrm{d}x}{\mathrm{d}\xi}\right]_y\frac{\mathrm{d}\xi}{\mathrm{d}x}, \tag{94}$$

where the symbol on the right must be developed on the assumption that the independent variable is ξ, i.e.

$$\left[\Phi\frac{\mathrm{d}x}{\mathrm{d}\xi}\right]_y = \frac{\mathrm{d}x}{\mathrm{d}\xi}\Phi_y - \frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\Phi_{\frac{\mathrm{d}y}{\mathrm{d}\xi}}\frac{\mathrm{d}x}{\mathrm{d}\xi}\right\}.$$

Formula (94) is completely analogous to (92) as we mentioned above, whilst Euler's equation $[\Phi\,\mathrm{d}x/\mathrm{d}\xi]_y = 0$ is obviously equivalent to the Euler equation $[F]_y = 0$. All this can be generalized to the case when the integrand contains several required functions.

Let us consider a functional for the case of two independent variables:

$$J = \iint\limits_B F(x, y, u, u_x, u_y)\,\mathrm{d}x, \mathrm{d}y.$$

We introduce instead of (x, y) the two new independent variables (ξ, η):

$$x = x(\xi, \eta); \quad y = y(\xi, \eta),$$

on the assumption that the functions written have continuous derivatives and that the functional determinant corresponding to

them does not vanish. We transform the integrand to the new independent variables:

$$F(x, y, u, u_x, u_y) = F\left[x(\xi, \eta),\ y(\xi, \eta),\ u,\ u_\xi \xi_x + u_\eta \eta_x,\ u_\xi \xi_y + u_\eta \eta_y\right]$$
$$= \Phi(\xi, \eta, u, u_\xi, u_\eta).$$

On introducing, as above, the neighbouring function $u + a\eta$, differentiating the integral with respect to a and putting $a = 0$, we have:

$$\iint\limits_B [F]_u \eta \,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{B_1} \left[\Phi \frac{D(x, y)}{D(\xi, \eta)}\right]_u \eta \,\mathrm{d}\xi\,\mathrm{d}\eta, \qquad (94_1)$$

where B_1 is the result of transforming domain B by the replacement of the variables, $\mathrm{D}(x, y)/\mathrm{D}(\xi, \eta)$ is the usual notation for the functional determinant, and the symbol $[\ ]_u$ denotes the left-hand side of Ostrogradskii's equation, for example:

$$[F]_u = F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y}.$$

On carrying out the change of variables in the integral on the right-hand side of (94_1) and using the fact that η is arbitrary, we obtain the following transformation formula for the left-hand side of Ostrogradskii's equation to the new independent variables:

$$[F]_u = \left[\Phi \frac{D(x, y)}{D(\xi, \eta)}\right]_u \frac{D(\xi, \eta)}{D(x, y)}.$$

A precisely analogous formula is obtained in the case of a larger number of independent variables. Ostrogradskii's equation $[F]_u = 0$ is equivalent to Ostrogradskii's equation $[\Phi \cdot \mathrm{D}(x, y)/\mathrm{D}(\xi, \eta)]_u = 0$ in the new independent variables.

It is possible to carry out a simultaneous change of the independent variables and the functions. For instance if we replace (x, y) by the new variables (ξ, η) in functional (93):

$$x = \varphi(\xi, \eta); \quad y = \psi(\xi, \eta),$$

we get the function $\eta = f_1(\xi)$ in the new variables in place of the function $y = f(x)$. On transforming functional (93) to the new variables, we obtain:

$$J = \int\limits_{\xi_0}^{\xi_1} F\left[\varphi(\xi, \eta), \psi(\xi, \eta), \frac{\psi_\xi + \psi_\eta \eta'}{\varphi_\xi + \varphi_\eta \eta'}\right] (\varphi_\xi + \varphi_\eta \eta')\,\mathrm{d}\xi = \int\limits_{\xi_0}^{\xi_1} \Phi(\xi, \eta, \eta')\,\mathrm{d}\xi,$$

and, as above, Euler's equation $[F]_y = 0$ will be equivalent to Euler's equation $[\Phi]_\eta = 0$.

We investigate in the next section the Euler equation when the functional relationship $y(x)$ is given in the parametric form.

72. Parametric forms. When we seek the extrema of a functional, the demand that the required curve have an explicit equation $y = y(x)$ may substantially restrict the problem, since it may happen that straight lines parallel to the y axis cut the curve yielding the solution in more than one point. We shall next consider the general case of a parametric form of the equation of the required curve. Assuming that x and y are functions of a parameter t, we can rewrite integral (93) as

$$J = \int_{t_0}^{t_1} F\left(x, y, \frac{y'}{x'}\right) x' \, \mathrm{d}t, \tag{95}$$

where x' and y' are the derivatives with respect to t, and t_0 and t_1 the values of the parameter corresponding to the ends of the curve. The integral J has form (95) for any choice of parameter t.

We observe that the integrand does not contain the independent variable t and is a homogeneous function of the first degree in x' and y'. Let us consider in general an integral:

$$J = \int_{t_0} F(x, y, x', y') \, \mathrm{d}t, \tag{96}$$

in which the integrand does not contain the independent variable t and is a homogeneous function of the first degree in x' and y', i.e.

$$F(x, y, kx', ky') = kF(x, y, x', y'). \tag{97}$$

We show that integral (96) now retains the same form whatever the subst tution for the parameter t. We introduce a new parameter τ instead of t by putting $\tau = \tau(t)$, with the assumption that $\tau'(t) > 0$, so that τ increases as t increases. On transforming (96) to the variable τ, we get:

$$J = \int_{\tau_0}^{\tau_1} F(x, \; y, \; x'_\tau \tau'_t, \; y'_\tau \tau'_t) \, t'_\tau \, \mathrm{d}\tau,$$

and we can write with the aid of (97):

$$\int_{\tau_0}^{\tau_1} F(x, \; y, \; x'_\tau \tau'_\tau, \; y'_t \tau'_t) \, t'_\tau \, \mathrm{d}\tau = \int_{\tau_0}^{\tau_1} F(x, \; y, \; x'_\tau, \; y'_\tau) \, \mathrm{d}\tau,$$

i.e. integral (96) retains its form on changing the parameter.

We remark that the role of k in formula (97) has been played by τ_t', so that it is sufficient to require that identity (97) holds for $k > 0$. We shall assume that condition (97) is fulfilled for integral (96).

We recall that, when defining the closeness of curves specified explicitly, we required the closeness of the ordinates of the curves corresponding to the same abscissa. In the general case of parametric forms, closeness can be defined independently of the choice of parameter, viz, we can say that a curve l lies in an ξ-neighbourhood of zero order of a curve l_1, if a continuous one-to-one correspondence can be established between points of l and l_1 such that the distance between corresponding points does not exceed ξ. A first order ξ-closeness can be similarly defined.

We now turn to deducing the necessary condition for an extremum. Let a curve l yield an extremum. We choose some parametric equation of the curve l, such that the equation of l is $x(t)$, $y(t)$. We take a neighbouring curve $x(t) + a\eta(t)$, $y(t) + a_1\eta_1(t)$, where corresponding points have the same value of the parameter. On substituting the equation of the neighbouring curve in integral (96) and equating to zero the derivatives with respect to a and a_1 at $a = a_1 = 0$, we can show as usual that the functions $x(t)$ and $y(t)$ must satisfy, whatever the choice of parameter t, the system of two Euler equations:

$$F_x - \frac{\mathrm{d}}{\mathrm{d}t} F_{x'} = 0; \quad F_y - \frac{\mathrm{d}}{\mathrm{d}t} F_{y'} = 0. \tag{98}$$

These equations do not contain the parameter explicitly. It may be noted, in addition, that it is in the nature of things that one of the functions $x(t)$ or $y(t)$ can be assumed arbitrary. For, if we carry out the change of parameter $t(\tau)$, we get $x[t(\tau)]$ and $y[t(\tau)]$ and, since the choice of $t(\tau)$ is arbitrary, we can assume one of these functions to be an arbitrary function of τ. If this is taken into account, we can justifiably expect that the two equations (98) reduce to one. Let us prove this.

On differentiating both sides of the identity

$$F = x' F_{x'} + y' F_{y'},$$

expressing the property of a homogeneous function F [I, 154], with respect to x, y, x', y', we get:

$$\begin{aligned} F_x &= x' F_{xx'} + y' F_{xy'}; \quad F_y = x' F_{yx'} + y' F_{yy'} \\ 0 &= x' F_{x'x'} + y' F_{x'y'}; \quad 0 = x' F_{x'y'} + y' F_{y'y'}. \end{aligned} \tag{99}$$

We find from the last two equations:

$$\frac{F_{x'x'}}{y'^2} = \frac{F_{x'y'}}{-x'y'} = \frac{F_{y'y'}}{x'^2} = F_1(x, y, x', y'), \tag{100}$$

where F_1 denotes the common value of the three ratios. On returning to equations (98) and carrying out the differentiations, we can write them as

$$F_x - x' F_{xx'} - y' F_{yx'} - x'' F_{x'x'} - y'' F_{x'y'} = 0,$$
$$F_y - x' F_{xy'} - y' F_{yy'} - x'' F_{x'y'} - y'' F_{y'y'} = 0.$$

On replacing $F_{x'x'}$, $F_{x'y'}$ and $F_{y'y'}$ in these equations in accordance with (100), and F_x, F_y in accordance with (99), they are transformed to the following:

$$y' T = 0; \quad x' T = 0,$$

where

$$T = F_1(x, y, x', y')(x' y'' - y' x'') + F_{xy'} - F_{yx'}.$$

We assume that x' and y' do not vanish simultaneously, so that the last two equations in fact reduce to one:

$$T = F_1(x, y, x', y')(x' y'' - y' x'') + F_{xy'} - F_{yx'} = 0. \tag{101}$$

To this single equation with two required functions, which is equivalent to system (98), a further equation may be added, characterizing the concrete choice of parameter t; for instance, if we choose as t the length of arc s of the required extremal, the additional equation will be $x'^2 + y'^2 = 1$. On recalling the expression for the radius of curvature of a plane curve [I, 71], equation (101) may be rewritten as

$$\frac{1}{R} = \frac{F_{xy'} - F_{yx'}}{(x'^2 + y'^2)^{3/2} F_1}. \tag{102}$$

Everything that has been said can be extended without difficulty to functionals depending on curves in n-dimensional space. Let us take the integral:

$$J = \int_{t_0}^{t_1} F(x_1, x_1', \ldots, x_n, x_n')\, dt, \tag{103}$$

where x_i are functions of t, and x_i' are derivatives. As above, we assume that the function F is a homogeneous function of the first degree in the x_i'. Integral (103) does not vary in this case however the parameter tis changed. As above, it is easily shown that the necessary conditions for a curve in n-dimensional space $(x_1, \ldots, x_n)$ to give an extremum

of integral (103) are represented by the Euler equations:

$$F_{x_i} - \frac{\mathrm{d}}{\mathrm{d}t} F_{x'_i} = 0 \tag{104}$$

or

$$F_{x_i} - \sum_{s=1}^{n} x'_s F_{x'_i x_s} - \sum_{s=1}^{n} x''_s F_{x'_i x'_s} = 0 \quad (i = 1, 2, \ldots, n).$$

It is readily seen that the left-hand sides of these equations are connected by the following relationships:

$$\sum_{i=1}^{n} x'_i \left(F_{x_i} - \frac{\mathrm{d}}{\mathrm{d}t} F_{x'_i}\right) =$$

$$= \sum_{i=1}^{n} x'_i F_{x_i} - \sum_{i,s=1}^{n} x'_i x'_s F_{x'_i x_s} - \sum_{i,s=1}^{n} x'_i x''_s F_{x'_i x'_s} \equiv 0. \tag{105}$$

For, since F is homogeneous, we can write by Euler's theorem:

$$F = \sum_{i=1}^{n} x'_i F_{x'_i};$$

differentiation of this identity with respect to x_s and x'_s gives:

$$F_{x_s} = \sum_{i=1}^{n} x'_i F_{x'_i x_s}; \quad 0 = \sum_{i=1}^{n} x'_i F_{x'_i x'_s}.$$

It follows at once from these identities that the sum in the middle of (105) in fact vanishes identically. Hence, one of the equations in system (104) is a consequence of the remainder, and we can add to system (104) a further equation characterizing the choice of parameter. It may be mentioned that the whole of the above theory can be extended to the case of multiple integrals.

73. Geodesics in n-dimensional space. Let the metric

$$\mathrm{d}s^2 = \sum_{i,k=1}^{n} a_{ik}\, \mathrm{d}x_i\, \mathrm{d}x_k \qquad (a_{ik} = a_{ki}) \tag{106}$$

be defined in n-dimensional real space, where the a_{ik} are given functions of arguments x_s. These functions are assumed continuous with their first order partial derivatives. The specification of metric (106) is equivalent to the fact that the length of any curve $x_s(t)$ $(s = 1, 2, \ldots, n)$ is given by the integral

$$J = \int \mathrm{d}s = \int_{t_0}^{t_1} \sqrt{\sum_{i,k=1}^{n} a_{ik} x'_i x'_k}\ \mathrm{d}t, \tag{107}$$

the expression under the radical being assumed positive for any values of the x_s and x'_s provided all the x'_s do not vanish, i.e. we assume that the quadratic form (106) is positive definite. We are evidently justified in assuming that the coefficients a_{ik} and a_{ki} included in the same products of differentials are the same, i.e. $a_{ik} = a_{ik}$.

Geodesics are defined as extremals of integral (107). This concept is a direct generalization of the concept of geodesic on a given surface as described above. For brevity, we write φ for the sum under the radical sign:

$$\varphi = \sum_{i,k=1}^{n} a_{ik} x'_i x'_k. \tag{108}$$

We have the following Euler equations for the extremals:

$$\frac{1}{2\sqrt{\varphi}} \varphi_{x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\sqrt{\varphi}} \varphi_{x'_i}\right) = 0 \qquad (i = 1, 2, \ldots, n). \tag{109}$$

One equation of this system is a consequence of the rest, and we can add a further equation, viz:

$$\varphi = \sum_{i,k=1}^{n} a_{ik} x'_i x'_k = 1; \tag{110}$$

let s be the value of the parameter t defined by this supplementary equation. It follows at once from (107) that (110) is equivalent to choosing as the parameter t the arc length s of the curve in n-dimensional space. As a result of (110), system (109) is simplified and becomes

$$\varphi_{x_i} - \frac{\mathrm{d}}{\mathrm{d}s} \varphi_{x'_i} = 0 \qquad (i = 1, 2, \ldots, n). \tag{111}$$

It is easily shown that this system has the solution

$$\varphi = \text{const}.$$

For:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \sum_{i=1}^{n} \varphi_{x_i} x'_i + \sum_{i=1}^{n} \varphi_{x'_i} x''_i$$

But φ is a homogeneous second degree polynomial in the x'_i, so that we have

$$\sum_{i=1}^{n} \varphi_{x'_i} x'_i = 2\varphi,$$

and hence:

$$2\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \sum_{i=1}^{n} \varphi_{x'_i} x''_i + \sum_{i=1}^{n} x'_i \frac{\mathrm{d}}{\mathrm{d}s} \varphi_{x'_i}.$$

We use this equation to rewrite the expression for $\mathrm{d}\varphi/\mathrm{d}s$ as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 2\frac{\mathrm{d}\varphi}{\mathrm{d}s} + \sum_{i=1}^{n} x'_i \left(\varphi_{x_i} - \frac{\mathrm{d}}{\mathrm{d}s} \varphi_{x'_i}\right),$$

and, by (111), we have $d\varphi/ds = 0$, i.e. $\varphi = \text{const.}$ is a solution of system (111), and supplementary condition (110) is obtained if we put the arbitrary constant equal to unity.

We now write out equations (111) in the developed form:

$$\varphi_{x_i} - \sum_{s=1}^{n} \varphi_{x_i' x_s} x_s' - \sum_{i=1}^{n} \varphi_{x_i' x_s'} x_s'' = 0,$$

or, on substituting expression (108) here:

$$\frac{1}{2} \sum_{p,q=1}^{n} \frac{\partial a_{pq}}{\partial x_i} x_p' x_q' - \sum_{p,s=1}^{n} \frac{\partial a_{pi}}{\partial x_s} x_s' x_p' - \sum_{s=1}^{n} a_{si} x_s'' = 0.$$

Let us consider the second sum. The coefficients of $x_s' x_p'$ and $x_p' x_s'$ are not equal but can be made so by replacing each by

$$\frac{1}{2}\left(\frac{\partial a_{pi}}{\partial x_s} + \frac{\partial a_{si}}{\partial x_p}\right).$$

By combining the first sum with the second and changing the sign, we reduce our system to the final form:

$$\sum_{s=1}^{n} a_{si} x_s'' + \sum_{p,q=1}^{n} \frac{1}{2}\left(\frac{\partial a_{pi}}{\partial x_q} + \frac{\partial a_{qi}}{\partial x_p} - \frac{\partial a_{pq}}{\partial x_i}\right) x_p' x_q' = 0 \quad (i = 1, 2, \ldots, n). \tag{112}$$

The derivatives are taken with respect to the arc length s in these equations. The expression in brackets in the second sum is called in differential geometry a *Christoffel symbol of the first kind* and is written as

$$\frac{1}{2}\left(\frac{\partial a_{pi}}{\partial x_q} + \frac{\partial a_{qi}}{\partial x_p} - \frac{\partial a_{pq}}{\partial x_i}\right) = \begin{bmatrix} pq \\ i \end{bmatrix}.$$

Equations (112) can be written as explicit equations for the x_s''. Let us write a^{ik} for the elements of the transpose of the matrix $\| a_{ik} \|^{-1}$, i.e.

$$a^{ik} = \frac{A_{ik}}{D} \quad \text{or} \quad \| a^{ik} \| = (\| a_{ik} \|^{-1})^*,$$

where D is the determinant of the matrix $\| a_{ik} \|$ and differs from zero by virtue of the positive definiteness of the quadratic form (106), whilst the A_{ik} are the cofactors of the elements a_{ik} of this determinant. We have the following fundamental equations for the elements a^{ik}:

$$\sum_{s=1}^{n} a^{is} a_{ks} = \begin{cases} 0 & (i \neq k) \\ 1 & (i = k). \end{cases} \tag{113}$$

On multiplying both sides of (112) by a^{ji}, summing over i and changing the order of summation in the second term, we obtain by (113):

$$x_j'' + \sum_{p,q=1}^{n} \begin{Bmatrix} pq \\ j \end{Bmatrix} x_p' x_q' = 0, \tag{114}$$

where

$$\begin{Bmatrix} pq \\ j \end{Bmatrix} = \sum_{i=1}^{n} a^{ji} \begin{bmatrix} pq \\ i \end{bmatrix}. \tag{115}$$

After using relationship (110), Euler's equations took the form (111), and these latter equations now become independent. We have succeeded in solving them for the x_s''.

Let us take as an example the problem of finding the geodesics on a cylinder. We take the Z axis parallel to the generators of the cylinder and let the equation of the directrix in the (x, y) plane be $x = \varphi(\sigma)$; $y = \psi(\sigma)$, where the parameter σ is the arc length of the directrix, so that

$$\varphi'^2(\sigma) + \psi'^2(\sigma) = 1.$$

We choose the parameter σ and coordinate z as the coordinate parameters defining the position of a point on the cylinder. Now,

$$ds^2 = d\sigma^2 + dz^2,$$

so that we have in this case:

$$a_{11} = a_{22} = 1; \quad a_{12} = a_{21} = 0.$$

Equations (114) give us $\sigma'' = 0$ (for $j = 1$) and $z'' = 0$ (for $j = 2$), the derivatives being taken with respect to the arc length s. We therefore obtain:

$$\sigma = As + B; \quad z = A_1 s + B_1.$$

If $A \neq 0$, we can write the equation of these curves as $z = C_1\sigma + C_2$, where C_1 and C_2 are arbitrary constants. These curves, the complete equations of which are

$$x = \varphi(\sigma); \quad y = \psi(\sigma); \quad z = C_1\sigma + C_2, \tag{116}$$

are the helices which we discussed in [II; 127]. The presence of the constant term in the expression for z is clearly of no importance.

74. Natural boundary conditions. In discussing the extrema of functional (93) we have as yet only taken as boundary conditions those imposed by fixing the ends of the required curve, i.e. we have specified $y(x_0)$ and $y(x_1)$. We shall now indicate another type of boundary condition. Suppose we are seeking the extremum of the integral

$$J = \int_{x_0}^{x_1} F(x, y, y')\, dx, \tag{117}$$

the left-hand end of the required curve being fixed, i.e. the boundary condition $y(x_0) = y_0$ holds for the left-hand end, whereas no boundary condition is imposed on the right-hand end apart from the self-

evident condition that this end lie on the straight line $x = x_1$ parallel to the y axis. We now show that a boundary condition must in fact be satisfied at this free end, which is obtained as a direct consequence of the condition for an extremum of integral (117). In fact, if a certain curve yields an extremum of the integral as compared with all the neighbouring curves with free right-hand end, it all the more gives an extremum when this end is fixed. But the curve must now satisfy Euler's equation, as we have shown above, i.e. it must be an extremal of integral (117). Let us now return to the general expression for the first variation of the integral [63]:

$$\delta J = [F_{y'}\,\delta y]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \delta y \,\mathrm{d}x \qquad (\delta y = \alpha\eta).$$

As above, this first variation must vanish. The term containing the integral vanishes, since the function $y(x)$ must satisfy Euler's equation in this case, as we have just shown. The term outside the integral must vanish at $x = x_0$, since this end is fixed. Thus the vanishing of the first variation amounts to having $F_{y'}\,\eta = 0$ for $x = x_1$. At the free end, η can be arbitrary so that we finally get the boundary condition for the free end:

$$F_{y'}\,|_{x=x_1} = 0. \tag{118}$$

This gives us a relationship between y and y' at the free end. It is easily seen that, for integral (2_1), condition (118) becomes $y' = 0$, i.e. it reduces here to the requirement that the extremal be perpendicular to the straight line $x = x_1$ at the end $x = x_1$. Boundary condition (118) is usually known as the *natural boundary condition*. On repeating the above discussion for the integral

$$J = \int_{x_0}^{x_1} F(x, y_1, y_1', \ldots, y_n, y_n')\,\mathrm{d}x,$$

we obtain the following n boundary conditions at the free end:

$$F_{y_i'} = 0 \quad (i = 1, 2, \ldots, n).$$

We now consider an integral containing second order derivatives:

$$J = \int_{x_0}^{x_1} F(x, y, y', y'')\,\mathrm{d}x.$$

By using (22) and (23), and the fact that $\eta(x)$ and $\eta'(x)$ are arbitrary at the free end, we obtain the following two natural boundary conditions at the free end:

$$F_{y'} - \frac{d}{dx} F_{y''} = 0; \quad F_{y''} = 0. \tag{119}$$

We remark that the first of these conditions gives the connection between y, y', y'', y''' at the free end. Similarly, in the case of the double integral

$$J = \int\int_B F(x, y, u, u_x, u_y)\, dx\, dy, \tag{120}$$

the natural boundary conditions on the contour l take the form:

$$F_{u_x} \frac{dy}{ds} - F_{u_y} \frac{dx}{ds} = 0, \tag{121}$$

where s is the length of arc on the contour l. This follows at once from (26) for the first variation of integral (120).

75. Functionals of a more general type. Let us now consider the first variations of functionals which, in addition to the usual integrals, contain auxiliary terms that depend on the values of functions at the ends of the interval of integration or on the contour of the domain of integration. When investigating the extrema of such functionals, we arrive at the previous Euler equations, and the extra terms only have an effect on the form of the natural boundary conditions. By introducing such extra terms we can obtain various forms of natural boundary conditions, of importance for applications of the calculus of variations to mathematical physics. We shall confine ourselves here to certain particular cases.

We take as a first example the functional:

$$J = \int_{x_0}^{x_1} F(x, y, y')\, dx - \varphi(y_0) + \psi(y_1), \tag{122}$$

where y_0 and y_1 are the values of the function $y(x)$ at the ends of the interval of integration, and $\varphi(y_0)$ and $\psi(y_1)$ are given functions of their arguments, the minus sign being taken in front of $\varphi(y_0)$ for the sake of convenience in future working. By considering the neighbouring curves $y(x) + \alpha\eta(x)$, substituting in the functional, differentiating with respect to α and putting $\alpha = 0$, we obtain the following

expression for the first variation:

$$\delta J = \int_{x_0}^{x_1} [F]_y \delta y \, \mathrm{d}x + \{ \psi' (y_1) + F_{y'} [x_1, y_1, y' (x_1)] \} \delta y_1 -$$
$$- \{ \varphi' (y_0) + F_{y'} [x_0, y_0, y' (x_0)] \} \delta y_0 . \qquad (123)$$

If a curve $y(x)$ gives functional (122) an extremum in the case of free ends, it must give an extremum all the more with fixed ends, i.e. we can take $\delta y_1 = \delta y_0 = 0$ in the last formula, and the fundamental lemma shows as usual that $y(x)$ must satisfy the ordinary Euler equation. If both ends are free, δy_1 and δy_0 are arbitrary in (123), and we obtain boundary conditions of the form:

$$\varphi' (y) + F_{y'} |_{x=x_0} = 0; \quad \psi' (y) + F_{y'} |_{x=x_1} = 0 .$$

On putting say $\varphi(y) = l(y - a)^2$, we obtain the natural boundary condition at $x = x_0$:

$$\frac{1}{2l} F_{y'} |_{x=x_0} + y_0 - a = 0,$$

and $y_0 = a$ in the limit as $l \to \infty$, i.e. we arrive at the case of a fixed end.

In the case of a double integral, let us take as the extra term the line integral over the contour l of the basic domain of integration B, the arc length s of the contour being taken as the independent variable in this line integral, s being measured from a fixed point of the contour. We shall assume that the following appear under the integral sign in the line integral: the independent variable s, the required function u and its tangential derivative u_s, i.e.

$$J = \iint_B F (x, y, u, u_x, u_y) \, \mathrm{d}x \, \mathrm{d}y + \int_l \Phi (s, u, u_s) \, \mathrm{d}s . \qquad (124)$$

After the usual working we arrive at the following expression for the first variation:

$$\delta J = \iint_B [F]_u \delta u \, \mathrm{d}x \, \mathrm{d}y +$$
$$+ \int_l \left(F_{u_x} \frac{\mathrm{d}y}{\mathrm{d}s} - F_{u_y} \frac{\mathrm{d}x}{\mathrm{d}s} + \Phi_u - \frac{\mathrm{d}}{\mathrm{d}s} \Phi_{u_s} \right) \delta u \, \mathrm{d}s . \qquad (125)$$

It can be shown by arguing as above that the necessary condition for $u(x, y)$ to give an extremum of functional (124) with the natural

boundary conditions is that u satisfy the ordinary Ostrogradskii equation and that the boundary condition be fulfilled on the contour:

$$F_{u_x} \frac{\mathrm{d}y}{\mathrm{d}s} - F_{u_y} \frac{\mathrm{d}x}{\mathrm{d}s} + \Phi_u - \frac{\mathrm{d}}{\mathrm{d}s} \Phi_{u_s} \Big|_l = 0. \qquad (126)$$

Let us take as an example the functional:

$$J = \int\int_B (u_x^2 + u_y^2) \, \mathrm{d}x \, \mathrm{d}y + \int_l p(s) \, u \, \mathrm{d}s,$$

where $p(s)$ is a function given on l. In this case Ostrogradskii's equation reduces to Laplace's equation, whilst the boundary condition becomes

$$2u_x \frac{\mathrm{d}y}{\mathrm{d}s} - 2u_y \frac{\mathrm{d}x}{\mathrm{d}s} + p(s) \Big|_l = 0.$$

If it is observed that $\mathrm{d}x/\mathrm{d}s$ and $\mathrm{d}y/\mathrm{d}s$ are the direction cosines of the tangent to l, i.e. $\mathrm{d}y/\mathrm{d}s$ and $(-\mathrm{d}x/\mathrm{d}s)$ are the direction cosines of the outward normal to l, we can write the boundary condition as:

$$\frac{\partial u}{\partial n} \Big|_l = -\frac{1}{2} p(s).$$

We have thus arrived at the problem of integrating Laplace's equation with given values of the normal derivative on the contour of the domain, i.e. at Neumann's problem. If we were to take

$$\Phi = p(s) \, u + q(s) \, u^2,$$

we should obtain the boundary condition:

$$\frac{\partial u}{\partial n} + q(s) \, u \Big|_l = -\frac{1}{2} p(s).$$

We remark that there is a further possibility for influencing the boundary conditions without changing the Euler and Ostrogradskii equations. Instead of adding extra terms to the functional, as above, we add an expression to the integrand of the fundamental integral in such a way as to leave the Euler and Ostrogradskii equations unaffected. Suitable expressions were found in [66]. If, for instance, we replace the integral

$$\int_{x_0}^{x_1} F(x, y, y') \, \mathrm{d}x$$

by the integral

$$\int_{x_0}^{x_1} (F + A(x, y) + B(x, y) \, y') \, \mathrm{d}x,$$

where $A_y = B_x$, Euler's equation is unchanged, whilst the natural boundary condition becomes $F_{y'} + B = 0$ instead of $F_{y'} = 0$.

A similar procedure can be used in the case of a multiple integral.

76. General form of the first variation. We have so far assumed, when discussing the first variation, that the interval or domain of integration remains unchanged. An expression will next be found for the first variation, without this assumption. This will enable us to consider the fundamental problem of the of calculus variations in the general case of movable ends. We start by considering the simplest integral, viz. (117). We assumed previously that the neighbouring curves $y(x) + a\eta(x)$ differ from the basic curve $y(x)$ in the addition of $a\eta(x)$. It will now be assumed that the neighbouring curves $y(x, a)$ contain the parameter a in any manner, the fundamental curve $y(x) = y(x, 0)$ for which the variation of the integral is calculated being obtained with $a = 0$. Thus, we shall consider

$$J = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x \tag{127}$$

and introduce a variable neighbouring curve into it by assuming that the limits of integration depend on a:

$$J(a) = \int_{x_0(a)}^{x_1(a)} F\left[x, y(x, a), y_x(x, a)\right]\mathrm{d}x, \tag{128}$$

the function and limits of integral (127) being obtained with $a = 0$:

$$y(x, 0) = y(x); \quad x_1(0) = x_1; \quad x_0(0) = x_0.$$

In view of the general definition of variation as the product of a and the derivative with respect to a, we can write:

$$\delta x_0 = \left.\frac{\mathrm{d}x_0(a)}{\mathrm{d}a}\right|_{a=0} a; \quad \delta x_1 = \left.\frac{\mathrm{d}x_1(a)}{\mathrm{d}a}\right|_{a=0} a; \quad \delta y = \left.\frac{\partial y(x, a)}{\partial a}\right|_{a=0} a,$$

$$\delta y' = \left.\frac{\partial}{\partial a}\left[\frac{\partial y(x, a)}{\partial x}\right]\right|_{a=0} a = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial y(x, a)}{\partial a}\right]_{a=0} a = \frac{\mathrm{d}}{\mathrm{d}x}\,\delta y,$$

on the assumption that $y(x, a)$ has continuous derivatives up to the second order. On differentiating integral (128) with respect to a, then setting $a = 0$ in it and multiplying by a, we obtain the following expression for the first variation of the integral:

$$\delta J = F(x_1, y_1, y_1')\,\delta x_1 - F(x_0, y_0, y_0')\,\delta x_0 + \int_{x_0}^{x_1} (F_y\,\delta y + F_{y'}\,\delta y')\,\mathrm{d}x,$$

or

$$\delta J = \left[F(x, y, y')\,\delta x\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} (F_y\,\delta y + F_{y'}\,\delta y')\,\mathrm{d}x. \qquad (129)$$

We transform the second term as usual by integrating by parts:

$$\int_{x_0}^{x_1} F_{y'}\,\delta y'\,\mathrm{d}x = \int_{x_0}^{x_1} F_{y'}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\,\mathrm{d}x =$$

$$= F_{y'}(x_1, y_1, y_1')(\delta y)_1 - F_{y'}(x_0, y_0, y_0')(\delta y)_0 -$$

$$- \int_{x_0}^{x_1} \delta y \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\,\mathrm{d}x, \qquad (130)$$

where $(\delta y)_1$ and $(\delta y)_0$ are the boundary values of the variation of function y:

$$(\delta y)_i = \left[\frac{\partial f(x_i, a)}{\partial a}\right]_{a=0} a \quad (i = 0, 1). \qquad (131)$$

We now find the first variation of the ordinates of the ends of the curve, the working being carried out only for the ordinate y_1 of the right-hand end. Obviously:

$$y_1 = f\left[x_1(a), a\right],$$

and when a varies both the arguments of the function f vary, and not merely the second as was the case when determining $(\delta y)_1$; thus the first variation δy_1 of the ordinate y_1 will be:

$$\delta y_1 = \left[\frac{\mathrm{d}}{\mathrm{d}a} f\left[x_1(a), a\right]\right]_{a=0} a = \left[\frac{\partial f}{\partial x_1}\cdot\frac{\mathrm{d}x_1}{\mathrm{d}a}\right]_{a=0} a + \left[\frac{\partial f}{\partial a}\right]_{a=0} a =$$

$$= y_1'\,\delta x_1 + (\delta y)_1, \qquad (132)$$

where y_1' is the slope of the tangent at the right-hand end of the curve. Similarly, we have for the variation δy_0 of the ordinate of the left-hand end:

$$\delta y_0 = y_0'\,\delta x_0 + (\delta y)_0. \qquad (133)$$

Substitution of expressions (132) and (133) for $(\delta y)_1$ and $(\delta y)_0$ in (130) gives us the following final expression for the first variation of integral (127):

$$\delta J = \left[F(x_1, y_1, y_1') - y_1' F_{y'}(x_1, y_1, y_1')\right]\delta x_1 +$$

$$+ F_{y'}(x_1, y_1, y_1')\,\delta y_1 - \left[F(x_0, y_0, y_0') - y_0' F_{y'}(x_0, y_0, y_0')\right]\delta x_0 -$$

$$- F_{y'}(x_0, y_0, y_0')\,\delta y_0 + \int_{x_0}^{x_1}\left(F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right)\delta y\,\mathrm{d}x, \qquad (134)$$

or

$$\delta J = \left[(F - y' F_{y'})\,\delta x + F_{y'}\,\delta y\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \delta y \,\mathrm{d}x. \tag{135}$$

The right-hand side of (135) is linear in δx_i and δy_i and retains its meaning when the neighbouring curves depend on several parameters, the first variation in this case being understood as the first total differential with respect to the parameters at their initial values, i.e.

$$\delta J = \sum_{i=1}^{n} \left(\frac{\partial J}{\partial a_i}\right)_{a_1 = \ldots a_n = 0} a_i \tag{136}$$

if the curve in question is obtained from a family depending on n parameters with $a_i = 0$ $(i = 1, 2, \ldots, n)$.

Working precisely similar to the above leads, in the case of an integral depending on n unknown functions:

$$J = \int_{x_0}^{x_1} F(x, y_1, y_1', \ldots, y_n, y_n')\,\mathrm{d}x, \tag{137}$$

to the following formula for the first variation:

$$\delta J = \left[F - \sum_{i=1}^{n} y_i' F_{y_i'}\right]_{x=x_1} \delta x_1 + \sum_{i=1}^{n} [F_{y_i'}]_{x=x_1} \delta y_i^{(1)} -$$
$$- \left[F - \sum_{i=1}^{n} y_i' F_{y_i'}\right]_{x=x_0} \delta x_0 - \sum_{i=1}^{n} [F_{y_i'}]_{x=x_0} \delta y_i^{(0)} +$$
$$+ \sum_{i=1}^{n} \int_{x_0}^{x_1} \left(F_{y_i} - \frac{\mathrm{d}}{\mathrm{d}x} F_{y_i'}\right) \delta y_i \,\mathrm{d}x,$$

or

$$\delta J = \left[\left(F - \sum_{i=1}^{n} y_i' F_{y_i'}\right) \delta x + \sum_{i=1}^{n} F_{y_i'}\,\delta y_i\right]_{x=x_0}^{x=x_1} +$$
$$+ \sum_{i=1}^{n} \int_{x_0}^{x_1} \left(F_{y_i} - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right) \delta y_i \,\mathrm{d}x, \tag{137$_1$}$$

where δx_0, δx_1, $\delta y_i^{(0)}$, $\delta y_i^{(1)}$ are the variations of the coordinates of the ends of the curve.

Let us explain the geometrical distinction between the magnitudes δy_1 and $(\delta y)_1$ appearing in (132). The coordinates of the right-hand end of the neighbouring curves $y = f(x, a)$ will be: $x_1(a)$ and $y_1(a) = f[x_1(a), a]$. When a varies, the right-hand end describes a certain curve λ. The initial value of a is $a = 0$, so that the increment of a form its initial value is a itself. By (132), δy_1 is the differential of $y_1(a) = f[x_1(a), a]$ with respect to the variable a, i.e. δy_1 is the principal part of the increment of the ordinate $y_1(a)$ of the right-hand end. This increment is represented in Fig. 2 by the segment CD. By (131), $(\delta y)_1$ is the differential of $f_1[x_1(0), a]$, where we have to put $a = 0$ in the first argument $x_1(a)$ before calculating the differential. Hence $(\delta y)_1$ is the principal part of the increment of the ordinate at the end $x_1(0)$ on passing from the basic curve $y(x)$ to the neighbouring curve $y = f(x, a)$. This increment is shown in Fig. 2 by the segment AB.

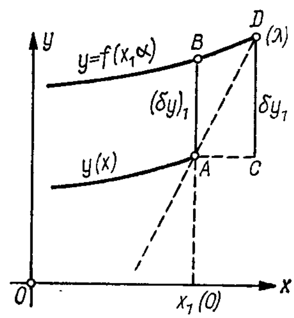

FIG. 2

77. Transversality condition. We assumed when discussing the natural boundary conditions that the end of the extremal can be displaced along the straight line $x = x_0$ or $x = x_1$ parallel to the y axis. Now let it be displaced along any given curve λ on the (x, y) plane.

We shall assume for definiteness that the left-hand end (x_0, y_0) is fixed, whilst the right-hand end can move along λ. By arguing as previously it can be shown that if a curve $y(x)$ yields an extremum of the integral it must satisfy Euler's equation, i.e. it must be an extremal. The first variation must vanish: the term containing the integral sign will vanish by virtue of Euler's equation, whilst the term outside the integral will vanish at $x = x_0$ by virtue of the fixing of the end. Equating the first variation to zero thus leads to the following condition at the moving end:

$$[F(x, y, y') - y' F_{y'}(x, y, y')]\,\delta x + F_{y'}(x, y, y')\,\delta y = 0, \qquad (138)$$

where δx and δy are the projections on the coordinate axes of an infinitesimal displacement along the curve λ. If we were to assume both ends movable, we should obtain condition (138) at both ends. It is sufficient to repeat the previous arguments, whilst recalling that, if

the curve yields an extremum of the integral with movable ends, it gives an extremum all the more with fixed ends or with one fixed end.

On writing $\overline{y'} = \delta y/\delta x$ for the slope of the tangent to the curve λ, we can rewrite condition (138) as

$$F(x, y, y') + (\overline{y'} - y') F_{y'}(x, y, y') = 0. \tag{139}$$

This condition, which is usually called the *transversality condition*, is thus seen to establish a connection between the slope $\overline{y'}$ of the tangent to the extremal and the slope y' of the tangent to the curve λ at every point of λ. If the equation of λ is given implicitly as $\varphi(x, y) = 0$, the transversality condition can be rewritten as

$$\frac{F - y' F_{y'}}{\varphi_x} = \frac{F_{y'}}{\varphi_y}. \tag{140}$$

Let us consider the transversality condition in three-dimensional space. The fundamental integral will have the form

$$J = \int_{x_0}^{x_1} F(x, y, y', z, z')\, dx. \tag{141}$$

On taking (137_1) into account and arguing precisely as above, we find that if one of the ends can move along a given surface S, the transversality condition must be satisfied at this end:

$$(F - y' F_{y'} - z' F_{z'})\, \delta x + F_{y'}\, \delta y + F_{z'}\, \delta z = 0, \tag{142}$$

where δx, δy, δz are the components of the infinitesimal displacement along the surface S. This condition is equivalent to saying that the coefficients of δx, δy, δz must be proportional to the direction cosines of the normal to S.

If the equation of the surface is given implicitly as $\varphi(x, y, z) = 0$, the transversality condition (142) obviously becomes

$$\frac{F - y' F_{y'} - z' F_{z'}}{\varphi_x} = \frac{F_{y'}}{\varphi_y} = \frac{F_{z'}}{\varphi_z}. \tag{143}$$

It gives us two relationships connecting x, y, z, z', y'. These relationships are replaced by the two conditions $y(x_0) = y_0$; $z(x_0) = z_0$ in the case of a fixed end.

In the general case of integral (137) the extremal is a curve in $(n + 1)$-dimensional space $(x, y_1, \ldots, y_n)$, and if its end can move over a given hypersurface $\varphi(x, y_1, \ldots, y_n) = 0$, the following transversality

condition must be observed at this end:

$$\left(F - \sum_{i=1}^{n} y_i' F_{y_i'}\right) \delta x + \sum_{i=1}^{n} F_{y_i'} \delta y_i = 0, \tag{144}$$

or

$$\frac{F - \sum_{i=1}^{n} y_i' F_{y_i'}}{\varphi_x} = \frac{F_{y_1'}}{\varphi_{y_1}} = \ldots = \frac{F_{y_n'}}{\varphi_{y_n}}. \tag{145}$$

A particular case must be mentioned. Suppose that the fundamental integral has the form:

$$J = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2 + z'^2}}{v(x, y, z)} \, dx = \int_{x_0}^{x_1} n(x, y, z) \sqrt{1 + y'^2 + z'^2} \, dx,$$

which corresponds to the problem of geometric optics. We show that in this case transversality condition (145) is the same as the orthogonality condition, i.e. the condition that the extremal be normal to the surface S. On substituting $F = n\sqrt{1 + y'^2 + z'^2}$ in condition (145) and doing some obvious cancelling, we get:

$$1 : \varphi_x = y' : \varphi_y = z' : \varphi_z.$$

But 1, y', z' are proportional to the direction cosines of the tangent to the extremal, whilst the partial derivatives of φ are proportional to the direction cosines of the normal to S, and the equations written express the above-mentioned orthogonality condition. A similar situation obtains for the integral

$$J = \int_{x_0}^{x_1} n(x, y) \sqrt{1 + y'^2} \, dx$$

in the plane case, except that a curve λ in the (x, y) plane replaces the surface S.

We observe further that, if we pass to the parametric form of the curve $y(x)$, $z(x)$ in integral (141), so that the integrand takes the form $\Phi(x, y, z, x', y', z')$, condition (145) becomes

$$\frac{\Phi_{x'}}{\varphi_x} = \frac{\Phi_{y'}}{\varphi_y} = \frac{\Phi_{z'}}{\varphi_z}. \tag{146}$$

78. Canonical variables. The transversality condition lies at the basis of a geometrical theory of extremal problems which is of great importance in the calculus of variations. We shall now discuss this theory. As a preliminary we carry out a change of variables in Euler's equations, to the so-called canonical variables. Let us start with the

case of three-dimensional space, when the basic integral has the form

$$J = \int_{x_0}^{x_1} F(x, y, y', z, z')\,\mathrm{d}x. \tag{147}$$

Euler's equations for this integral are

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0; \quad F_z - \frac{\mathrm{d}}{\mathrm{d}x} F_{z'} = 0 \tag{148}$$

i.e. a system of two second order equations. We replace y' and z' by new variables v and w in accordance with the formulae

$$v = F_{y'}; \quad w = F_{z'}, \tag{149}$$

these equations being assumed soluble with respect to y' and z', i.e. the corresponding functional determinant is assumed non-zero:

$$\frac{D(F_{y'}, F_{z'})}{D(y', z')} \neq 0.$$

Instead of F we introduce the new function H:

$$H(x, y, z, v, w) = y'v + z'w - F = y'F_{y'} + z'F_{z'} - F, \tag{150}$$

and assume this new function H to be expressed in terms of the new variables v and w. Let us find the partial derivatives of $H(x, y, z, v, w)$ with respect to the last four variables:

$$H_y = \frac{\mathrm{d}y'}{\mathrm{d}y} v + \frac{\mathrm{d}z'}{\mathrm{d}y} w - F_y - F_{y'} \frac{\mathrm{d}y'}{\mathrm{d}y} - F_{z'} \frac{\mathrm{d}z'}{\mathrm{d}y},$$

or, by (149):

$$H_y = -F_y. \tag{151}$$

Similarly, simple differentiation gives us:

$$H_z = -F_z; \quad H_v = y'; \quad H_w = z'. \tag{152}$$

We can thus write, instead of the two second order equations (148), a system of four first order equations in the new variables for functions y, z, v, w of the independent variable x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = H_v; \quad \frac{\mathrm{d}z}{\mathrm{d}x} = H_w; \quad \frac{\mathrm{d}v}{\mathrm{d}x} = -H_y; \quad \frac{\mathrm{d}w}{\mathrm{d}x} = -H_z. \tag{153}$$

System (153) is usually known as the canonical system. An expression for the integrand F of the functional in terms of the function H follows at once from (150) and (152):

$$F = vH_v + wH_w - H. \tag{154}$$

The general solution of system (148) or (153) will contain four arbitrary constants. Given the usual conditions of the existence and uniqueness theorem of the theory of differential equations, we can draw through every point (x, y, z) of space a pencil of extremals by assigning arbitrary initial values to the derivatives y' and z'. Such a pencil of extremals will consist of a family of curves depending on two arbitrary constants, i.e. on the two initial values of these derivatives. In general, we describe as a *family of extremals* a set of solutions of Euler's equations depending on two arbitrary constants and filling a certain part of space without mutual intersections, i.e. such that one, and only one, extremal of the family passes through every point this part of space. We obtain, with such a family of extremals, definite values of y' and z' at every point, and at every point of the part of space filled by the family we get definite values of v and w, i.e. we can assume that v and w are defined as functions of the coordinates (x, y, z) at every point of this part of space. These functions $v(x, y, z)$ and $w(x, y, z)$ are called the *slope functions of the family of extremals.* Let us now show that these functions must satisfy certain equations containing the partial derivatives of the functions. In fact, the four functions

$$y(x),\ z(x),\ v[x, y(x), z(x)],\quad w[x, y(x), z(x)]$$

of the independent variable x must satisfy system (153). On replacing the total derivatives $\mathrm{d}v/\mathrm{d}x$ and $\mathrm{d}w/\mathrm{d}x$ by the expressions for them, the last two equations of (153) can be rewritten as

$$v_x + v_y \frac{\mathrm{d}y}{\mathrm{d}x} + v_z \frac{\mathrm{d}z}{\mathrm{d}x} = -H_y; \quad w_x + w_y \frac{\mathrm{d}y}{\mathrm{d}x} + w_z \frac{\mathrm{d}z}{\mathrm{d}x} = -H_z. \tag{155}$$

If we now use the other two equations of the system, we in fact obtain a system of partial differential equations which must be satisfied by the slope functions $v(x, y, z)$ and $w(v, y, z)$:

$$v_x + v_y H_v + v_z H_w = -H_y; \quad w_x + w_y H_v + w_z H_w = -H_z. \tag{156}$$

Let us now suppose, conversely, that $v(x, y, z)$ and $w(x, y, z)$ appear simply as solutions of system (156) instead of as slope functions of a family of extremals. On substituting these functions $c(x, y, z)$ and $w(x, y, z)$ in the right-hand sides of the first two equations of system (153), we get a system of two equations of the first order for y and z. When this system is solved y and z become functions of x and two arbitrary constants, say $y(x, C_1, C_2)$ and $z(x, C_1, C_2)$; on sub-

stituting these latter in $v(x, y, z)$ and $w(x, y, z)$, expressions in x and two arbitrary constants are also obtained for v and w.

It may easily be shown that the last two equations of system (153) will now also be satisfied. In fact, by using the rule for differentiation of functions of a function and the first two equations of system (153), we can write:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = v_x + v_y H_v + v_z H_w,$$

whence, by the first of equations (156), we in fact obtain $\mathrm{d}v/\mathrm{d}x = -H_y$. The validity of the last of equations (153) can be proved similarly.

If the extremals $y(x, C_1, C_2)$ and $z(x, C_1, C_2)$ fill a part of space without intersections, i.e. form a family of extremals, the functions v and w which we have taken as arbitrary solutions of system (156) must be slope functions for the family. We have thus shown that, *given a solution of system* (156), *we can obtain the corresponding family of extremals for which this solution is the set of slope functions.* We naturally confine ourselves here to the part of space for which $y(x, C_1, C_2)$ and $z(x, C_1, C_2)$ represent the family of extremals, i.e. which they fill without mutual intersections.

It is worth noticing how the transversality condition looks in the canonical variables. This was condition (142) in the original variables. By using (150) and (152) we can write the transversality condition as

$$-H\,\delta x + v\,\delta y + w\,\delta z = 0. \tag{157}$$

79. Field of extremals in three-dimensional space. We now turn to the geometrical theory for the case of integral (147).

We shall discuss special families of extremals, defined as follows. Let l be a curve in space. We shall describe the value of integral (147) taken over this curve as its *quasi-* or *J-length*. For instance, in the case of integral (2), corresponding to the problem of geometrical optics, the quasi-length expressed the time in which the point traverses the curve l, moving with a given velocity $v(x, y, z)$ in space.

Let us take the pencil of extremals departing from a given point M_0 in space, and let this pencil form a family in some neighbourhood of M_0, i.e. the extremals of the pencil are assumed not to intersect in this neighbourhood except at M_0. We mark off an arc M_0M along each extremal from M_0 such that the quasi-length of the arc is equal to the same number ϱ for all the extremals. The locus of the points M will yield a certain surface which we shall call the *quasi-sphere* with centre

M_0. When the number ϱ varies, a family of quasi-spheres is obtained, depending on a single parameter and filling a certain neighbourhood of the point M_0. It is easily seen that the extremals of the pencil will cut the quasi-spheres transversally, i.e. at every point belonging to some neighbourhood of M_0 the slope functions $v(x, y, z)$ and $w(x, y, z)$ of the pencil will satisfy the transversality condition (157), where δx, δy, δz are the components of an infinitesimal displacement over the quasi-sphere passing through the point concerned.

For, on returning to the expression for the variation of functional (147) in the general case:

$$\delta J = [-H\,\delta x + v\,\delta y + w\,\delta z]_{x=x_0}^{x=x_1} + \\ + \int_{x_0}^{x_1} \left[\left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\right)\delta y + \left(F_z - \frac{\mathrm{d}}{\mathrm{d}x} F_{z'}\right)\mathrm{d}z\right]\delta x, \tag{158}$$

let the end M of an extremal of the pencil move over the surface of the quasi-sphere. The value of the functional J here remains constant, by construction, so that $\delta J = 0$. The integrand on the right-hand side of (158) vanishes, since the curve taken is an extremal; the term outside the integral sign vanishes at the lower limit, since M_0 is fixed, and hence $\delta x = \delta y = \delta z = 0$ at this point; thus the term outside the integral sign must also vanish at the upper limit, i.e. the transversality condition must be fulfilled at the point M, as it moves over the surface of the quasi-sphere. We observe that the whole of our pencil of extremals depends on two arbitrary constants, and motion of the point M over the surface of the quasi-sphere reduces to variation of the values of these constants, which play in the present case the role of the parameters which we mentioned in [77].

Let M be a point belonging to the neighbourhood of M_0. There is a definite extremal joining M_0 and M, and the value of integral (147) along the arc M_0M of this extremal is a definite function $\theta(x, y, z)$ of the coordinates of M. The family of quasi-spheres now obviously has the equation

$$\theta\,(x, y, z) = \varrho, \tag{159}$$

where ϱ is the parameter mentioned above. The pencil of extremals issuing from M_0 is usually said to form a *central field of extremals.* The above-mentioned quasi-spheres are called the *transversal surfaces* of this field and θ is the basic function of the field.

We now turn to constructing the general field of extremals. Let S_0 be a surface in three-dimensional space. Transversality condition (142)

defines y' and z' at every point of this surface, or else (157) defines v and w at this point. On taking these values of y' and z' as the initial values of the derivatives, an extremal can be produced from every point of S_0 cutting S_0 transversally. If this construction is carried out for every point of S_0, a set of extremals is obtained, depending on two parameters, which intersect S_0 transversally. Let these extremals form a family in some neighbourhood of the surface, i.e. they are non-intersecting. We mark off on each extremal of the family, from the point M_0 on the surface S_0, an arc M_0M such that the value of integral (147) along this arc of the extremal has a definite value ϱ. The locus of the ends M of these arcs gives us a surface S.

It may easily be seen that the extremals of our family cut the surface S transversally. For, we only need to repeat the previous arguments as used in the case of the central field. The point M_0 is admittedly not fixed in the present case, and moves on the surface S_0, but the extremals of our family cut S_0 transversally by construction, so that the term outside the integral sign on the right-hand side of (158) vanishes at the lower limit precisely as in the case of a central field. Hence the surfaces S which fill the part of space in the neighbourhood of S_0 are cut transversally by the extremals of the present family. We also call the family a field of extremals in this case, whilst the surfaces S are the transversal surfaces of the field. Thus a family of extremals is a field of extremals if there exists a family of surfaces depending on a single parameter which are cut transversally by the extremals of the family. The value of integral (147), taken over the above-mentioned arc M_0M of an extremal of the field is a function $\theta(x, y, z)$ of the coordinates of the point M, and equation (159) is the equation of the family of transversal surfaces of the field. In particular, when $\varrho = 0$ we get the surface S_0. In the case of the integral corresponding to the problem of geometrical optics, the quasi-spheres of the central field are the wave-fronts of a local perturbation at M_0 at different instants. In the general case the transversal surfaces S also give the wave-fronts at different instants, on the assumption that S_0 is the wave-front at the initial instant.

At every point of the transversal surface S the coefficients of δx, δy, δz in the transversality condition (157) must be proportional to the direction cosines of the normal to surface S. On the other hand, we know that these direction cosines are proportional to the partial derivatives of the left-hand sides of (159) with respect to the coordinates, i.e. these partial derivatives must be proportional to the coeffi-

cients in transversality condition (157). But, as we shall now show, the striking fact appears that in the present case we have strict identity, and not merely proportionality, i.e.

$$\frac{\partial\theta}{\partial x} = - H(x, y, z, v, w); \quad \frac{\partial\theta}{\partial y} = v; \quad \frac{\partial\theta}{\partial z} = w, \tag{160}$$

where v and w must obviously be taken as functions of (x, y, z). These are the slope functions of our field such as were discussed in the previous section. This assertion follows at once from the basic formula (158) as we shall now show.

For the sake of clarity we shall first consider the central field. As already mentioned $\theta(x, y, z)$ is in this case the value of integral (147) over the arc M_0M of an extremal of our central field.

Suppose that M is displaced in an arbitrary manner in space, and not along the quasi-sphere as above. Obviously the extremal field connecting M_0 to the moving point M also changes in this case, generally speaking. The motion of M will depend here on a set of three parameters which we shall not fix, instead of on two as for the previous motion on the quasi-sphere. We shall write δ for the differential relative to a change in these parameters. We return to the fundamental formula (158) where the value of integral J can be replaced, by what has been said, by the function $\theta(x, y, z)$. The integral term falls out on the right-hand side of this formula, since we are integrating along an extremal. The term outside the integral sign also vanishes at the lower limit, since M_0 is fixed. But this last term no longer vanishes at the upper limit, since M moves arbitrarily instead of over the quasi-sphere, and we get the equation:

$$\delta\theta(x, y, z) = - H\,\delta x + v\,\delta y + w\,\delta z, \tag{161}$$

whence expressions (160) follow.

The proof of these expressions is precisely the same for any field. Instead of the quasi-spheres we have surfaces S, and the term outside the integral sign on the right-hand side of (158) vanishes as before at the lower limit, since the surface S_0 cuts the extremals transversally.

On eliminating v and w from the three equations (160), we get a first order partial differential equation for the basic function of the field

$$\theta_x + H(x, y, z, \theta_y, \theta_z) = 0. \tag{162}$$

It may be seen from this that *the basic function must satisfy the same equation* (162) *for any field*. Let us now prove, conversely, that every solution of (162) is the basic function for a field.

Let $\theta^{(0)}$ be a solution of (162). We define functions v and w from the expressions:

$$v = \theta_y^{(0)}; \quad w = \theta_z^{(0)}. \tag{163}$$

On differentiating the identity

$$\theta_x^{(0)} + H\left(x, y, z, \theta_y^{(0)}, \theta_z^{(0)}\right) = 0 \tag{164}$$

with respect to y and z, we get two equations (156), i.e. as we saw above, for the constructed functions v and w there is a corresponding family of extremals, for which v and w are the slope functions. By (163) and (164), the left-hand side of (157) is the total differential of $\theta^{(0)}$, i.e. $\theta^{(0)}(x, y, z) = C$ is a family of transversal surfaces for the above-mentioned family of extremals, and this latter family consequently forms a field. We observe further that it follows from the above that the necessary and sufficient condition for a family of extremals to give a field is that the left-hand side of (157) be a total differential, i.e. that the line integral of the left-hand side be independent of the path.

In the case of the integral corresponding to the fundamental problem of geometrical optics, transversality condition (142) has the form:

$$\left(n\sqrt{1 + y'^2 + z'^2} - n\frac{y'^2}{\sqrt{1 + y'^2 + z'^2}} - n\frac{z'^2}{\sqrt{1 + y'^2 + z'^2}}\right)\delta x +$$
$$+ n\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\,\delta y + n\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\,\delta z = 0,$$

or, after obvious simplifications:

$$\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z = 0,$$

whence it follows at once that the transversality condition here coincides with the orthogonality condition, and the transversal surfaces of any field are cut orthogonally by the extremals of the field.

The canonical variables and function H are given in the present case by the equations:

$$v = \frac{ny'}{\sqrt{1 + y'^2 + z'^2}}; \quad w = \frac{nz'}{\sqrt{1 + y'^2 + z'^2}};$$
$$H = \frac{ny'^2}{\sqrt{1 + y'^2 + z'^2}} + \frac{nz'^2}{\sqrt{1 + y'^2 + z'^2}} - n\sqrt{1 + y'^2 + z'^2},$$

or, after simplifying:

$$H = -\frac{n}{\sqrt{1 + y'^2 + z'^2}} = -\sqrt{n^2 - v^2 - w^2},$$

whilst (161) becomes:

$$\theta_x - \sqrt{n^2 - \theta_y^2 - \theta_z^2} = 0 \quad \text{or} \quad \theta_x^2 + \theta_y^2 + \theta_z^2 = n^2(x, y, z).$$

If $n = \text{const.}$, the space is homogeneous, and the extremals become straight lines. They form a field when and only when they are normal to some surface S_0. The remaining transversal surfaces S_0 of the field are obtained by marking off segments of equal length along these normals. The surfaces can be obtained by drawing a family of spheres with centre on S_0 and fixed radius and taking the envelope of this family (Huyghens' construction). This construction is also valid for a non-homogeneous space, provided the spheres are replaced by quasi-spheres. A further point is to recall that [II, 128] explains the conditions in which a family of straight lines is a family of normals to a surface.

80. Theory of fields in the general case. The geometrical theory described above still holds in the case of a plane, when the basic integral has the form:

$$J = \int_{x_0}^{x_1} F(x, y, y')\, dx. \tag{165}$$

We replace y' by the new variable u in accordance with the formula $u = F_{y'}$, and bring in $H(x, y, u) = y'F_{y'} - F$. Instead of Euler's equations for integral (165) we get a system of two first order equations:

$$\frac{dy}{dx} = H_u; \quad \frac{du}{dx} = -H_y. \tag{166}$$

The transversality condition:

$$(F_y - y' F_{y'})\, \delta x + F_{y'}\, \delta y = 0 \tag{167}$$

becomes in the new variables:

$$-H\, \delta x + u\, \delta y = 0. \tag{168}$$

A family of extremals on a plane must contain a single parameter, and will be assumed to cover part of the plane without mutual intersections. In this part of the plane y' and the new variable u are definite functions of the coordinates (x, y) of a point (u is the slope function of the family). On returning to transversality condition (168), we see that it can be interpreted as a first order differential equation for the transversal curves of the family of extremals, i.e. the curves which cut

the extremals transversally:

$$\frac{\delta y}{\delta x} = \frac{H}{u}. \tag{169}$$

We have the special feature here that every family of extremals forms a field. We naturally assume here that the conditions required by the existence and uniqueness theorem for equation (169) are fulfilled.

We now turn to the theory of fields in the general case of any number of dimensions. The proofs will be omitted, since they are entirely analogous to those already given for the case of three-dimensional space. The basic integral will now contain n functions $q_1, \ldots, q_n$ of the independent variable x and their derivatives q_k':

$$J = \int_{x_0}^{x_1} F(x, q_1, q_1', \ldots, q_n, q_n')\,\mathrm{d}x. \tag{170}$$

The corresponding extremals are given by a system of n second order equations:

$$F_{q_k} - \frac{\mathrm{d}}{\mathrm{d}x} F_{q_k'} = 0 \qquad (k = 1, 2, \ldots, n). \tag{171}$$

Instead of the q_k' we introduce new variables p_k:

$$p_k = F_{q_k'}, \tag{172}$$

the functional determinant

$$\frac{D(F_{q_1'}, \ldots, F_{q_n'})}{D(q_1', \ldots, q_n')} \tag{173}$$

being assumed non-zero, i.e. equations (172) are soluble for the q_k'.

The function H, which we take to be expressed in terms of the variables (x, q_k, p_k), is given by

$$H(x, q_k, p_k) = \sum_{s=1}^{n} q_s' p_s - F. \tag{174}$$

We obtain by direct differentiation and the use of (172):

$$H_{q_k} = -F_{q_k}; \quad H_{p_k} = q_k',$$

whilst system (171) can be rewritten as $2n$ first order equations (the canonical system):

$$\frac{\mathrm{d}q_k}{\mathrm{d}x} = H_{p_k}; \quad \frac{\mathrm{d}p_k}{\mathrm{d}x} = -H_{q_k}. \tag{175}$$

We can define with the aid of integral (170) the concept of quasi-length of any curve in $(n + 1)$-dimensional space with coordinates $(x, q_1, \ldots, q_n)$. If a set of extremals, which depends on n arbitrary constants, fills part of $(n + 1)$-dimensional space without mutual intersections, we say that the extremals form a *family*. The q_k' and hence the p_k are definite functions of a point in this part of space, i.e. are functions of $(x_1, q_1, \ldots, q_n)$ (the slope functions of the family). The central field is defined exactly as in the case of three-dimensional space. The general field is obtained by taking some hypersurface S_0: $\varphi(x, q_1, \ldots, q_n) = 0$. The transversality conditions give us n relationships for finding the derivatives q_k' at every point of S_0; on taking the values obtained as the initial values in the solution of system (171), we in general obtain a family of extremals intersecting S_0 transversally. Precisely as in three-dimensional space, we can construct further surfaces S which are cut transversally by extremals of the family, and this family forms a field. There exists at every point a basic function $\theta(x, q_1, \ldots, q_n)$, which, say for a central field, gives the value of the solution from the central point M_0 to a variable point on an extremal of the field. The basic function is similarly defined for any other field. Whatever the field, we have for the basic function:

$$\theta_x = -H; \quad \theta_{q_k} = F_{q_k'} = p_k,$$

and it must satisfy the partial differential equation:

$$\theta_x + H\left(x, q_1, \ldots, q_n, \theta_{q_1}, \ldots, \theta_{q_n}\right) = 0. \tag{176}$$

Conversely, any solution of this last equation is in general the basic function of some field, the functions (172) corresponding to this field being defined by the formulae $p_k = \theta_{q_k}$. The expression

$$-H\,\delta x + \sum_{k=1}^{n} p_k\,\delta q_k$$

will be a total differential when and only when the p_k are the slope functions of a field, and in this case the expression will be the total differential of the basic function $\theta(x, q_1, \ldots, q_n)$ of this field.

81. A singular case. An important singular case may be mentioned when the transformation to the canonical variables is carried out. Suppose that F is a first degree homogeneous function in the derivatives q_k', as is the case, for instance, in the parametric form of the

variational problem. We have by Euler's formula for homogeneous functions:

$$\sum_{s=1}^{n} q'_s F_{q'_s} = F. \tag{177}$$

Differentiation of this identity with respect to the q'_k gives

$$\sum_{s=1}^{n} q'_s F_{q'_s q'_k} = 0,$$

and the determinant of this homogeneous system must vanish. But this is precisely determinant (173), which must be non-zero for a transition to the canonical variables to be possible. It follows at once from identity (177) that the function H vanishes identically in the present case. By the foregoing, we can define a field of extremals, and for every field we have the basic function $\theta(x, q_1, \ldots, q_n)$, the partial derivatives of which are given by the equations:

$$\theta_x = F - \sum_{s=1}^{n} q'_s F_{q'_s} \equiv 0; \quad \theta_{q_k} = F_{q'_k}. \tag{178}$$

The first of these equations shows that the basic function does not contain x. The right-hand sides of the equations $\theta_{q_k} = F_{q'_k}$ are homogeneous functions of zero degree in the $q'_{k'}$ and with the aid of these equations the ratios q'_k/q'_1 $(k = 2, \ldots, n)$ can be expressed in terms of the derivatives θ_{q_k}. On substituting these expressions in equation (177), we obtain a partial differential equation which here replaces (176).

Let us go through all the working for the integral expressing the length of a curve in n-dimensional space:

$$J = \int_{x_0}^{x_1} \sqrt{\sum_{i,k=1}^{n} a_{ik} q'_i q'_k}\; dx. \tag{179}$$

The coefficients a_{ik} satisfy the relationships $a_{ik} = a_{ki}$ and are given functions of the variables q_k. We have here:

$$\theta_{q_k} = F_{q'_k} = \sum_{s=1}^{n} \frac{a_{ks} q'_s}{F} \qquad \left(F = \sqrt{\sum_{i,k=1}^{n} a_{ik} q'_i q'_k}\right),$$

whence

$$\frac{q'_k}{F} = \sum_{s=1}^{n} A_{ks} \theta_{q_s},$$

where the A_{ik} denote the elements of the inverse of matrix a_{ik} [III$_1$, 25].

On substituting the expressions q_k'/F in equation (177), we obtain the required partial differential equation which has to be satisfied by the basic function of any field of extremals for integral (179):

$$\sum_{i,k=1}^{n} A_{ik}\,\theta_{q_i}\,\theta_{q_k} = 1. \tag{180}$$

The value of integral (179), over an extremal of the field between points M_0 and M, gives the geodesic distance between these points, and we obtain for the square of this distance $\Gamma = \theta^2$ the partial differential equation in any field:

$$\sum_{i,k=1}^{n} A_{ik}\,\Gamma_{q_i}\,\Gamma_{q_k} = 4\Gamma. \tag{181}$$

The independent variable in this problem is a parameter which can be chosen quite arbitrarily and does not appear in the coefficients a_{ik} or the function θ. It is possible here to discuss a field and basic function in n-dimensional space $(q_1, q_2, \ldots, q_n)$; one of the variables in this space can be taken as an independent variable, whilst equation (180) amounts here to a symmetric way of writing equation (176).

In the case of the basic problem of geometrical optics, we have in the parametric form:

$$F = n\,(x, y, z)\sqrt{x'^2 + y'^2 + z'^2},$$

and equation (180) becomes

$$\theta_x^2 + \theta_y^2 + \theta_z^2 = n^2\,(x, y, z).$$

We obtained this equation earlier, by starting from the form of the basic integral in which the role of independent variable is played by x.

Throughout the above treatment, no assumption has been made that the independent variable is absent from the integrand F. In the case of the problem of geodesics, corresponding to which we have integral (179), the a_{ik} do not contain the independent variable, and another procedure can be adopted. On writing φ, as in [73], for the expression under the radical sign in (179), and taking the arc length as parameter, i.e. on introducing the relationships

$$\varphi = \sum_{i,k=1}^{n} a_{ik}\,q_i'\,q_k' = 1,$$

we obtain the system of differential equations (111):

$$\varphi_{q_i} - \frac{\mathrm{d}}{\mathrm{d}s}\,\varphi'_{q_i} = 0 \qquad (i = 1, 2, \ldots, n),$$

and we can now pass to the canonical variables in the usual way, i.e. instead of the q_i' we introduce the new variables $p_i = \varphi'_{q_i}$.

The function H is given by: $H(q_k, p_k) = \sum\limits_{s=1}^{n} q_s'\,p_s - \varphi$, and it follows at once from the fact that φ is a homogeneous second degree polynomial in the q_s' that $H = \varphi$. On writing φ in terms of q_k and p_k and substituting $p_k = \theta_{q_k}$

in the relationship $\varphi = 1$, we obtain a partial differential equation for θ. If, for clarity, we write ψ for the function φ expressed in terms of q_k and p_k, we get the canonical system:

$$\frac{dq_k}{ds} = \psi_{p_k}; \quad \frac{dp_k}{ds} = -\psi_{q_k} \qquad (k = 1, 2, \ldots, n),$$

In view of the fact that $\psi(q_k, p_k)$ is a homogeneous second degree polynomial in the p_k, we can say that the last equations retain their form if we simultaneously substitute in them ap_k for p_k and as for s, where a is an arbitrary constant. Let $q_k^{(0)}$ and $p_k^{(0)}$ be the initial values of q_k and p_k at $s = 0$. By what has been said above, we can assert that p_k, $p_k^{(0)}$ and s appear in the solution of the canonical system only in the combinations sp_k, $sp_k^{(0)}$, i.e. this solution has the form

$$q_k = \varphi_k\left(r_k, q_k^{(0)}\right); \quad t_k = \psi_k\left(r_k, q_k^{(0)}\right) \qquad (k = 1, 2, \ldots, n),$$

where $t_k = sp_k$ and $r_k = sp_k^{(0)}$. On taking into account the relationship $\psi(q_k, p_k) = 1$ and the fact that $t_k = sp_k$, we can say that the square of the geodesic distance from the point $(q_1^{(0)}, \ldots, q_n^{(0)})$ to the point $(q_1, q_2, \ldots q_n)$ can be expressed as

$$s^2 = \Gamma = \psi\,(q_k, t_k) = \psi\left[q_k, \psi_k\left(r_k, q_k^{(0)}\right)\right].$$

By using the equations $q_k = \varphi_k(r_k, q_k^{(0)})$, we can express r_k in terms of q_k and $q_k^{(0)}$, so that the right-hand side of the last formula is expressed in terms of q_k and $q_k^{(0)}$.

82. Jacobi's theorem. If we can find the general solution of the system of ordinary differential equations (175), we can naturally obtain all the possible fields corresponding to a given variational problem, and hence find any solution of equation (176). We shall return to this question in the second half of the present volume, when dealing with the theory of first order partial differential equations. Conversely, if we are in a position to find a solution of equation (176), we shall shortly prove that the general solution of system (175) can be obtained. We first need to see precisely what is meant by saying that we are in a position to find a solution of (176). This equation must define a function θ of the independent variables $(x, q_1, \ldots, q_n)$. It does not contain θ itself, so that, on adding an arbitrary constant a to any solution, we again obtain a solution. We shall define the *complete integral* of this equation as the solution of it which, in addition to the arbitrary constant a already mentioned, also contains n arbitrary constants:

$$\theta = \theta\,(x, q_1, \ldots, q_n, a_1, \ldots, a_n) + a, \tag{182}$$

the determinant whose elements are the second order partial derivatives $\theta_{q_i a_k}$ being assumed non-zero. As a matter of fact, a knowledge of the complete integral of equation (176) enables us to construct

with the aid of simple differentiations the general solution of system (175), i.e. the following theorem of Jacobi holds:

If the complete integral (182) *of equation* (176) *is known, the equations*

$$\theta_{a_k} = b_k; \tag{183}$$

$$\theta_{q_k} = p_k \qquad (k = 1, \ldots, n), \tag{183$_1$}$$

where a_k and b_k are arbitrary constants, give a solution of system (175) *depending on $2n$ arbitrary constants.*

In view of our assumption that the determinant $\| \theta_{q_i a_k} \|$ differs from zero, we can solve equation (183) for the q_i, the variables q_k being expressed in terms of the independent variable x and the arbitrary constants a_s and b_s $(s = 1, 2, \ldots, n)$. On substituting these expressions for the q_k in the left-hand sides of equation (183_1), we obtain expressions for the p_k also in terms of $x, a_1, \ldots, a_n$, and we have to show that the expressions thus obtained for the q_k and p_k satisfy system (175). On differentiating equations (183) with respect to x and equation (176) with respect to a_i, we obtain the $2n$ equations:

$$\frac{\partial^2 \theta}{\partial x\, \partial a_i} + \sum_{s=1}^{n} \frac{\partial^2 \theta}{\partial q_s\, \partial a_i} \cdot \frac{dq_s}{dx} = 0, \quad \frac{\partial^2 \theta}{\partial x\, \partial a_i} + \sum_{s=1}^{n} H_{p_s} \frac{\partial^2 \theta}{\partial q_s\, \partial a_i} = 0$$

$$(i = 1, \ldots, n),$$

whence we have the n equations:

$$\sum_{s=1}^{n} \frac{\partial^2 \theta}{\partial q_s\, \partial a_i} \left(\frac{dq_s}{dx} - H_{p_s} \right) = 0 \qquad (i = 1, \ldots, n).$$

By hypothesis, $\| \theta_{q_s a_i} \| \neq 0$, whence it follows at once that $dq_s/dx = H_{p_s}$. To prove that the remaining equations of system (175) hold, we differentiate equations (183_1) with respect to x and equation (176) with respect to q_i:

$$\frac{dp_i}{dx} = \frac{\partial^2 \theta}{\partial x\, \partial q_i} + \sum_{s=1}^{n} \frac{\partial^2 \theta}{\partial q_i\, \partial q_s} \frac{dq_s}{dx}, \quad 0 = \frac{\partial^2 \theta}{\partial x\, \partial q_i} + \sum_{s=1}^{n} H_{p_s} \frac{\partial^2 \theta}{\partial q_s\, \partial q_i} + H_{q_i}.$$

On subtracting term by term and using the equations already proved, we in fact obtain the remaining equations of system (175).

We thus see that finding the complete integral of equation (176) gives the general solution of system (175), defining the extremals of our problem. The relationship between system (175) and equation (176) corresponds to the geometrical fact that every field of extremals of the problem can be described with the aid either of the extremals themselves or of the transversal surfaces of the field.

83. Discontinuous solutions. It happens in certain cases that there is no curve having a continuously varying tangent which yields an extremum of the functional, and the question arises as to the possibility of finding a solution among curves of a more general class, say those which have no tangent at individual points, but always have a definite tangent from the left or right (curves with vertices). We shall give the broad outlines of the argument for the elementary functional

$$J = \int_{x_0}^{x_1} F(x, y, y')\, dx, \tag{184}$$

without dwelling on detailed proofs.

We shall first take a particular case, viz. the functional

$$J = \int_{-1}^{1} y^2 (1 - y')^2\, dx, \tag{185}$$

where the required extremal must pass through the points $M_0(-1, 0)$ and $M_1(1, 1)$. Functional (185) will obviously be positive for any such curve. We construct a curve consisting of two straight segments joining M_0 and M_1, viz. we form the step-line M_0OM_1, where O is the origin on the (x, y) plane. It is easily seen that functional (185) vanishes for this step-line, since $y = 0$ along M_0O and $y' = 1$ along OM_1. This step-line having a vertex at the origin will obviously give an extremum of integral (185).

We now take the general case. Suppose that some curve that joins (x_0, y_0) and (x_1, y_1) and has a vertex at (x_2, y_2) gives an extremum of functional (184) in comparison with others sufficiently close to it; these others may also have a vertex and must pass through the given end-points (x_0, y_0) and (x_1, y_1). We can assume that (x_2, y_2) is fixed as well as the end-points, i.e. the vertex of the curve under investigation is fixed. This curve must all the more give an extremum of the functional with this hypothesis. Hence it follows at once that the pieces of curve corresponding to the intervals $[x_0, x_2]$ and $[x_2, x_1]$ of the x axis, must be extremals of the problem, i.e. must satisfy the corresponding Euler equation. It is essential to consider the conditions which have to be satisfied by the ordinates and slopes of the tangents at the vertex. We define the variation of integral (174) by taking our curve as the initial curve and splitting the total interval $[x_0, x_1]$ into two parts: $[x_0, x_2]$ and $[x_2, x_1]$.

On recalling that the ends of the curve are fixed and that both pieces of the curve satisfy Euler's equation, we obtain the following expression for the first variation:

$$\delta J = [F - y' F_{y'}]_{x_2-0}\,\delta x_2 - [F - y' F_{y'}]_{x_2+0}\,\delta x_2 + \\ + [F_{y'}]_{x_2-0}\,\delta y_2 - [F_{y'}]_{x_2+0}\,\delta y_2 .$$

In view of the arbitrariness of δx_2 and δy_2, we obtain the following two conditions, which must be fulfilled at the vertex of the curve if it is to give an extremum of integral (184):

$$[F - y' F_{y'}]_{x_2-0} = [F - y' F_{y'}]_{x_2+0}; \quad [F_{y'}]_{x_2-0} = [F_{y'}]_{x_2+0}. \tag{186}$$

These conditions are usually known as the *Weierstrass–Erdmann conditions.* We suggest that the reader prove that they are in fact satisfied at the origin for the step-line giving the extremum of integral (185).

We remark that conditions (186) reduce to requiring the continuity of the expressions $F - y' F_{y'}$ and $F_{y'}$ at the point $x = x_2$ at which y' has a jump. These expressions will clearly be continuous at the remaining points, where y' is continuous. Suppose that we have obtained the general solution of Euler's equations. The values of the two arbitrary constants appearing in this solution will in general be different for the intervals $[x_0, x_2]$ and $[x_2, x_1]$. Let

$$y = \omega_1 (x, C_1, C_2)$$

be the general solution for the interval $[x_0, x_2]$ and

$$y = \omega_2 (x, C_3, C_4)$$

for the interval $[x_2, x_1]$. We have to find five constants, viz. the four arbitrary constants C_1, C_2, C_3, C_4, and the abscissa x_2 of the vertex. We have two boundary conditions at $x = x_0$ and $x = x_1$, and two conditions (186). The missing fifth equation is obtained from the continuity of the curve at $x = x_2$:

$$\omega_1 (x_2, C_1, C_2) = \omega_2 (x_2, C_3, C_4).$$

The case when the curve has several vertices could have been considered in precisely the same way.

Similar conditions can be found for the discontinuous problem in the case of a multiple integral:

$$J = \iint_B F (x, y, u, u_x, u_y)\, \mathrm{d}x\, \mathrm{d}y. \tag{187}$$

Suppose that a surface $u(x,y)$ having a fixed contour and containing a line of discontinuity yields an extremum of this integral. In other words, $u(x, y)$ must be defined in the domain B of the (x, y) plane and must have given values on the contour of B: yet there can exist inside the domain a curve λ along which the first order derivatives of $u(x, y)$ have discontinuities, though when taken from either side of the curve these partial derivatives have definite limits, there being possibly points at which these limits differ. The function that gives a relative extremum of integral (187) is sought within this class of functions.

Suppose that a function of this class actually yields an extremum and has a discontinuity curve λ inside B, dividing B into two parts: B_1 and B_2. It can be seen by precisely the same arguments as above that $u(x, y)$ must be a solution of Ostrogradskii's equations in the domains B_1 and B_2. The essential problem is to find the conditions which must be satisfied by $u(x, y)$ and its first order partial derivatives at points of λ. Let φ be a function containing $u(x,y)$ and its first order partial derivatives. Such a function will in general have different limits, say φ_1 and φ_2, on approaching points of λ from domains B_1 and B_2. We introduce a special notation for the difference between these limits, i.e. for the jump in the function φ:

$$[\varphi] = \varphi_2 - \varphi_1 .$$

We return to formula (26), giving the variation of a double integral. The first term on the right-hand side can be written as:

$$\int_l \delta u \cdot \left(F_{u_x} \frac{dy}{ds} - F_{u_y} \cdot \frac{dx}{ds} \right) ds,$$

or, on observing that dy/ds and $(-dx/ds)$ give the direction cosines of the outward normal n to the contour l we can write this term as

$$\int_l \delta u \left[F_{u_x} \cos(n, x) + F_{u_y} \cos(n, y) \right] ds .$$

We now apply formula (26) to integral (187) by splitting the domain B into B_1 and B_2. In each of these latter domains $u(x, y)$ must satisfy Ostrogradskii's equation, so that the double integrals will vanish. The contours of B_1 and B_2 are made up of parts of the contour l and the curve λ. We have $\delta u = 0$ on the contour l, whilst the direction cosines of the outward normal differ in sign for domains B_1 and B_2 on λ. Hence we finally have:

$$\delta J = \int_\lambda \delta u \left\{ [F_{u_x}] \cos(n, x) + [F_{u_y}] \cos(n, y) \right\} ds,$$

where n is the direction of the normal to λ, outwards with respect to B_2. The condition $\delta J = 0$ and the fact that δu is arbitrary lead to one of the conditions that must hold along λ:

$$[F_{u_x}] \cos(n, x) + [F_{u_y}] \cos(n, y) = 0. \tag{188}$$

Only one condition has been obtained because the curve λ was regarded as fixed when considering the first variation of integral (187). A more detailed consideration of the variation of the integral leads to a second condition of the form†:

$$[F] = (F_{u_x})_2 [u_x] + (F_{u_y})_2 [u_y], \tag{189}$$

where the subscript 2 outside the curved brackets indicates that we have to take the value of the quantity in the brackets along λ from the side of domain B_2. Conditions (188) and (189) are analogous to conditions (186) for the functional (184).

84. One-sided extrema. We have considered above [67] the problem: to find the curve, among those joining the points M_0 and M_1 on the (x, y) plane, such that it generates the surface of least area on revolution about OX.

The functional corresponding to this problem has the form

$$J = \int_{x_0}^{x_1} y \sqrt{1 + y'^2}\, dx.$$

Strictly speaking, we have to impose here the condition that the curve $y(x)$ lie above OX, i.e. that the inequality $y(x) \geqslant 0$ is fulfilled. Those problems of the variational calculus in which the required functions (or their derivatives) must be subjected to certain inequalities are usually called one-sided extremum problems.

Let us take the elementary problem of the extremum of the functional

$$J = \int_{x_0}^{x_1} F(x, y, y')\, dx \tag{190}$$

subject to a supplementary condition of the form

$$y - \varphi(x) \geqslant 0,$$

† N. M. Günther, *Leçons sur le calcul des variations.*

where $\varphi(x)$ is a given function having a continuous derivative. In other words, the curve $y(x)$ must lie above the curve $y = \varphi(x)$. In addition, the required curve must pass through the given points $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$. The required curve may consist of pieces lying above $y = \varphi(x)$ and of pieces of the latter curve. In Fig. 3 there are two pieces (M_0A and BM_1) lying above the curve, and the piece AB of the curve itself. Two-sided variation is possible for the pieces M_0A and BM_1, and as usual, these pieces must be extremals of integral (190). Only one-sided variation, for which $\delta y \geqslant 0$, is possible on the piece AB. On taking (13) into account for the variation of integral (190), we can say that the necessary condition for a minimum of this integral is that, along AB:

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} \geqslant 0 .$$

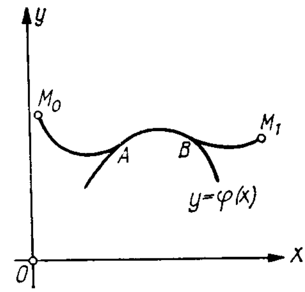

FIG. 3

In addition, certain conditions must be fulfilled at points A and B in order for an extremum to exist. We shall merely mention, without going into a discussion of the question, that this condition reduces in the elementary case to the fact that the curves M_0A and BM_1 have a tangent in common with the curve AB at points A and B.

85. Second variation. We have so far only investigated the first variation of functionals of various types. Putting this first variation equal to zero has given us the necessary condition for a given curve or surface to extremize the functional. This necessary condition is precisely analogous to the situation in the differential calculus, where the necessary condition for a function of several variables to attain an extremum at a point is that its first order total differential vanish at this point. In the differential calculus sufficient conditions were obtained in certain cases; in order to state these, we had to find the second order partial derivatives of the function concerned. The problem of establishing sufficient conditions is far more difficult in the variational calculus. We shall only consider the elementary functional

$$J = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x \tag{191}$$

in the case of fixed ends. We consider as usual the neighbouring curves $y(x) + a\eta(x)$ and define the second variation of functional (191) as the term in the expansion of $J(a)$ in powers of a which contains a^2, i.e. we put

$$\delta^2 J = \frac{a^2}{2}\left[\frac{d^2 J}{da^2}\right]_{a=0}.$$

This leads directly to the formula:

$$\delta^2 J = \frac{a^2}{2}\int_{x_0}^{x_1} (P\eta^2 + 2Q\eta\eta' + R\eta'^2)\,dx, \tag{192}$$

where

$$P = F_{yy}; \quad Q = F_{yy'}; \quad R = F_{y'y'}. \tag{193}$$

Since $2Q\eta\eta' = Qd(\eta^2)/dx$, we obtain on integrating by parts and using the fact that $\eta(x_0) = \eta(x_1) = 0$:

$$\delta^2 J = \frac{a^2}{2}\int_{x_0}^{x_1} (S\eta^2 + R\eta'^2)\,dx \quad \text{where} \quad S = P - \frac{dQ}{dx}. \tag{194}$$

We assume that the necessary condition for an extremum is fulfilled i.e. that $y(x)$ is an extremal. We shall talk about a minimum of integral (191) for the sake of definiteness. The function $J(a)$ must have a minimum at $a = 0$, so that the necessary condition for a minimum is that $\delta^2 J \geqslant 0$ for any choice of $\eta(x)$. We show that it follows directly from this that the inequality $R \geqslant 0$ must hold along our curve. In fact, suppose that we had $R(c) < 0$ at some point $x = c$ on our curve. Since $R(x)$ is assumed continuous, this inequality will hold over some sufficiently small interval $[c - \varepsilon, c + \varepsilon]$. We now define the function $\eta(x)$ so that it vanishes outside and at the ends of this interval, has all the necessary derivatives, is sufficiently small in absolute value in the interval but performs fairly rapid oscillations. With this choice of $\eta(x)$, integral (194) reduces to an integral over the interval $[c - \varepsilon, c + \varepsilon]$, in which $R(x)$ has negative values by hypothesis. The dominant term under the integral sign is that containing $\eta'^2(x)$, and the integral turns out negative, which contradicts the above-mentioned necessary condition for a minimum of integral (191). Thus, the necessary condition for the extremal $y(x)$ to minimize the integral (191) is that

$$F_{y'y'} \geqslant 0 \tag{195}$$

along this extremal. Similarly, the necessary condition for the extremal to maximize of integral (191) is that

$$F_{y'y'} \leqslant 0$$

along this extremal.

These conditions are generally known as *Legendre's conditions.*

86. Jacobi's condition. Before continuing our investigation of the second variation, some important remarks must be made in connection with the zeros of the solutions of a second order linear equation:

$$y'' + p(x)\,y' + q(x)\,y = 0, \tag{196}$$

where the coefficients $p(x)$ and $q(x)$ are assumed continuous in a closed interval $[x_0, x_1]$, to which all further discussion will relate. Let x_2 be the zero of some solution $y(x)$ of equation (196), i.e. $y(x_2) = 0$. It is easily shown that $y'(x_2) \neq 0$. For otherwise the solution $y(x)$ would satisfy at x_2 the initial conditions $y(x_2) = y'(x_2) = 0$, and by the existence and uniqueness theorem, must vanish identically. We can thus assert that *every solution of equation* (196) *changes sign when it passes through a zero.* Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of equation (196). They cannot have common zeros. For, if they did, the Wronskian of these solutions [II, 24] would vanish at the point, and hence throughout the interval $[x_0, x_1]$, which contradicts the linear independence. If the solutions are linearly dependent, i.e. only differ by a constant factor, they obviously have common zeros. As above, let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions. We have the formula [II, 24]:

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \Delta_0 \frac{e^{-\int_{x_0}^{x} p(x)\,dx}}{y_1^2},$$

where Δ_0 is a definite non-zero number. The right-hand side of this formula retains a constant sign, viz. the sign of Δ_0. Hence it follows that the ratio $y_2 : y_1$ must vary monotonically. Let $\Delta_0 > 0$ say. The ratio $y_2 : y_1$ must increase as x increases, and must jump from $(+\infty)$ to $(-\infty)$ on passing through the zero $y_1(x)$. On further increase of x we can arrive at the following zero of $y_1(x)$ only by passing through a zero of $y_2(x)$. We thus see that the zeros of any two linearly independent solutions of equation (196) must alternate, i.e. *between two zeros of one solution there must be one and only one zero of any other solution linearly independent of the first.*

Returning to our investigation of the second variation, we shall assume that a stronger condition than (195), namely $F_{y'y'} > 0$, holds along extremals of the field (called the *strengthened Legendre condition*).

We take the integral appearing in (194) with the letter η replaced by the letter u:

$$K(u) = \int_{x_0}^{x_1} (Su^2 + Ru'^2)\,\mathrm{d}x. \tag{197}$$

We observe that Euler's equation for this integral has the form

$$L(u) = \frac{\mathrm{d}}{\mathrm{d}x}(Ru') - Su = 0. \tag{198}$$

On the other hand, if we use the fact that $Ru'^2\,\mathrm{d}x = Ru'\,\mathrm{d}u$, and integrate by parts, we get:

$$K(u) = -\int_{x_0}^{x_1} uL(u)\,\mathrm{d}x, \tag{199}$$

where u must be assumed to vanish at the ends. Let us consider the solution of linear equation (198) which vanishes at the end $x = x_0$. Every such solution differs only by a constant factor from the solution $u_0(x)$ of (198) satisfying the initial conditions:

$$u_0(x_0) = 0; \quad u_0'(x_0) = 1, \tag{200}$$

the uniqueness and existence theorem being applicable for equation (198) throughout the interval $[x_0, x_1]$ by virtue of the strengthened Legendre condition. An essential point for what follows is whether the solution $u_0(x)$ has zeros inside the interval. If there are such zeros, our extremal cannot give a minimum of integral (191). The proof of this statement is as follows. Suppose that $u_0(x)$ has a zero $x = x_2$ inside the interval, i.e. $u_0(x_2) = 0$. As we saw above, we must have $u_0'(x_2) \neq 0$. We draw a curve $u_1(x)$ consisting of two extremals of the integral $K(u)$, i.e. we put $u_1(x) = u_0(x)$ for $x_0 \leqslant x \leqslant x_2$ and $u_1(x) = 0$ for $x_2 \leqslant x \leqslant x_1$. We thus obtain a curve which has a vertex at $x = x_2$. On expressing the integral $K(u_1)$ over the interval $[x_0, x_2]$ in accordance with (199) (the values of $u_1(x)$ at the ends of this interval are zero), we see that the value of this integral over $[x_0, x_2]$ is zero, since $u_0(x)$ satisfies equation (198) in $[x_0, x_2]$. Similarly, the value of the integral $K(u_1)$ over the interval $[x_2, x_1]$ is also zero, since $u_1(x)$ vanishes

identically in this interval. Hence integration over the whole of $[x_0, x_1]$ gives us $K(u_1) = 0$.

We now see if the first of the Weierstrass–Erdmann conditions holds for integral (197) at the vertex x_2. To the right we have $u_1(x_2 + 0) = u'(x_2 + 0) = 0$, and to the left: $u_1(x_2 - 0) = 0$ and $u_0'(x_2 - 0) = u_0(x_2) \neq 0$, whence it is clear that the condition in question is not fulfilled, so that $u_1(x)$ cannot yield an extremum for the integral $K(u)$. Hence it follows, since $K(u_1) = 0$, that there are curves, possibly with a vertex, as close as desired to $u_1(x)$ and satisfying the boundary conditions $u(x_0) = u(x_1) = 0$, for which $K(u) < 0$. By smoothing away the vertices, it can be assumed that this last inequality still holds for certain curves $u_2(x)$ having continuously varying tangents. On putting $\eta = u_2$ in (194), we have $\delta^2 J < 0$ for curves $y(x) = a\eta(x) = y(x) + a\, u_2(x)$ as close as desired to the curve under investigation, i.e. with a close to zero, so that this curve cannot minimize integral (191). Hence, *if* $u_0(x)$ *has zeros inside the interval* $[x_0, x_1]$, *the extremal* $y(x)$ *satisfying the strengthened Legendre condition, cannot minimize integral* (191).

Equation (198) is usually known as *Jacobi's equation*, and if $u_0(x) \neq 0$ for $x_0 < x < x_1$, the extremal $y(x)$ is said to satisfy *Jacobi's condition* in the interval $[x_0, x_1]$. If $u_0(x) \neq 0$ for $x_0 < x \leqslant x_1$, $y(x)$ is said to satisfy the *strengthened Jacobi condition.*

We observe that coefficients S and R of equation (198) depend by definition on the choice of the extremal $y(x)$, so that the condition stated above is in fact a condition imposed on the extremal $y(x)$. On recalling what was said above about the alternation of the zeros of solutions of a second order linear equation, we can say that, if Jacobi's condition is satisfied, no solution of (198) can have more than one zero inside $[x_0, x_1]$.

Suppose that the strengthened Legendre and Jacobi conditions are fulfilled along the extremal $y(x)$. By (200), the solution $u_0(x)$ of (198) is positive for x close to x_0, and hence is positive throughout $[x_0, x_1]$, and in particular, at $x = x_1$. We slightly modify the first of conditions (200) by putting $u_0(x) = a$, where a is a sufficiently small positive number. We have $u_0(x_1) > 0$ as before. But, by what has been said above, $u_0(x)$ cannot have more than one zero inside $[x_0, x_1]$, and if there were such a zero, $u_0(x)$ would have to change sign on passing through it, which contradicts the fact that $u_0(x)$ has the same sign at the ends of $[x_0, x_1]$. Thus, when the strengthened Legendre and Jacobi conditions hold, the solution $u_0(x)$ of equation (198), defined

by the initial conditions: $u_0(x) = a$ and $u_0'(x_0) = 1$, where a is a fairly small positive number, will be positive throughout the closed interval $[x_0, x_1]$.

By using this solution of equation (198), we can now reduce expression (194) to a form from which it will follow at once that $\delta^2 J \geqslant 0$.

Let $\omega(x)$ be any function with a continuous derivative. We have the obvious equation:

$$\int_{x_0}^{x_1} (2\eta\eta' \omega + \eta^2 \omega') \, dx = 0,$$

since the integrand is the total derivative of the function $\eta^2 \omega$, vanishing at the ends of the interval. On multiplying this integral by $\alpha^2/2$ and adding to the right-hand side of (194), we obtain:

$$\delta^2 J = \frac{\alpha^2}{2} \int_{x_0}^{x_1} \left[(S + \omega') \eta^2 + 2\omega\eta\eta' + R\eta'^2\right] dx.$$

We require that the integrand be a perfect square, i.e. that

$$\omega^2 - R(S + \omega') = 0.$$

On setting $\omega = -Ru'/u$ in this equation, we arrive at once at (198), i.e. we can take as ω the function $\omega = -Ru_0'/u$, where it is essential that $u_0(x)$ should not vanish throughout the closed interval $[x_0, x_1]$. With this choice of ω we reduce formula (194) to the form

$$\delta^2 J = \frac{\alpha^2}{2} \int_{x_0}^{x_1} R\left(\eta' + \frac{\omega}{R}\eta\right)^2 dx,$$

whence it follows at once that $\delta^2 J \geqslant 0$.

Therefore, *if an extremal satisfies the strengthened Legendre and Jacobi conditions, the second variation* (194) *must be non-negative for this extremal.*

87. Weak and strong extrema. An extremal $y(x)$ is said to yield a *weak extremum* of integral (191) if it gives the integral an extremum in comparison with all the curves lying in some first order ε-neighbourhood of it, i.e. the curves sufficiently close in respect to the ordinate and slope of the tangent. If an extremal gives the integral an extremum in comparison with all the curves lying on a zero order ε-neighbourhood of it (close only as regards the ordinate), the extremal

is said to give the integral a *strong extremum*. Obviously, every strong extremum is also a weak extremum, but the converse does not always hold.

It is shown in courses of variational calculus, that the strengthened Legendre and Jacobi conditions discussed above are sufficient conditions for an extremal to give integral (191) a weak minimum†.

Let us take a new approach to Jacobi's equation and give a geometrical interpretation of Jacobi's condition. Suppose we have a family of extremals $y(x, a)$ depending on one parameter, and let

$$u(x) = \left.\frac{\partial y(x, a)}{\partial a}\right|_{a=a_0}, \tag{201}$$

where a_0 is a particular value of the parameter a.

For any a, the functions $y(x, a)$ satisfy Euler's equation:

$$F_y[x, y(x,a), y'(x,a)] - \frac{d}{dx} F_{y'}[x, y(x,a), y'(x,a)] = 0.$$

On differentiating both sides with respect to a and changing the order of the differentiations with respect to a and x, we obtain:

$$F_{yy}[x, y(x,a), y'(x,a)] \frac{\partial y(x,a)}{\partial a} + F_{yy'}[\quad] \frac{\partial y'(x,a)}{\partial a} -$$

$$- \frac{d}{dx}\left\{F_{yy'}[\quad] \frac{\partial y(x,a)}{\partial a} + F_{y'y'}[\quad] \frac{\partial y'(x,a)}{\partial a}\right\} = 0.$$

But it follows from (201) that

$$\left.\frac{\partial y(x,a)}{\partial a}\right|_{a=a_0} = u(x); \qquad \left.\frac{\partial y'(x,a)}{\partial a}\right|_{a=a_0} = u'(x),$$

and we thus get the following equation for u:

$$F_{yy} u + F_{yy'} u' - \frac{d}{dx}\left(F_{yy'} u + F_{y'y'} u'\right) = 0,$$

which can be written as

$$\frac{d}{dx}(Ru') - Su = 0, \tag{202}$$

where

$$R = F_{y'y'}[x, y(x, a_0), y'(x, a_0)]; \quad S = F_{yy}[\quad] - \frac{d}{dx} F_{yy'}[\quad]. \tag{203}$$

† M. A. Lavrent'ev and L. A. Lyusternik, *Course of Variational Calculus* (Kurs variatsionnogo ischisleniya), 1938, p. 168.

Equation (202) is the same as Jacobi's equation obtained above. We now take as the family of extremals a pencil issuing from a fixed point (x_0, y_0). If the slope of the extremal at (x_0, y_0) is taken as the parameter a, we obtain the initial conditions for extremals of the family: $y(x_0, a) = y_0$, $y'(x_0, a) = a$, so that the $u(x)$ defined by (201) will satisfy the initial conditions $u(x_0) = 0$; $u'(x_0) = 1$, i.e. will coincide with the solution $u_0(x)$ of Jacobi's equation which we introduced above. The equation $u_0(x) = 0$ thus coincides with the equation $\partial y(x, a)/\partial a\,|_{a=a_0} = 0$. By the theory of envelopes, this latter equation gives the abscissae of the points of contact of the envelope of the pencil with the extremals of the pencil. Let $y(x, a_0)$ be an extremal of the pencil and let this extremal touch the envelope at the point with abscissa $x = x_2$. The point of contact is said to be conjugate to the initial point with abscissa $x = x_0$ with respect to the extremal $y(x, a_0)$. It follows from the foregoing discussion that the equation $u_0(x) = 0$ gives the abscissa x_2 of the conjugate point. We can therefore find the conjugate point on a given extremal without knowing the equation of the entire pencil, if we are able to construct the solution $u_0(x)$ of Jacobi's equation (202) corresponding to the extremal taken. We can therefore say that the strengthened Jacobi condition has the geometrical meaning that the interval $[x_0, x_1]$ does not contain points conjugate to the initial point of the extremal. It may be remarked that the above discussion is not altogether precise because the equation $\partial y(x, a)/\partial a = 0$ does not always give the points of contact of the envelope with the enveloped curves. But it can be proved strictly that, if R is non-zero along an extremal $y(x)$, the necessary and sufficient condition for x_2 to be conjugate to x_0 is that x_2 be a root of the equation $u_0(x) = 0$.

We remark further that the strengthened Legendre and Jacobi conditions are simultaneously sufficient conditions for a given extremal $y(x)$ to be surrounded by a field, i.e. for it to be possible to construct a field of extremals such that $y(x)$ belongs to the field and its arc lies inside the field for $x_0 \leqslant x \leqslant x_1$. This fact plays an important part in the strict deduction of the sufficient conditions for an extremum.

88. Weierstrass's function. The present section will deal with some results relating to the sufficient conditions for a strong extremum. Suppose we have a field of extremals on the (x, y) plane covering a domain B of this plane. The slope y' of an extremal of the field will be a function of a point in domain B, as already mentioned. We introduce the special notation $y' = t(x, y)$ for this function (the slope function for the field). Let $\theta(x, y)$ be the basic function of the field; its total differential is given by:

$$\mathrm{d}\theta\,(x, y) = [F\,(x, y, t) - tF_{y'}\,(x, y, t)]\,\mathrm{d}x + F_{y'}\,(x, y, t)\,\mathrm{d}y. \tag{204}$$

where the previous letter δ has been replaced by the ordinary letter d. It follows at once from this that the line integral of the right-hand side of (204) does not depend on the path inside B. This integral can be written as

$$\int\limits_{\lambda} \left\{F\,(x, y, t) + \left[\frac{\mathrm{d}y}{\mathrm{d}x} - t\,(x, y)\right] F_{y'}\,(x, y, t)\right\} \mathrm{d}x, \tag{205}$$

which is usually known as *Hilbert's invariant integral.* If an extremal of the field is taken as the curve λ, we have $\mathrm{d}y/\mathrm{d}x = t(x, y)$ along this extremal, and integral (205) reduces to the fundamental integral

$$J = \int_{\lambda} F(x, y, t)\,\mathrm{d}x. \tag{206}$$

We follow these introductory remarks by deducing the basic formula for the increment of the basic functional J. Let λ be an extremal of this functional, connecting points (x_0, y_0) and (x_1, y_1), and suppose that this extremal can be surrounded by a field covering a domain B of the (x, y) plane. Let l be some other curve with continuously varying tangent joining the same points (x_0, y_0) and (x_1, y_1), and lying in the domain B. Let $J(l)$ and $J(\lambda)$ denote the values of the basic functional (206) for curves l and λ. As we saw above, $J(\lambda)$ is the same as integral (205) over λ, whilst this latter integral does not depend on the path, so that we can take it over l instead of λ. We thus have:

$$J(\lambda) = \int_{l} \left\{F(x, y, t) + \left[\frac{\mathrm{d}y}{\mathrm{d}x} - t(x, y)\right] F_{y'}(x, y, t)\right\} \mathrm{d}x,$$

so that the following expression is obtained for the difference:

$$J(l) - J(\lambda) = \int_{l} \left\{F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) - F(x, y, t) - \left[\frac{\mathrm{d}y}{\mathrm{d}x} - t(x, y)\right] F_{y'}(x, y, t)\right\} \mathrm{d}x. \tag{207}$$

We recall that $t(x, y)$ in this expression is the slope of the field, whilst $\mathrm{d}y/\mathrm{d}x$ is the slope of the tangent to the curve l. We introduce the following function of four variables:

$$E(x, y, \xi, \eta) = F(x, y, \eta) - F(x, y, \xi) - (\eta - \xi) F_{y'}(x, y, \xi), \tag{208}$$

which is generally known as *Weierstrass's function* for functional (206). By using this function formula (208) can be rewritten as

$$J(l) - J(\lambda) = \int_{l} E\left(x, y, t, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \mathrm{d}x. \tag{209}$$

This last is the fundamental formula for investigating the sufficient conditions for an extremum. In particular, by using this formula it can be shown that the necessary condition for an extremal $y(x)$ to minimize strongly functional (206) is that, for any values of the variable η, we have

$$E(x, y, y', \eta) \geqslant 0 \tag{210}$$

along this extremal.

The following theorem giving the sufficient condition for a strong minimum is a direct consequence of (209): *the sufficient condition for an extremal $y(x)$*

with fixed ends to yield a strong minimum is that it can be surrounded by a field and that there exists a neighbourhood of $y(x)$ at every point of which we have, for any value of the variable η:

$$E\,[x, y, t\,(x, y), \eta] \geqslant 0. \tag{210_1}$$

where $t(x, y)$ is the slope function of the field as above. Since we are using the explicit equations of the curves, we have to require, when surrounding the extremal $y(x)$ by a field, that the family of extremals forming the field has an explicit equation $y = y(x, a)$, where the function $y(x, a)$ has continuous derivatives up to the second order.

On expanding the difference $F(x, y, \eta) - F(x, y, \xi)$ appearing in Weierstrass's function by Taylor's formula up to the second power of the difference $(\eta - \xi)$, we can write Weierstrass's function as

$$E\,(x, y, \xi, \eta) = \frac{1}{2}\,(\eta - \xi)^2\, F_{y'y'}\,(x, y, \eta_1),$$

where η_1 lies between ξ and η. It follows at once from this that the sufficient condition for Weierstrass's function to be positive is that $F_{y'y'}(x, y, \eta) \geqslant 0$ for any value of η. This leads to a simpler sufficient condition for a strong minimum, viz the sufficient condition for an extremal $y(x)$ with fixed ends to yield a strong minimum is that it can be surrounded by a field at every point of which

$$F_{y'y'}\,(x, y, \eta) \geqslant 0 \tag{210_2}$$

for any η.

Proofs of the theorems of this section may be found in the course cited above of M. A. Lavrent'ev and L. A. Lyusternik.

89. Examples. 1. Let us take the functional corresponding to the fundamental problem of geometrical optics on a plane:

$$J = \int\limits_{x_0}^{x_1} n\,(x, y)\,\sqrt{1 + y'^2}\,dx.$$

Here:

$$F_{y'y'}\,(x, y, \eta) = \frac{n\,(x, y)}{(1 + \eta^2)^{3/2}} > 0$$

for any value of η, i.e. condition (210_2) is fulfilled; hence, if an extremal passing through the points M_0 and M_1 can be surrounded by a field it gives the functional a strong minimum. In the case $n(x, y) = y^{-1}$ the extremals in the half-plane $y > 0$ are semicircles orthogonal to OX. If M_0 and M_1 of the upper half-plane do not lie on a straight line perpendicular to OX, one definite extremal passes through these points, and it can be surrounded by a field.

2. We take the case $n(x, y) = \sqrt{y + h}$, i.e. we consider

$$J = \int\limits_{x_0}^{x_1} \sqrt{y + h}\,\sqrt{1 + y'^2}\,dx \qquad (h \text{ is a constant} > 0).$$

The integrand does not contain x, and Euler's equation has the solution:

$$\sqrt{y + h}\,\sqrt{1 + y'^2} - \frac{\sqrt{y + h}\,y'^2}{\sqrt{1 + y'^2}} = C_1 \quad \text{or} \quad \frac{\sqrt{y + h}}{\sqrt{1 + y'^2}} = C_1.$$

On solving for y' and integrating, we obtain the general solution of Euler's equation:

$$y + h - C_1^2 = \left(\frac{x}{2C_1} + C_2\right)^2,$$

which is a family of parabolas.

With $C_1 = 0$ we obtain as extremals straight lines parallel to OY.

We take the pencil of extremals issuing from the origin, i.e. we take the initial conditions:

$$y\,|_{x=0} = 0; \quad y'\,|_{x=0} = a.$$

Having found C_1 and C_2 from these initial conditions, we obtain:

$$y = \frac{(1 + a^2)\,x^2}{4h} + ax.$$

Differentiation with respect to a and elimination of a gives the envelope of the family of parabolas:

$$y = \frac{x^2}{4h} - h.$$

This is a parabola with vertex $A(0, -h)$ and axis $x = 0$ (Fig. 4). On the part of an extremal from the origin to any point prior to the point of contact of this parabola with the envelope, Jacobi's strengthened condition is fulfilled. In addition, since

$$F_{y'y'} = \frac{\sqrt{y + h}}{(1 + y'^2)^{3/2}} > 0,$$

the strengthened Legendre condition is also fulfilled, i.e. this part of the extremal can be surrounded by a field and gives our functional a strong minimum by virtue of what was said in the previous section. We remark that the condition $y + h \geqslant 0$ follows from the form of our functional, i.e. we are here concerned with a problem on a one-sided extremum. Everything is as usual in the half-plane $y + h > 0$.

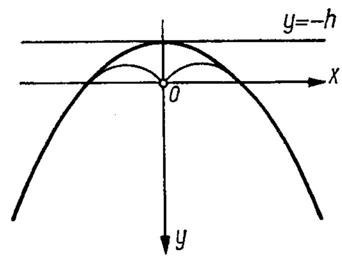

FIG. 4

3. We take the integral

$$J = \int_0^1 y'^3\,dx$$

and pose the problem of drawing the extremal through $M_0(0, 0)$ and $M_1(1, 1)$.

Euler's equation has the general solution $y = C_1 x + C_2$, and the extremal $y = x$ passes through the given points. Here, $F_{yy} = F_{yy'} = 0$ and $F_{y'y'} = 6y'$, i.e. we have $F_{y'y'} = 6 > 0$ on the extremal $y = x$, and the strengthened Legendre condition is fulfilled. Jacobi's equation (198) here becomes $u'' = 0$, and its solution satisfying initial conditions (200) is $u_0(x) = x$. It has no zeros apart from the initial zero at $x_0 = 0$. Thus the strengthened Legendre and Jacobi

conditions are fulfilled along the segment $M_0 M_1$ of the extremal $y = x$, and this segment gives our functional a weak minimum.

Weierstrass's function (208) becomes

$$E(x, y, \xi, \eta) = \eta^3 - \xi^3 - 3(\eta - \xi)\xi^2.$$

Along our extremal the left-hand side of inequality (210) becomes:

$$E(x, y, y', \eta) = \eta^3 - 3\eta + 2,$$

and there exist η for which (210) is not fulfilled, i.e. the extremal $y = x$ cannot give a strong minimum.

4. The problem of finding the geodesics on a given surface leads to the functional [67]:

$$J = \int_{u_0}^{u_1} \sqrt{E + 2Fv' + Gv'^2}\, du,$$

where E, F and G are given functions of (u, v), and the expression under the radical can take only positive values, i.e. $EG - F^2 > 0$ and $E > 0$.

We have:

$$F_{v'v'} = \frac{EG - F^2}{(E + 2Fv' + Gv'^2)^{3/2}} > 0,$$

and condition (210_2) is satisfied, i.e. if a geodesic can be surrounded by a field of geodesics, it gives the functional (with fixed ends) a strong minimum. In particular, on a sphere the arc of a great circle, less than π in radian measure, can be surrounded by a field consisting of arcs of great circles.

90. The Ostrogradskii–Hamilton principle. The variational calculus plays a fundamental role in establishing the equations of mechanics and mathematical physics. These equations can be obtained in a unified manner from a variational principle with the aid of the energy concept. This latter concept, familiar from the mechanics of systems of points, can be extended to other physical processes and leads, as we shall see later, when used in conjunction with the basic principles of the calculus of variations, to a general scheme for forming the equations of mathematical physics. We shall start with the mechanics of systems of material particles.

Suppose we have a system of n material particles, of mass m_k and coordinates (x_k, y_k, z_k). Let the motion of the system be subject to the constraints:

$$\varphi_s = 0 \quad (s = 1, 2, \ldots, m) \tag{211}$$

and occur under the action of forces possessing the force functions

$$X_k = \frac{\partial U}{\partial x_k}; \quad Y_k = \frac{\partial U}{\partial y_k}; \quad Z_k = \frac{\partial U}{\partial z_k}, \tag{212}$$

φ_s and U being given functions of the coordinates of the particles and time. The kinetic energy of the system is given by

$$T = \frac{1}{2} \sum_{k=1}^{n} m_k \left(x_k'^2 + y_k'^2 + z_k'^2\right).$$

Suppose that the system is displaced from the position I, corresponding to the instant $t = t_0$, to the position II corresponding to $t = t_1$. From all the possible methods by which this displacement might be realized, we choose the class of admissible motions of the system, viz. the motions which are compatible with the given constraints and displace the system from position I to II in the given time interval $[t_0, t_1]$. The Ostrogradskii–Hamilton principle states that *the actual motion of the system is distinguished from all the admissible motions in that it satisfies the necessary condition* $\delta J = 0$ *for an extremum of the integral*

$$J = \int_{t_0}^{t_1} (T + U)\, \mathrm{d}t. \tag{213}$$

To each admissible motion there corresponds a set of $3n$ functions $x_k(t)$, $y_k(t)$, $z_k(t)$, defined in the interval $[t_0, t_1]$, satisfying equations (211), and having given values at the ends of this interval. We thus have a variational problem with holonomic constraints (211) and fixed boundaries. To solve it, we have to form an auxiliary function using the method of Lagrange multipliers:

$$F = T + U + \sum_{s=1}^{m} \lambda_s(t)\, \varphi_s,$$

and then write the usual Euler equation for this function. We have here:

$$F_{x_k'} = m_k x_k'; \quad F_{x_k} = U_{x_k} + \sum_{s=1}^{m} \lambda_s(t) \frac{\partial \varphi_s}{\partial x_k},$$

and similarly for coordinates y_k and z_k, and Euler's equations become:

$$m_k x_k'' - X_k - \sum_{s=1}^{m} \lambda_s(t) \frac{\partial \varphi_s}{\partial x_k} = 0,$$

$$m_k y_k'' - Y_k - \sum_{s=1}^{m} \lambda_s(t) \frac{\partial \varphi_s}{\partial y_k} = 0,$$

$$m_k z_k'' - Z_k - \sum_{s=1}^{m} \lambda_s(t) \frac{\partial \varphi_s}{\partial z_k} = 0,$$

i.e. they are the same as the differential equations of the actual motion of the system, which is what we wanted to prove.

If the position of the system is defined with the aid of independent parameters $q_1, \ldots, q_k$, where $k = 3n - m$, instead of in rectilinear rectangular coordinates, the functions T and U will be functions of these parameters:

$$T(q_1, q_1', \ldots, q_k, q_k', t); \quad U(q_1, \ldots, q_k, t),$$

the equations of the constraints fall out, and we obtain the problem of the extremum of integral (213) with fixed boundary values of the q_k and no constraints. Euler's equations become

$$T_{q_i} + U_{q_i} - \frac{\mathrm{d}}{\mathrm{d}t}(T_{q_i'} + U_{q_i'}) = 0,$$

or, since U does not depend on q_k' [II, 19]:

$$T_{q_i} + U_{q_i} - \frac{\mathrm{d}}{\mathrm{d}t} T_{q_i'} = 0 \qquad (i = 1, \ldots, k). \tag{214}$$

The canonical variables here are q_i and p_i, where the p_i, usually known as the *generalized momenta*, are given by

$$p_i = \frac{\partial}{\partial q_i'}(T + U) = T_{q_i'}.$$

The function H [78] becomes:

$$H = \sum_{i=1}^{k} q_i' p_i - (T + U) = \sum_{i=1}^{k} q_i' T_{q_i'} - T - U.$$

If T is a homogeneous second degree polynomial in q_i' [II, 19], we obtain by Euler's theorem on homogeneous functions [I, 154]: $H = 2T - T - U = T - U$, i.e. H represents the total energy of the system.

91. Principle of least action. Suppose that neither the force function U nor the functions φ_s contain t. In this case, we have the familiar energy integral

$$T - U = h \tag{215}$$

which expresses the fact that the sum of the kinetic energy T and the potential energy $(-U)$ remains constant throughout the motion. Further, in the present case the expressions for the Cartesian coordi-

nates in terms of the coordinate parameters q_s will not contain t. The kinetic energy will be a quadratic form in the derivatives q_i':

$$2T = \sum_{s=1}^{n} m_s x_s'^2 = \sum_{i,j=1}^{n} a_{ij} q_i' q_j' \qquad (a_{ij} = a_{ji}), \tag{216}$$

where the a_{ij} are functions of q_s. Using relationship (215), we can rewrite the integrand of (213) as

$$T + U = 2T - h.$$

If we neglect the added constant, write $2T$ as $\sqrt{2U + 2h}\,\sqrt{2T}$ and replace $2T$ in one of the factors by its expression from (216), we arrive at an integral of the form

$$\int_{t_0}^{t_1} \sqrt{2U + 2h} \sqrt{\sum_{i,j=1}^{k} a_{ij} q_i' q_j'}\; dt. \tag{217}$$

We show that Euler's equations for this integral lead us back to the Lagrange equations (214) obtained above. In fact, Euler's equations for integral (217) have the form:

$$U_{q_i} \sqrt{\frac{2T}{2U + 2h}} + T_{q_i} \sqrt{\frac{2U + 2h}{2T}} - \\ - \frac{d}{dt}\left[\sqrt{\frac{2U + 2h}{2T}}\, T_{q_i'}\right] = 0 \quad (i = 1, \ldots, k). \tag{218}$$

Notice that the integrand in integral (217) does not contain the independent variable and is a homogeneous function of the first degree in the derivatives q_i'. Consequently, as we saw above [72], one of the Euler's equations written will be a consequence of the remainder, and we can add to our Euler equations a further equation fixing the choice of independent variable (parameter). In order to have time as the independent variable, we associate with equations (218) the equation

$$\sqrt{\frac{2U + 2h}{2T}} = 1,$$

which is obviously equivalent to the law of conservation of energy (215). Equations (218) now reduce to Lagrange's equations (214). Thus the equations of the actual motion are obtained in the present case from the necessary condition for an extremum of integral (217) with fixed ends. This assertion represents the *principle of least action in Jacobi's form*.

We introduce into k-dimensional space with coordinates $q_1, \ldots, q_k$ a metric defined by the following expression for the differential of the arc:

$$\mathrm{d}s^2 = (2U + 2h) \sum_{i,j=1}^{k} a_{ij} q_i' q_j'.$$

Integral (217) can now be written in the form

$$\int \mathrm{d}s,$$

and the fundamental problem of the mechanics of systems of particles turns out to be equivalent to the problem of the geodesics in the k-dimensional space. It can be shown that, given sufficiently small pieces of trajectory of the actual motion, the action integral along the pieces has a weak minimum. Let us consider the motion of a single particle over some surface S under the action of inertia. In this case we can take $U = 0$, and integral (217) becomes simply

$$\int_{t_0}^{t_0} \sqrt{T}\,\mathrm{d}t, \tag{217$_1$}$$

or, if we introduce rectilinear rectangular coordinates,

$$\int_{l} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}.$$

The trajectories of the motion will be the geodesics of this surface.

The integral of Example 2 of [70] is obtained by applying the principle of least action to the case of a single particle under the influence of gravity, where OY coincides with the direction of the gravity force.

The principle of least action can be put in another form; this will now be indicated, though without dwelling on the detailed proofs. By using the energy integral (215) and throwing away the added constant, we can rewrite integral (213) as

$$\int_{t_0}^{t_1} T\,\mathrm{d}t. \tag{219}$$

We shall take as admissible motions those which satisfy the equations of the constraints and equation (215) with the same value of the constant h as for the actual motion, and which have a fixed initial and final position and fixed initial instant t_0. The final instant is not fixed for them. The actual motion is distinguished from all these admissible motions by the fact that it must satisfy the necessary condition for

an extremum of integral (219). This is *the principle of least action in Lagrange's form.* We remark that the potential energy appears now in the auxiliary condition (215) and not in the integral.

92. Strings and membranes. Before establishing the variational principle in the general theory of elasticity, we shall consider a number of particular cases of elastic bodies, whose sizes in one or two dimensions are substantially greater than in the remaining dimensions. The establishment of the variational principle in essence reduces here to certain propositions regarding the potential energy, i.e. regarding the work done by the forces of deformation as a function of the shape of the deformed body.

Let a string be stretched along the x axis and perform plane transverse vibrations in the (x, u) plane [II, 163]. The kinetic energy of the vibrating string is given by

$$\frac{1}{2}\int_0^l \varrho u_t^2\,\mathrm{d}x,$$

where ϱ is the linear density of the string and $x = 0$, $x = l$ are the abscissae of its ends. We shall assume that the work done by of the forces of deformation is given by the product of the string tension T_0 with its elongation

$$\int_0^l \sqrt{1+u_x^2}\,\mathrm{d}x - l.$$

On expanding the radical by the binomial formula and confining ourselves to the first two terms, we get the following expression for the potential energy of deformation:

$$\frac{T_0}{2}\int_0^l u_x^2\,\mathrm{d}x.$$

In the case of an external force $F(x, t)$ per unit length, we have to add to the potential energy the further term

$$-\int_0^l Fu\,\mathrm{d}x.$$

Finally, the Ostrogradskii–Hamilton principle leads to the necessary condition $\delta J = 0$ for an extremal of the integral

$$J = \frac{1}{2}\int_{t_0}^{t_1}\int_0^l (\varrho u_t^2 - T_0 u_x^2 + 2Fu)\,\mathrm{d}x\,\mathrm{d}t. \tag{220}$$

The integration is performed over the rectangle $0 \leqslant x \leqslant l$; $t_0 \leqslant t \leqslant t_1$ on the (x, t) plane. In the case of a constrained string we have the boundary condition $u = 0$ along the sides $x = 0$ and $x = l$ of this rectangle, and on the sides

$t = t_0$ and $t = t_1$ the function u must coincide with the functions $u(x, t_0)$ and $u(x, t_1)$, giving the shape of the string at the beginning and end of the interval $[t_0, t_1]$.

If elastic forces act at the ends of the string, and we use the fact that the potential of the elastic force is proportional to the square of the deviation, we have to add to integral (220) a term of the form

$$-\int_{t_0}^{t_1} [h_1 u^2 (0, t) + h_2 u^2 (l, t)]\, dt.$$

In essence, this additional term is an integral over the contour of the above-mentioned rectangle, the integrand being zero on the sides $t = t_0$ and $t = t_1$, and equal to $h_1 u^2(0, t)$ and $h_2 u^2(l, t)$ on the sides $x = 0$ and $x = l$. On taking into account what was said in [75], and the fact that the outward normal on the side $x = 0$ is in the opposite direction to the x axis, we have natural boundary conditions on the sides $x = 0$ and $x = l$ of the form

$$u_x - \frac{2h_1}{T_0} u \Big|_{x=0} = 0; \qquad u_x + \frac{2h_2}{T_0} u \Big|_{x=l} = 0.$$

Ostrogradskii's equation for double integral (220) gives us the usual equation for the vibrations of a string.

Precisely similar arguments are used to find the equations of vibration of a membrane [II, 176]. Let the membrane be stretched in the (x, y) plane in the natural state with tension T_0, measured per unit length. The work done by the force of deformation will be given by the product of T_0 and the increment of the area, viz.

$$\iint_B \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy - \iint_B dx\, dy,$$

where $u(x, y, t)$ is the deviation of the point (x, y) of the membrane at the instant t from the equilibrium position and B is the domain of the (x, y) plane occupied by the membrane. On confining ourselves to small vibrations, we obtain the following expression for integral (213):

$$\frac{1}{2} \int_{t_0}^{t_1} \iint_B [\varrho u_t^2 - T_0 (u_x^2 + u_y^2) + 2Fu]\, dt\, dx\, dy. \tag{221}$$

Ostrogradskii's equation for this last integral leads us to the familiar equation of the vibrations of the membrane. If there is an elastic constraint on the boundary with modulus $q(s)$, we have to add to integral (221) the term

$$-\int_l q(s)\, u^2\, ds,$$

where l is the contour of the membrane. The natural boundary conditions here take the form:

$$\frac{\partial u}{\partial n} - \frac{2}{T_0} q(s)\, u \Big|_l = 0,$$

where n is the direction of the outward normal to l. In the case of a constrained membrane they obviously become $u\,|_l = 0$.

93. Rods and plates. We understand by a rod a body of linear dimensions that operates under flexure. The potential energy arising on deformation is assumed proportional to the integral of the square of the curvature. In the case of small vibrations we replace the curvature by the second derivative u_{xx} and obtain for the potential energy of deformation:

$$\frac{\mu}{2}\int_0^l u_{xx}^2\,\mathrm{d}x\,,$$

where the coefficient of proportionality $\mu = EJ$ [II, 16]. The boundary conditions were given by us in [II, 189]. Integral (213) takes the following form in this case:

$$\frac{1}{2}\int_{t_0}^{t_1}\int_0^l (\varrho u_t^2 - \mu u_{xx}^2 + 2Fu)\,\mathrm{d}t\,\mathrm{d}x,$$

and the corresponding Euler equations leads us to the following equation for the transverse vibrations of the rod:

$$\varrho\frac{\partial^2 u}{\partial t^2} + \mu\frac{\partial^4 u}{\partial x^4} = F.$$

We observe that, if the end of the rod is free, the boundary conditions [II, 189] can be obtained as the natural boundary conditions discussed in [74].

By analogy with a rod [70, 90], we shall assume that the potential energy of a plate which has a plane shape in the natural state is a homogeneous quadratic form in the reciprocals of the principal radii of curvature of the plate in the deformed state, i.e.

$$-U = \iint_B \left[a\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{2b}{R_1 R_2}\right]\mathrm{d}x\,\mathrm{d}y,$$

where a and b are constants and B is the domain of the (x, y) plane occupied by the plate. We have for the curvatures of the principal normal sections the equation [II, 134]:

$$(EG - F^2)\frac{1}{R^2} + (2FM - EN - GL)\frac{1}{R} + (LN - M^2) = 0.$$

In the case of the surface having the explicit equation $u = u(x, y)$, we obtain by neglecting second order terms in u_x and u_y:

$$E = G = 1; \quad F = 0; \quad L = r = u_{xx}; \quad M = s = u_{xy}; \quad N = t = u_{yy},$$

whence

$$\frac{1}{R_1 R_2} = u_{xx}u_{yy} - u_{xy}^2; \quad \frac{1}{R_1} + \frac{1}{R_2} = u_{xx} + u_{yy},$$

and consequently:

$$\frac{1}{R_1^2} + \frac{1}{R_2^2} = (u_{xx} + u_{yy})^2 - 2\,(u_{xx}u_{yy} - u_{xy}^2)\,.$$

We can finally write:

$$-U = \frac{D}{2} \iint\limits_B [(u_{xx} + u_{yy})^2 - 2(1-\sigma)(u_{xx} u_{yy} - u_{xy}^2)] \, dx \, dy,$$

where D and σ are two new constants composed of constants a and b. The coefficient D is called the flexural rigidity of the plate and σ is the familiar Poisson coefficient. To this expression for the energy of deformation we must add the potential of the external forces acting on the surface of the plate. We finally get the following expression for integral (213), assuming the plate to be fixed along the edge and confining ourselves to the case of statical bending:

$$\frac{D}{2} \iint\limits_B \left[-(u_{xx} + u_{yy})^2 + 2(1-\sigma)(u_{xx} u_{yy} - u_{xy}^2) + \frac{2}{D} pu \right] dx \, dy,$$

where p is the load per unit area. By (30) of [65], Ostrogradskii's equation in the case when the integrand contains derivatives up to the second order of the required function u with respect to the two independent variables x and y has the form

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} + \frac{\partial^2}{\partial x^2} F_{u_{xx}} + 2 \frac{\partial^2}{\partial x \, \partial y} F_{u_{xy}} + \frac{\partial^2}{\partial y^2} F_{u_{yy}} = 0. \quad (222)$$

Taking $D = 1$, we can write:

$$F = -\frac{1}{2}(u_{xx} + u_{yy})^2 + (1-\sigma)(u_{xx} u_{yy} - u_{xy}^2) + pu, \quad (223)$$

and we arrive at the following equation for statical bending:

$$\Delta\Delta u = p.$$

In the case of a vibrating plate, addition of the kinetic energy gives us

$$\varrho u_{tt} + \Delta\Delta u = p.$$

It is characteristic that the term appearing in (223) containing the factor $(1-\sigma)$ gives identically zero on substitution in the left-hand side of Ostrogradskii's equation (222) and has no effect on this latter equation. But it must be noticed that this term plays an essential role in establishing the natural boundary conditions.

94. The fundamental equations of the theory of elasticity. Let (u, v, w) be the components of the displacement vector on deformation of a continuous medium. The picture of the stresses in the medium is provided by the six components of the stress tensor:

$$\sigma_x, \ \sigma_y, \ \sigma_z, \ \tau_{xy} = \tau_{yx}, \ \tau_{xz} = \tau_{zx}, \ \tau_{yz} = \tau_{zy},$$

where σ_x is the component along the x axis of the stress acting on an area perpendicular to the axis (σ_y and σ_z are similarly defined). $\tau_{xy} = \tau_{yx}$ is the component along the x axis of the stress acting on an area perpendicular to

the y axis or vice versa. τ_{xz} and τ_{yz} have similar meanings. The deformation of the medium is characterized by the following six components of the deformation tensor when the deformation is small:

$$\left.\begin{aligned} \varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_y = \frac{\partial v}{\partial y}; \quad \varepsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad \gamma_{xz} = \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \\ \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{224}$$

The quantities ε_x, ε_y, ε_z characterize the relative elongations of the linear elements in the axial directions, and γ_{xy} the change of the right-angle formed by the x and y axes. We introduce two further quantities, viz.

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z,$$

characterizing the relative change in volume, and

$$s = \sigma_x + \sigma_y + \sigma_z.$$

It can be shown that the last two quantities do not depend on the choice of axes. In the classical theory of elasticity for an isotropic homogeneous body the components of the deformation and stress tensors are assumed to be connected by a linear relationship which expresses a generalized Hooke's law:

$$\varepsilon_x = \frac{1}{2G}\left(\sigma_x - \frac{s}{m+1}\right); \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \text{or} \quad \sigma_x = 2G\left(\varepsilon_x + \frac{\theta}{m-2}\right); \quad \tau_{xy} = G\gamma_{xy},$$

$$\varepsilon_y = \frac{1}{2G}\left(\sigma_y - \frac{s}{m+1}\right): \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \qquad \sigma_y = 2G\left(\varepsilon_y + \frac{\theta}{m-2}\right); \quad \tau_{yz} = G\gamma_{yz},$$

$$\varepsilon_z = \frac{1}{2G}\left(\sigma_z - \frac{s}{m+1}\right); \quad \gamma_{zx} = \frac{1}{G}\tau_{zx}, \qquad \sigma_z = 2G\left(\varepsilon_z + \frac{\theta}{m-2}\right); \quad \tau_{zx} = G\gamma_{zx},$$

where G and m are constants, characteristic of the given material, G being called the shear modulus and m the coefficient of transverse compression (Poisson's constant). The following relationship between θ and s is a direct consequence of Hooke's law:

$$\theta = \frac{1}{2G}\frac{m-2}{m+1}s.$$

Further, let A denote the work of the forces of deformation, referred to unit volume; this can be expressed in terms of the components either of the deformation or the stress tensor:

$$\begin{aligned} A &= G\left[\frac{m-1}{m-2}\theta^2 - 2(\varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x) + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)\right] \\ &= \frac{1}{4G}\left[\frac{m}{m+1}s^2 - 2(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\right], \end{aligned} \tag{225}$$

where the following relationships hold, as may be shown by using the formulae written:

$$\left.\begin{array}{llll}\sigma_x = \dfrac{\partial A}{\partial \varepsilon_x}; & \tau_{xy} = \dfrac{\partial A}{\partial \gamma_{xy}}; & \varepsilon_x = \dfrac{\partial A}{\partial \sigma_x}; & \gamma_{xy} = \dfrac{\partial A}{\partial \tau_{xy}}; \\ \sigma_y = \dfrac{\partial A}{\partial \varepsilon_y}; & \tau_{yz} = \dfrac{\partial A}{\partial \gamma_{yz}}; & \varepsilon_y = \dfrac{\partial A}{\partial \sigma_y}; & \gamma_{yz} = \dfrac{\partial A}{\partial \tau_{yz}}; \\ \sigma_z = \dfrac{\partial A}{\partial \varepsilon_z}; & \tau_{zx} = \dfrac{\partial A}{\partial \gamma_{zx}}; & \varepsilon_z = \dfrac{\partial A}{\partial \sigma_z}; & \gamma_{zx} = \dfrac{\partial A}{\partial \tau_{zx}}. \end{array}\right\} \tag{226}$$

It can be shown that three mutually perpendicular directions exist at every point of an elastic body such that, if they are taken as the axes, we get the equations $\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$ at this point. On choosing these directions as the coordinate axes and writing ε_1, ε_2, ε_3 for the new values of what we previously wrote as ε_x, ε_y, ε_z, we obtain, by (225), the following expression for A at this point:

$$A = G\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \frac{1}{m-2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2\right].$$

The condition that A be positive leads us to the following inequality for the constant m: $2 < m \leqslant \infty$.

We shall first discuss the equilibrium conditions for an elastic body D bounded by a surface S. Let the body be acted on by forces with components

$$X(x, y, z, t); \quad Y(x, y, z, t); \quad Z(x, y, z, t)$$

per unit mass, and let the displacement vector be given on a part S_1 of the surface S, and the stress on a part S_2, where X_1, Y_1, Z_1 denote the components of this stress. These latter are given functions of the variable point M on S_2. The sum of the integral of A over D and the total work, done by the external forces, taken with the opposite sign, gives the potential energy of the elastic body:

$$\iiint\limits_D [A - (Xu + Yv + Zw)]\, dv - \iint\limits_{S_2} (X_1 u + Y_1 v + Z_1 w)\, d\sigma. \tag{227}$$

This potential energy is a functional of the three functions u, v, w of the coordinates (x, y, z) of points of the body. We get the equilibrium equation by writing Ostrogradskii's equation for the functional, where it must be borne in mind that the surface integral plays no part in the composition of the Ostrogradskii equation. On observing that A depends only on the derivatives of u, v, w and not on the functions themselves, we arrive at the following Ostrogradskii equation for functional (227) with respect to the function u:

$$-X - \frac{\partial}{\partial x} A_{u_x} - \frac{\partial}{\partial y} A_{u_y} - \frac{\partial}{\partial z} A_{u_z} = 0. \tag{228}$$

Since A depends on u_x only via ε_x, on u_y only via γ_{xz} and on u_z only via γ_{xz}, we can use (226) to rewrite the last equation as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0. \tag{229}$$

Two analogous equilibrium equations can be written. The values of u, v, w are fixed on the part S_1 of the boundary surface, and the natural boundary conditions [75] on the part S_2 take the form:

$$\begin{aligned}
&\sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{zx} \cos(n, z) - X_1 = 0,\\
&\tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz} \cos(n, z) - Y_1 = 0,\\
&\tau_{zx} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_z \cos(n, z) - Z_1 = 0,
\end{aligned}$$

where n is the direction of the outward normal to S_2.

On replacing the components of the stress tensor in (229) by their expressions in terms of the deformation tensor, we obtain the following three equilibrium equations:

$$\begin{aligned}
&G\left(\Delta u + \frac{m}{m-2}\frac{\partial\theta}{\partial x}\right) + X = 0,\\
&G\left(\Delta v + \frac{m}{m-2}\frac{\partial\theta}{\partial y}\right) + Y = 0,\\
&G\left(\Delta w + \frac{m}{m-2}\frac{\partial\theta}{\partial z}\right) + Z = 0,
\end{aligned}$$

or in vector form:

$$G\left(\Delta\mathbf{u} + \frac{m}{m-2}\operatorname{grad}\operatorname{div}\mathbf{u}\right) + \mathbf{F} = 0.$$

We remark that formula (225) for the elastic potential can be written as

$$A = G\left\{\frac{m-1}{m-2}\theta^2 + \frac{1}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2) + 2\left[\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right) + \ldots\right]\right\}, \quad (230)$$

where ω_x, ω_y, ω_z are the components of the curl of the displacement vector, i.e.

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \quad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

We notice the fact that the expression in square brackets in (230) has no influence whatever on Ostrogradskii's equation, i.e. we obtain the equilibrium equations for an elastic body if we write Ostrogradskii's equation for the integral:

$$\iiint\limits_D \left[G\frac{m-1}{m-2}\theta^2 + \frac{G}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2) - (Xu + Yv + Zw)\right]dv.$$

In order to obtain the equation of motion, we utilize the Ostrogradskii–Hamilton principle, and simply add to the above integral a term corresponding to the kinetic energy (with reversed sign), viz.

$$-\frac{\varrho}{2}(u_t^2 + v_t^2 + w_t^2),$$

where ϱ is the density of the body and integration is over the finite time interval $[t_0, t_1]$. The problem thus amounts to forming the Ostrogradskii equations for the integral:

$$\int_{t_0}^{t_1}\iiint_D \left[-\frac{\varrho}{2}(u_t^2+v_t^2+w_t^2)+A-(Xu+Yv+Zw)\right] dt\, dv$$

with respect to the functions u, v, w of the independent variables (x, y, z, t). This leads us, as may readily be seen, to the following fundamental dynamical equation of the theory of elasticity, written in the vector form:

$$\varrho \frac{\partial^2 \mathbf{u}}{\partial t^2} = G\left(\Delta\mathbf{u} + \frac{m}{m-2}\operatorname{grad}\operatorname{div}\mathbf{u}\right).$$

Here, as usual, it is assumed that at the initial and final instants $t = t_0$ and $t = t_1$ the displacements coincide with the actual displacements [87].

95. Absolute extrema. We introduced in [63] the concept of absolute extremum. We now consider some particular examples and discuss the existence of an absolute extremum in connection with these.

Suppose we have the functional

$$J(y) = \int_0^l [p(x)y'^2 + q(x)y^2 + 2f(x)y]\, dx, \tag{231}$$

where $p(x)$, $q(x)$ and $f(x)$ are functions continuous in the closed interval $[0, l]$, $p(x)$ has a continuous derivative and

$$p(x) > 0; \quad q(x) \geqslant 0. \tag{232}$$

We want to find, in the class D of functions $y(x)$, continuous with the derivative $y'(x)$ in the interval $[0, l]$ and satisfying the boundary conditions:

$$y(0) = a; \quad y(l) = b, \tag{233}$$

the function which minimizes functional (231). Euler's equation for this functional has the form

$$\frac{d}{dx}[p(x)y'] - q(x)y = f(x). \tag{234}$$

We show that, given conditions (232), this equation has a solution satisfying conditions (233) in the interval $[0, l]$, and that this solution is unique. Let $z_0(x)$ and $z_1(x)$ be solutions of the homogeneous equation

$$\frac{d}{dx}[p(x)z'] - q(x)z = 0, \tag{235}$$

satisfying the initial conditions:

$$z_0(0) = 0; \quad z_0'(0) = 1; \quad z_1(l) = 0; \quad z_1'(l) = 1.$$

Bearing in mind that $p(x) > 0$, the existence and uniqueness theorem guarantees the existence of such solutions existing throughout $[0, l]$. We show further that $z_0(b) \neq 0$. This is equivalent to the fact that homogeneous equation (235) has no solutions, apart from identical zero, which are zero at $x = 0$ and $x = l$. Now obviously, $z_1(0) \neq 0$, and the solutions $z_0(x)$ and $z_1(x)$ are linearly independent.

The general solution of (234) has the form:

$$y_0(x) = c_0 z_0(x) + c_1 z_1(x) + g(x),$$

where c_0 and c_1 are arbitrary constants and $g(x)$ is any particular solution of (234), the existence of which in the interval $[0, l]$ is guaranteed by the existence theorem with $p(x) > 0$. The boundary conditions (233) lead us to the equations:

$$c_1 z_1(0) + g(0) = a; \quad c_0 z_0(l) + g(l) = b,$$

from which c_0 and c_1 are uniquely determined. We therefore obtain a unique solution $y(x)$ of equation (234), satisfying the boundary conditions (233), this solution being continuous together with its derivatives up to the second order in $[0, l]$. It remains to show that $z_0(l) \neq 0$. We rewrite (235) by substituting $z = z_0(x)$:

$$\frac{d}{dx}[p(x) z_0'] = q(x) z_0. \tag{236}$$

By virtue of the conditions $z_0(0) = 0; z_0'(0) = 1$, the function $z_0(x)$ and its derivative are positive for x sufficiently close to $x = 0$. Hence both sides of (236) are non-negative for these values of x, so that $p(x) z_0'$, which is positive for $x = 0$, cannot decrease for x sufficiently near $x = 0$, and by (236), can only begin to decrease after $z_0(x)$ becomes negative. But for $z_0(x)$ to be negative, its derivative $z_0'(x)$ must be negative, i.e. $p(x) z_0'$ would be negative. We have arrived at a contradiction and it can be asserted that $z_0(x) > 0$ for $0 < x \leqslant l$.

The solution $y_0(x)$ of equation (234) obtained above belongs to class C_2. *We show that it minimizes integral* (231), *or to be more precise, we show that* $J_0(y_0) \leqslant J(y)$, *where* y *is any function of class* D, *the sign of equality being obtained when and only when* $y(x)$ *is identically equal to* $y_0(x)$.

Every function $y(x)$ of D can be written as $y(x) = y_0(x) + \eta(x)$, where $\eta(x)$ is continuous with its derivative in $[0, l]$ and vanishes at the ends of this interval. We have:

$$J(y) - J(y_0) = 2\int_0^l [p(x)\, y_0'\, \eta' + q(x)\, y_0\, \eta + f(x)\, \eta]\, \mathrm{d}x + \\ + \int_0^l [p(x)\, \eta'^2 + q(x)\, \eta^2]\, \mathrm{d}x.$$

On taking into account the properties of $y_0(x)$ and $\eta(x)$, we can integrate by parts in the first integral:

$$J(y) - J(y_0) = 2\int_0^l \Big[-\frac{\mathrm{d}}{\mathrm{d}x}[p(x)\, y_0'] + q(x)\, y_0 + f(x)\Big]\eta\, \mathrm{d}x + \\ + \int_0^l [p(x)\, \eta'^2 + q(x)\, \eta^2]\, \mathrm{d}x + p(x)\, y_0'\, \eta \Big|_{x=0}^{x=l},$$

whence, in view of the fact that $y_0(x)$ is a solution of equation (234) and that $\eta(0) = \eta(l) = 0$, we obtain by (232):

$$J(y) - J(y_0) = \int_0^l [p(x)\, \eta'^2 + q(x)\, \eta^2]\, \mathrm{d}x \geqslant 0,$$

where the sign of equality only holds when $\eta(x) \equiv 0$. For if we have the sign of equality, we must have $\eta'(x) \equiv 0$, i.e. $\eta(x)$ is constant in $[0, l]$. But $\eta(0) = 0$, so that $\eta(x) \equiv 0$ in the interval. Our assertion is therefore proved, i.e. functional (231) is minimized in class D by and only by $y_0(x)$. We remark that only the requirements of existence and continuity of the derivative are imposed on functions of class D, whilst the function $y_0(x)$ which minimizes functional (231) also has a continuous second order derivative.

We take as a second example the problem of the least value of the functional:

$$J(y) = \int_{-1}^{1} x^2 y'^2\, \mathrm{d}x \tag{237}$$

in the class D of continuous functions $y(x)$ having a continuous derivative in the interval $[-1, 1]$ and satisfying the boundary conditions:

$$y(-1) = a; \quad y(1) = b, \tag{238}$$

where $a \neq b$. In view of the last condition, class D does not contain any constants, so that $J(y) > 0$ for any function of D. The set of numbers $J(y)$ must have a strict lower bound [I, 42]. We show that this is zero.

It may easily be seen that the function

$$y = \frac{a+b}{2} + \frac{b-a}{2} \frac{\arctan \frac{x}{\varepsilon}}{\arctan \frac{1}{\varepsilon}} \tag{239}$$

belongs to class D for any positive ε. We have:

$$y' = \frac{b-a}{2 \arctan \frac{1}{\varepsilon}} \cdot \frac{\varepsilon}{\varepsilon^2 + x^2},$$

so that, for function (239):

$$J(y) < \int_{-1}^{1} (\varepsilon^2 + x^2) y'^2 \, dx = \frac{\varepsilon^2 (b-a)^2}{4 \arctan^2 \frac{1}{\varepsilon}} \int_{-1}^{1} \frac{dx}{\varepsilon^2 + x^2} = \frac{\varepsilon (b-a)^2}{2 \arctan \frac{1}{\varepsilon}}.$$

The right-hand side tends to zero as $\varepsilon \to 0$, whence it is clear that the strict lower bound of the values of function (237) in class D is zero. But, since class D does not contain a constant, $J(y) > 0$ for any function of D, as we indicated above. Hence the strict lower bound of $J(y)$ is not attained in class D, and functional (237) has no least value in this class.

96. Absolute extrema (continued). We take as a third example the functional

$$J(u) = \iint_B (u_x^2 + u_y^2) \, dx \, dy, \tag{240}$$

where B is a circle with centre the origin and unit radius. This integral is usually known as *Dirichlet's integral.* We shall consider this functional in the class D of functions $u(x, y)$, continuous in the closed circle $x^2 + y^2 \leqslant 1$, having continuous first order derivatives inside this circle and satisfying on the boundary l of the circle the boundary condition

$$u|_l = f(\theta), \tag{241}$$

where $f(\theta)$ is a given continuous function of the polar angle θ on the circumference l. Since we are not assuming continuity of the partial derivatives u_x and u_y in the closed circular domain, we must interpret

integral (240) as an improper integral, i.e. as the limit of the integrals over circles $B_\varrho(x^2 + y^2 \leqslant \varrho^2)$ of radius ϱ as $\varrho \to 1$. Since the integrand is non-negative, the integral over B_ϱ does not decrease as ϱ increases, and the limit mentioned is either finite or $(+\infty)$. In the former case, we say as usual that the integral is convergent, and in the latter, that it is divergent. It can be assumed that the value of the integral is $(+\infty)$ in the latter case. Ostrogradskii's equation for functional (240) is Laplace's equation [67]:

$$u_{xx} + u_{yy} = 0,$$

and we are justified in expecting that the function harmonic in the circle B and taking boundary values (241) on l minimizes functional (240) in the class D. We know that such a harmonic function exists and is unique [II, 195]. We write it as $v(x, y)$.

We show first of all that the continuous function $f(\theta)$ appearing in condition (241) can be specified so that functional (240) for $u = v$:

$$J(v) = \iint_B (v_x^2 + v_y^2)\, \mathrm{d}x\, \mathrm{d}y \tag{242}$$

becomes equal to $(+\infty)$. For suppose we put

$$f(\theta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(2^{2n}\theta). \tag{243}$$

This series is clearly absolutely and uniformly convergent with respect to θ and defines a continuous 2π-periodic function $f(\theta)$. The solution of Dirichlet's problem with boundary values (243) has the form [II, 195]:

$$v(x, y) = v(r, \theta) = \sum_{n=1}^{\infty} \frac{r^{2^{2n}}}{2^n} \cos(2^{2n}\theta).$$

We change to polar coordinates in integral (240):

$$J(v) = \iint_{r<1} \left(u_r^2 + \frac{1}{r^2} u_\theta^2\right) r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{244}$$

We have:

$$v_r = \sum_{n=1}^{\infty} 2^n r^{2^{2n}-1} \cos(2^{2n}\theta); \quad v_\theta = -\sum_{n=1}^{\infty} 2^n r^{2^{2n}} \sin(2^{2n}\theta),$$

where the series are absolutely and uniformly convergent with respect to r and θ in any circle $r \leqslant \varrho$, where $\varrho < 1$. Since the sines and cosines of multiple arcs are orthogonal in an interval of length 2π, we get:

$$\iint\limits_{r \leqslant \varrho} \left(v_r^2 + \frac{1}{r^2} v_\theta^2\right) r \, \mathrm{d}r \, \mathrm{d}\theta = \sum_{n=1}^{\infty} \iint\limits_{r \leqslant \varrho} 2^{2n} r^{2^{2n+1}-1} \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= 2\pi \int_0^\varrho 2^{2n} r^{2^{2n+1}-1} \, \mathrm{d}r = \pi \sum_{n=1}^{\infty} \varrho^{2^{2n+1}},$$

and as $\varrho \to 1$ the sum of the latter series increases indefinitely, whence it follows that the value of integral (242) is $(+\infty)$ under condition (243).

Thus the harmonic function $v(x, y)$ does not minimize functional (240) in the present case. It can be shown that, if integral (240) is equal to $(+\infty)$ with $u = v$, it is equal to $(+\infty)$ for any function of class D satisfying the boundary condition (241). This follows directly from the theorem:

THEOREM. *If with boundary condition* (241) *integral* (240) *has a finite value for any function of class D, it has a finite value for the harmonic function v of class D, and we have here for any function of D:*

$$J(u) \geqslant J(v), \tag{245}$$

the sign of equality being obtained only when $u \equiv v$.

The proof of this theorem is extremely simple if we assume that the harmonic function $v(x, y)$ has bounded first order partial derivatives inside B. Integral (240) now obviously has a finite value. It is enough for us to show that, if the integral $J(u)$ has a finite value for any function w of D, then $J(w) \geqslant J(v)$, the sign of equality being obtained only when $w \equiv v$. We can write: $w = v + \eta$, where $\eta(x, y)$ has continuous first order partial derivatives inside B, is continuous in the closed circle B and vanishes on the circumference l. We have:

$$J_\varrho(v + \eta) = J_\varrho(v) + J_\varrho(\eta) + 2J_\varrho(v, \eta), \tag{246}$$

where $J_\varrho(u)$ denotes the integral $J(u)$ taken over the circle B_ϱ, and

$$J_\varrho(v, \eta) = \iint\limits_{B_\varrho} (v_x \eta_x + v_y \eta_y) \, \mathrm{d}x \, \mathrm{d}y.$$

The function $v(x, y)$ has continuous second order partial derivatives inside B, and, on applying Green's formula, we obtain

$$J_\varrho(v, \eta) = \iint_{B_\varrho} (v_x \eta_x + v_y \eta_y)\, dx\, dy$$
$$= -\iint_{B_\varrho} \eta \Delta v\, dx\, dy + \int_{l_\varrho} \eta \frac{\partial v}{\partial n} \varrho\, d\theta,$$

where l_ϱ is the circle with centre the origin and radius ϱ, and $\partial v/\partial n$ is the derivative with respect to the normal to this circle. Since v is a harmonic function, the double integral on the right-hand side vanishes, whilst in the line integral, as ϱ tends to unity, η tends to zero uniformly with respect to the polar angle, and $\partial v/\partial n$ remains bounded by hypothesis, so that the line integral clearly tends to zero. Expression (246) therefore gives in the limit:

$$J(w) - J(v) = \iint_B (\eta_x^2 + \eta_y^2)\, dx\, dy,$$

whence it follows that $J(w) \geqslant J(v)$, the sign of equality being obtained only when $\eta_x \equiv 0$ and $\eta_y \equiv 0$, i.e. when η is constant in the circle B. But $\eta = 0$ on l, so that $\eta \equiv 0$.

We now prove the theorem in the general case. As before, let w be a function of D for which the integral $J(w)$ has a finite value, and let:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{247}$$

be the Fourier series of the function $f(\theta)$ appearing in condition (241)· The function v is defined inside B by the series

$$v(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n. \tag{248}$$

We put:

$$v_m(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos n\theta + b_n \sin n\theta)\, r^n \tag{249}$$

and define the function $\eta_m(r, \theta)$ by the equation: $w = v_m + \eta_m$. This function $\eta_m(r, \theta)$ has continuous first order partial derivatives inside B, is continuous in the closed circle, and has boundary values $\eta_m(1, 0)$ on l with the Fourier series:

$$\sum_{n=m+1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

This follows directly from (249) and the fact that the boundary values w have Fourier series (247). Hence it follows that:

$$\int_0^{2\pi} \eta_m(1,\theta)\cos k\theta\, d\theta = 0; \quad \int_0^{2\pi} \eta_m(1,\theta)\sin k\theta\, d\theta = 0. \tag{250}$$

$$(k = 0,\ 1,\ 2,\dots,\ m) \qquad (k = 1,\ 2,\dots, m)$$

As above, we have:

$$J_\varrho(v_m, \eta_m) = -\iint\limits_{B_\varrho} \eta_m \Delta v_m\, dx\, dy + \int\limits_{l_\varrho} \eta_m(\varrho,\theta)\frac{\partial v_m(\varrho,\theta)}{\partial\varrho}\varrho\, d\theta.$$

The double integral vanishes, whilst the integrand in the line integral tends uniformly with respect to θ to

$$\eta_m(1,\theta)\frac{\partial v_m(\varrho,\theta)}{\partial\varrho}\bigg|_{\varrho=1}$$

and the integral of this product vanishes by (249) and (250). Hence $J_\varrho(v_m, \eta_m) \to 0$ as $\varrho \to 1$. On passing to the limit in the expression

$$J_\varrho(v_m + \eta_m) = J_\varrho(v_m) + J_\varrho(\eta_m) + 2J_\varrho(v_m, \eta_m),$$

we obtain

$$J(w) = J(v_m) + J(\eta_m). \tag{251}$$

$J(w)$ has a finite value by hypothesis, and it is clear from the last expression that $J(\eta_m)$ also has a finite value. This is obvious for $J(v_m)$ from (249).

It follows from (251) that:

$$J(v_m) \leqslant J(w) \tag{252}$$

and all the more for any $\varrho < 1$:

$$J_\varrho(v_m) \leqslant J(w). \tag{253}$$

But series (248) can be differentiated term by term in the circle B_ϱ and the series obtained are uniformly convergent in B_ϱ, i.e. the derivatives of v_m tend uniformly in the circle B_ϱ as $m \to \infty$ to the corresponding derivatives of v. Hence inequality (253) gives, as $m \to \infty$:

$$J_\varrho(v) \leqslant J(w),$$

whence, as $\varrho \to 1$, $J(v) \leqslant J(w)$.

Let the sign of equality hold, and let us show that $w \equiv v$. We put $w = v + \eta$, where η, as usual, has continuous first order derivatives inside B, is continuous in the closed circle and vanishes on the circumference l.

We have:

$$J_\varrho(\eta) = J_\varrho(w) + J_\varrho(v) - 2J_\varrho(w, v). \tag{254}$$

On observing that

$$|2(w_x v_x + w_y v_y)| \leqslant w_x^2 + w_y^2 + v_x^2 + v_y^2,$$

we have:

$$|2J_\varrho(w, v)| \leqslant J_\varrho(w) + J_\varrho(v),$$

whence it follows that, for all $\varrho < 1$:

$$|2J_\varrho(w, v)| \leqslant J(w) + J(v),$$

i.e. $J_\varrho(w, v)$ remains bounded as $\varrho \to 1$.

The first two terms on the right-hand side of (254) have finite limits as $\varrho \to 1$, so that $J_\varrho(\eta)$ remains bounded, i.e. has a finite limit as $\varrho \to 1$. We can write further:

$$J_\varrho(w) = J_\varrho(v) + J_\varrho(\eta) + 2J_\varrho(v, \eta),$$

and it follows from the existence of the finite limits of $J(w)$, $J_\varrho(v)$ and $J_\varrho(\eta)$ that $J_\varrho(v, \eta)$ also has a finite limit as $\varrho \to 1$. We write:

$$J(v, \eta) = \lim_{\varrho \to 1} J_\varrho(v, \eta).$$

On introducing the arbitrary real parameter ε, we can put

$$J_\varrho(v + \varepsilon\eta) = J_\varrho(v) + 2\varepsilon J_\varrho(v, \eta) + \varepsilon^2 J_\varrho(\eta).$$

As $\varrho \to 1$ all the terms on the right-hand side have a finite limit, so that the same is true for the left-hand side. We obtain in the limit:

$$J(v + \varepsilon\eta) = J(v) + 2\varepsilon J(v, \eta) + \varepsilon^2 J(\eta). \tag{255}$$

Therefore the Dirichlet integral (240) has a finite value for the function $u = v + \varepsilon\eta$ for any real ε, and we have, by what has been proved above:

$$J(v) \leqslant J(v + \varepsilon\eta), \tag{256}$$

the sign of equality being obtained with $\varepsilon = 0$ and $\varepsilon = 1$, since $J(w) = J(v)$ by hypothesis. Hence it follows that the right-hand side

of (255) attains its least values with $\varepsilon = 0$ and $\varepsilon = 1$, which is only possible when $J(\eta) = 0$, i.e. $\eta \equiv 0$, and since $w = v + \eta$, it follows that $w = v$. The theorem is fully proved.

In the chapter on boundary value problems we shall return to the problem of the absolute extremum in isoperimetric problems, i.e. with supplementary conditions in the form of integrals.

97. Direct methods of the calculus of variations. At the present time different methods of approach have been widely developed for the solution of problems on absolute extrema — methods which get round the use of differential equations. An attempt is made to construct the required function by giving the functional an absolute extremum with the aid of some convergence process, starting directly from the form of the integral whose extremum is sought.

As already mentioned, the present problem is far more difficult than the corresponding problem of the differential calculus. In the latter case, by virtue of Weierstrass's fundamental theorem on continuous functions, we know that every function continuous in a closed domain certainly attains its greatest (or least) value at some point of this domain. In problems of the variational calculus we no longer have such a simple theorem, and hence the actual existence of a solution of the problem comes in question.

Let $J(y)$ be a functional of the required function $y(x)$, and let us seek this function so that the functional is minimized in some class C of functions $y(x)$. For any choice of the function $y(x)$ of class C, the functional J obtains a definite numerical value. By using all the functions of class C, we obtain an infinite set of numbers — the values of functional J. Let d be the strict lower bound of this number set. We do not know in advance whether or not there exists in class C a function $y(x)$ which gives our functional this least value a, but by definition of strict lower bound, we can always find a sequence $y_n(x)$ of functions of class C such that the numbers $J(y_n)$ have d as their limit on indefinite increase of n. The sequence of $y_n(x)$ is usually termed the *minimal sequence*. One of the possibilities for carrying out the methods of the variational calculus directly is as follows: a method is given for constructing minimal sequences in such a way that the minimal sequence constructed leads by a passage to the limit to the required function giving our functional its least value. If a problem can be carried to a conclusion with the aid of such a method, we finish up by solving a convergence problem for a differential equation which expresses the necessary condition for an extremum of the functional. Such a method is applicable, not merely for proving the existence of a solution, but also for devising a practically convenient means of calculating it approximately. The principle just described forms the basis of the celebrated Ritz method for solving boundary value problems. It may be mentioned that a wide generalization of this method to the case of differential equations not connected with the calculus of variations is due to Galerkin (*Vestnik Inzhenerov I Tecknikov*, 1915). Ritz's work was published in 1908 (*Journal für die Reine und Angew. Mathem.*, Bd. 135). A natural theoretical basis of direct methods is obtained, especially for partial differential equations, from an employment of the theory of functions of a real variable. This must be postponed until Vol. V. For the present, some

simple examples of the employment of direct methods will be considered. We shall return to the subject in Chapter IV.

A number of works by Soviet mathematicians have investigated in detail the convergence of the Ritz and Galerkin method and the question of estimating the error. A discussion of these questions and references to the relevant literature may be found in L. V. Kantorovich and V. I. Krylov: *Approximation Methods of Higher Analysis* (Priblizhennye metody vysshego analiza). A substantial part of S. G. Mikhlin's *Direct Methods in Mathematical Physics* (Pryamye metody v matematicheskoi fizike) (1950) is devoted to a study of the convergence of the Ritz–Galerkin method.

S. L. Sobolev's monograph *Some Applications of Functional Analysis in Mathematical Physics* (Nekotorye primeneniya funktsional'nogo analiza v matematicheskoi fizike) (1950) discusses the theoretical bases of direct methods as regards existence theorems for the relevant extremal functions and their properties.

98. Examples. 1. Let us take the functional discussed in [95]:

$$J(y) = \int_0^l [p(x)\, y'^2 + q(x)\, y^2 + 2f(x)\, y]\, dx \tag{257}$$

and seek its minimum in the above-mentioned class D [95] under the homogeneous boundary conditions

$$y(0) = y(l) = 0, \tag{258}$$

where we assume, as in [95], that $p(x) > 0$ and $q(x) \geqslant 0$.

We know that a solution of this problem is given by the function $y_0(x)$ which satisfies equation (234) and boundary conditions (258). Let

$$u_1(x), u_2(x), \ldots \tag{259}$$

be a sequence of functions continuous together with their first derivatives in the interval $[0, l]$, satisfying conditions (258) and linearly independent.

We form a linear combination of the first n functions of the sequence with as yet undetermined constant coefficients $y_n = a_1^{(n)} u_1 + \ldots + a_n^{(n)} u_n$ and substitute in integral (257). After carrying out the quadratures, we obtain a result of the form:

$$J_n = J(y_n) = \sum_{i,j=1}^{n} a_{ij}\, a_i^{(n)} a_j^{(n)} + \sum_{i=1}^{n} \beta_i\, a_i^{(n)} \qquad (a_{ij} = a_{ji}). \tag{260}$$

We determine coefficients $a_i^{(n)}$ from the condition that they satisfy the necessary condition for an extremum of the expression J_n, i.e. to put it more simply, we equate to zero the partial derivatives of J_n with respect to $a_i^{(n)}$. We thus obtain n equations of the first degree for the $a_i^{(n)}$:

$$\sum_{k=1}^{n} a_{ik}\, a_k^{(n)} + \frac{1}{2}\beta_i = 0 \qquad (i = 1, \ldots, n). \tag{261}$$

The determinant of this system is at the same time the discriminant of the quadratic form appearing in expression (260) and resulting from the integration of the expression $(py_n'^2 + qy_n^2)$. In view of our assumptions, this quadratic

form will be positive definite. For, it can only be zero when $y_n \equiv 0$, which, in view of the linear independence of the functions $u_k(x)$, leads to the vanishing of all the a_k^n. But the discriminant of a positive definite quadratic form, equal to the product of its eigenvalues, will certainly be positive. Hence the determinant of system (261) will differ from zero and we can find from the system definite values of the $a_i^{(n)}$, i.e. we can form the nth approximation $y_n(x)$. We remark that, as the number n increases, the coefficients already calculated in general change. This is why we use the superscript n for the coefficients, indicating the number of the approximation.

On using the fact that the quadratic form appearing in (260) is positive definite, and that system (261) has a unique solution, we can say that the solution $(a_1^{(n)}, \ldots, a_n^{(n)})$ of the system minimizes the expression (260). As the number n increases, the least value of the functional is sought in a wider class of functions, and we can therefore say that

$$J(y_q) \leqslant J(y_p) \quad \text{for} \quad q > p. \tag{262}$$

Moreover, for any linear combination $z(x)$ of functions (259):

$$z(x) = \sum_{k=1}^{m} a_k u_k(x), \tag{263}$$

we have [95]:

$$J(Z) \geqslant J(y_0).$$

We show that, given certain assumptions regarding the system of functions (259), functions $y_n(x)$ tend uniformly in the interval $[0, l]$ to the above-mentioned function $y_0(x)$.

Let us state these assumptions. For any function $y(x)$, continuous together with its derivative in the interval $[0, l]$, and given any positive ε, there exists a finite linear combination of functions (259) such that:

$$\left| y(x) - \sum_{k=1}^{m} a_k u_k(x) \right| \leqslant \varepsilon; \ \left| y'(x) - \sum_{k=1}^{m} a_k u_k'(x) \right| \leqslant \varepsilon \quad (0 \leqslant x \leqslant l). \tag{264}$$

We show first all that

$$J(y_n) \to J(y_0). \tag{265}$$

We have [95]:

$$J(y_n) \geqslant J(y_0) \quad (n = 1, 2, 3, \ldots). \tag{266}$$

On applying (264) to the function $y(x) = y_0(x)$ and using the fact that ε is arbitrary, we can say that, given any positive δ, there exists a linear combination (263) of functions (259) such that $J(z) - J(y_0) \leqslant \delta$. Further, by the construction of $y_m(x)$, we have $J(y_m) - J(y_0) \leqslant \delta$, and by (262), we can write: $J(y_n) - J(y_0) \leqslant \delta$ for $n \geqslant m$, whence (265) follows, since δ is arbitrary. We have further, as may easily be seen:

$$J(y_n) - J(y_0) = 2\int_0^l [p y_0' (y_n' - y_0') + q y_0 (y_n - y_0) + f(y_n - y_0)]\, dx +$$
$$+ \int_0^l [p(y_n' - y_0')^2 + q(y_n - y_0)^2]\, dx.$$

On carrying out the first integration by parts and using the fact that $y = y_n - y_0$ satisfies conditions (258), we obtain for this integral:

$$\int_0^l \left[-\frac{d}{dx}(py_0') + qy_0 + f\right](y_n - y_0)\,dx,$$

whence it is clear that the first integral vanishes, so that

$$J(y_n) - J(y_0) = \int_0^l [p(y_n' - y_0')^2 + q(y_n - y_0)^2]\,dx,$$

whence

$$J(y_n) - J(y_0) \geqslant \int_0^l p(y_n' - y_0')^2\,dx. \tag{267}$$

On writing a for the least value of the positive function $p(x)$ in the interval $[0, l]$, we obtain by (267):

$$\int_0^l (y_n' - y_0')^2\,dx \leqslant \frac{J(y_n) - J(y_0)}{a}. \tag{268}$$

Further, Buniakowski's inequality gives us:

$$|y_n - y_0|^2 = |\int_0^x (y_n' - y_0')\,dx|^2 \leqslant \int_0^x (y_n' - y_0')^2\,dx \int_0^x 1^2\,dx \leqslant l \int_0^l (y_n' - y_0')^2\,dx$$

and we obtain, in view of (268):

$$|\,y_n(x) - y_0(x)| \leqslant \sqrt{\frac{l}{a}} \cdot \sqrt{J(y_n) - J(y_0)} \quad (0 \leqslant x \leqslant l),$$

whence, by (265), it follows that $y_n(x) \to y_0(x)$ uniformly in the interval $[0, l]$.

Let us show that conditions (264) are also fulfilled for the functions

$$u_k(x) = \sin\frac{k\pi x}{l} \qquad (k = 1, 2, 3, \ldots), \tag{269}$$

satisfying conditions (258).

We make an even continuation of the function $y'(x)$ given in the interval $[0, l]$, into the interval $[-l, 0]$. Given any positive η, we can find a trigonometric polynomial:

$$T(x) = c_0 + \sum_{k=1}^{m} c_k \cos\frac{k\pi x}{l},$$

such that [II, 164]

$$|y'(x) - T(x)| \leqslant \eta \qquad (-l \leqslant x \leqslant l), \tag{270}$$

where

$$c_0 = \frac{1}{l}\int_0^l T(x)\,dx.$$

But it follows from (270) that $T(x) = y'(x) + f(x)$, where $|f(x)| \leqslant \eta$, so that we have, in view of the fact that $y(0) = y(l) = 0$:

$$c_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x,$$

whence $|c_0| \leqslant \eta$, and it follows from (270) that

$$\left| y'(x) - \sum_{k=1}^{m} c_k \cos\frac{k\pi x}{l} \right| \leqslant \eta + |c_0| \leqslant 2\eta.$$

Integration of this difference from 0 to x gives us

$$\left| y(x) - \sum_{k=1}^{m} \frac{l}{k\pi} c_k \sin\frac{k\pi x}{l} \right| \leqslant 2\eta l$$

and, on taking $\eta = \varepsilon/2(l+1)$, we obtain the linear combination of functions (269):

$$\sum_{k=1}^{m} \frac{l}{k\pi} c_k \sin\frac{k\pi x}{l},$$

satisfying conditions (264).

It can be shown in precisely the same way, by using the theorem on the approximation of a continuous function with the aid of polynomials, that the functions

$$u_k(x) = (l - x)\, x^k \quad (k = 1, 2, 3, \ldots)$$

also satisfy conditions (264). All these obviously also satisfy conditions (258).

2. The proof of the convergence of the Ritz method for partial differential equations is more difficult, and we shall confine ourselves to particular examples.

We take Poisson's equation:

$$u_{xx} + u_{yy} = g(x, y) \tag{271}$$

for the case of a square B defined by the inequalities $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$, and we take as our boundary condition the vanishing of the required function φ on the contour l of the square. We shall assume that the given function $g(x, y)$ can be expanded in a Fourier series in the square B:

$$g(x, y) = \sum_{p,q=1}^{\infty} c_{pq} \sin px \sin qy, \tag{272}$$

where the series $\sum_{p,q=1}^{\infty} |c_{pq}|$ is convergent.

Equation (271) is the Euler equation for the integral:

$$J(u) = \iint_B (u_x^2 + u_y^2 + 2gu)\,\mathrm{d}x\,\mathrm{d}y. \tag{273}$$

The required function is in this case a function of two independent variables, so that its approximate solution is naturally sought as a segment of a double series in sines of multiple arcs:

$$u_{mn} = \sum_{p=1}^{m} \sum_{q=1}^{m} a_{pq} \sin px \sin qy. \tag{274}$$

The basic functions in which we expand the approximate solutions obviously satisfy the boundary condition. On substituting expression (274) in integral (273) and using the familiar formulae for the integrals of products of trigonometric functions, we obtain:

$$\frac{4}{\pi^2} J(u_{mn}) = \sum_{p=1}^{m} \sum_{q=1}^{n} (p^2 + q^2)\, a_{pq}^2 + 2 \sum_{p=1}^{m} \sum_{q=1}^{n} c_{pq}\, a_{pq}.$$

On equating to zero the derivatives of the last expression with respect to the a_{pq}, we arrive at the following expressions for the required coefficients:

$$a_{pq} = -\frac{c_{pq}}{p^2+q^2}.$$

The values of the coefficients do not depend in this case on the number of the approximation, so that, as this number increases, the coefficients already calculated remain unchanged, and passage to the limit leads us to the function:

$$u(x, y) = -\sum_{p,q=1}^{\infty} \frac{c_{pq}}{p^2 + q^2} \sin px \sin qy. \tag{275}$$

On taking into account the convergence of the series $\sum_{p,q=1}^{\infty} | c_{pq} |$, it is easily seen that series (275) can be twice differentiated term by term with respect to the independent variables. Hence it follows at once that function (275) in fact satisfies equation (271). The fulfillment of the boundary conditions is immediately obvious. This present problem can only have one solution. For, if two functions u_1 and u_2 exist, satisfying equation (271) and the boundary condition, their difference must satisfy Laplace's equation and must vanish on the contour of the square B. But this implies at once [II, 194] that the difference is identically zero.

3. We now take the case when the right-hand side of the equation is a given number:

$$u_{xx} + u_{yy} = c,$$

and seek the solution under the previous boundary conditions for the square B defined by $-a \leqslant x \leqslant a$; $-a \leqslant y \leqslant a$.

Integral (273) here becomes

$$J(u) = \iint_B (u_x^2 + u_y^2 + 2cu)\, dx\, dy. \tag{276}$$

We shall seek approximate expressions for the required function in the form of a polynomial in variables x and y, and to satisfy the boundary conditions

we put the factor $(x^2 - a^2)(y^2 - a^2)$ in front of the polynomial, i.e. we put

$$u_n = (x^2 - a^2)(y^2 - a^2) \sum_{p+q \leqslant n} a_{pq} x^p y^q.$$

In view of the symmetry of the square B, we only need terms in even powers of the independent variables in the polynomials. In addition, by virtue of the symmetry, the polynomial must be assumed symmetrical with respect to both independent variables. Taking the polynomial of zero degree, we get:

$$u_0 = a_{00}^{(0)} (x^2 - a^2)(y^2 - a^2).$$

On substituting this expression in integral (276), carrying out the quadrature and equating to zero the derivative with respect to $a_{00}^{(0)}$, we obtain for $a_{00}^{(0)}$:

$$a_{00}^{(0)} = \frac{5}{16} \frac{c}{a^2}.$$

We take as our second approximation:

$$u_1 = (x^2 - a^2)(y^2 - a^2)[a_{00}^{(1)} + a_{02}^{(1)}(x^2 + y^2)].$$

On performing the above-mentioned operations, we get the following values for the coefficients of the polynomial:

$$a_{00}^{(1)} = \frac{5}{16} \cdot \frac{259}{277} \cdot \frac{c}{a^2}; \quad a_{02}^{(1)} = \frac{15}{32} \cdot \frac{35}{277} \cdot \frac{c}{a^4}.$$

CHAPTER III

FUNDAMENTAL THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

§ 1. First order equations

99. Linear equations with two independent variables. We have encountered fairly often differential equations containing partial derivatives of the required function. These have always been equations of a strictly specialized type, originating from concrete problems of mathematical physics. The aim of the present chapter is to lay the foundations of the general theory of partial differential equations, and we start by considering the theory of first order equations.

A first order equation with a single required function u of the independent variables $x_1, \ldots, x_n$ has the form:

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0,$$

where the x_k are the independent variables and the $p_k = u_{x_k}$ are the partial derivatives of the required function with respect to these independent variables. We shall start by investigating equations which are linear in the partial derivatives p_k, i.e. equations of the form:

$$a_1(x_1, \ldots, x_n, u)\, p_1 + \ldots + a_n(x_1, \ldots, x_n, u)\, p_n = c(x_1, \ldots, x_n, u), \qquad (1)$$

where the coefficients a_k and the right-hand side c are given functions of the independent variables x_k and the required function u. Since the function u itself can be introduced in any manner into the coefficients and right-hand side, such equations are sometimes called *quasi-linear* instead of linear. We shall consider in the present section equations of type (1) in the case of two independent variables. In this particular case the independent variables are usually written as x and y, whilst the partial derivatives are denoted as usual by: $p = u_x$ and $q = u_y$.

Thus the present discussion will be concerned with equations of the form:

$$a(x, y, u)\,p + b(x, y, u)\,q = c(x, y, u). \tag{2}$$

It may be recalled that linear partial differential equations have already been discussed in [II, 21], where we saw that the problem of integrating an equation of type (2) is equivalent to the problem of integrating a system of ordinary differential equations. We shall add to the results already obtained certain new facts which will be useful later when investigating more complex problems.

The given functions $a(x, y, u)$, $b(x, y, u)$ and $c(x, y, u)$ define a tangent field in space (x, y, u), i.e. at each fixed point of this space we have a direction whose direction-cosines are proportional to a, b, c. This direction field defines a family of curves such that the tangent at any point of a curve of the family coincides with the direction of the field at this point. This family of curves is obtained as a result of integrating the system of ordinary equations:

$$\frac{\mathrm{d}x}{a(x, y, u)} = \frac{\mathrm{d}y}{b(x, y, u)} = \frac{\mathrm{d}u}{c(x, y, u)}, \tag{3}$$

or, if we write $\mathrm{d}s$ for the common value of these ratios:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x, y, u); \quad \frac{\mathrm{d}y}{\mathrm{d}s} = b(x, y, u); \quad \frac{\mathrm{d}u}{\mathrm{d}s} = c(x, y, u). \tag{4}$$

The quantities p, q and (-1) are proportional to the direction-cosines of the normal to the required surface $u = u(x, y)$, and equation (2) expresses the condition for the normal to the required surface: $ap + bq + c(-1) = 0$ to be perpendicular to the direction of the field, i.e. (2) amounts to the requirement that the direction defined by the direction field lies in the tangent plane at every point of the required surface. We describe the curves defined by system (4) as *characteristics* of equation (2). If a surface $u = u(x, y)$ represents the locus of characteristics of (2), i.e. is formed by the curves l' which satisfy system (4), at every point of the surface the tangent to the curve l' through this point lies in the tangent plane to the surface, so that the surface satisfies equation (2), i.e. is an integral surface of the equation. Thus, *if a surface $u = u(x, y)$ is formed by characteristics of equation* (2), *it is an integral surface of the equation.*

Let us assume that the surface $u = u(x, y)$ has a tangent plane at every point and that the direction of the normal to the surface

changes continuously on displacement over the surface. This amounts to the existence and continuity of the first order derivatives of $u(x, y)$.

In future, when speaking of an integral surface, we shall assume that the surface has the properties just mentioned. For brevity, we shall in general describe such surfaces as *smooth.*

We proved above that a smooth surface having the equation $u = u(x, y)$ and formed by characteristics is an integral surface. It can be shown that, conversely, *if a smooth surface satisfies equation* (2), *i.e. is an integral surface, it can be covered by characteristics.*

In fact, if a surface S satisfies equation (2), the direction (a, b, c) at every point lies in the tangent plane to S, and we thus have a direction field on S. On integrating the ordinary first order differential equation corresponding to this field, we in fact find the curves l' lying on S and satisfying system (4). We can take as this first order equation e.g.

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)},$$

in which u is replaced by its expression $u = u(x, y)$ from the equation of the surface S. Suppose that the existence and uniqueness theorem is applicable to the equation obtained, and that integration of it leads to an expression for y in terms of x and an arbitrary constant C; on substituting this expression in the formula $u = u(x, y)$, we in fact obtain for u an expression in terms of x and C, and thus obtain the equation of a family of curves l' covering S. We shall return later to this question.

We saw in [II, 50, 51], when investigating ordinary first order differential equations, that the required functions are fully defined by specifying the initial values of the functions for a given value of the independent variable. The arbitrary constants appearing in the general solution are defined from these initial data, provided the general solution can be found. But the determination of the solution satisfying given initial data can be carried through without a knowledge of the general solution, by using the method of successive approximations which we used when proving the existence and uniqueness theorem [II, 51]. The general solution of equation (2) contains arbitrary functions [II, 22] instead of arbitrary constants, and the problem of finding the solution satisfying given initial data may be formulated as follows in this case: *to find the integral surface of equation* (2) *which passes through a given curve l in space* (x, y, u). If we write λ for the projection of

the curve l on the (x, y) plane, this problem amounts to seeking the solution of (2) which takes given values at points of the curve λ. Let us first note a solution of this problem [II, 22]. Let M_0 be a point of the curve l. We take its coordinates as the initial values of the functions defined by system (4). By the existence and uniqueness theorem, we obtain a completely defined characteristic passing through this point M_0. If this is done for each point of l, we get a family of characteristics; suppose that they form some surface S with the equation $u = u(x, y)$. It passes through the curve l, and from, what was said above, is an integral surface of equation (2).

A strict proof of the existence and uniqueness of the solution of the problem requires certain assumptions regarding the right-hand sides of equations (4) and certain important provisos regarding the curve l. If, for instance, the given curve l is itself a characteristic, the above method of drawing the characteristics from points of l leads to l itself instead of a surface. In this case there may be an infinite set of solutions [II, 23]. For, let us draw through a point of l a curve l_1 which is not a characteristic. On drawing the characteristics from points of this latter curve (which will include among them the given curve l) we obtain, given certain conditions, an integral surface through l. It will be seen, since l_1 is chosen arbitrarily, that the problem has an infinite set of solutions when the given curve l is a characteristic. It may happen that the problem has no solutions at all. This will be the case when the characteristics issuing from points of l do not form in the neighbourhood of l a surface having an explicit equation $u = u(x, y)$, where $u(x, y)$ is single-valued and continuous and has continuous first order derivatives. This happens, for instance, if the characteristics in question form a cylindrical surface with generators parallel to the u axis. The next section deals with the conditions in which the problem has a single definite solution.

100. Cauchy's problem and characteristics. A Cauchy problem is generally understood to be a problem as stated above, of finding the integral surface of equation (2) passing through a given curve l. To carry out an exact investigation of the question of the existence and uniqueness of the solution of this problem we need a theorem from the theory of ordinary differential equations, viz.

THEOREM. *If the right-hand sides of the system of differential equations*

$$\frac{dy_k}{dx} = f_k(x, y_1, \ldots, y_n) \quad (k = 1, 2, \ldots, n) \tag{5}$$

are continuous functions of their arguments in some domain defined by the inequalities

$$|x - a| \leqslant A; \quad |y_k - b_k| \leqslant B \quad (k = 1, 2, \ldots, n) \tag{6}$$

and if, in addition, continuous partial derivatives $\partial f_k/\partial y_s$ *exist in this domain, then the solution of system* (5), *defined by virtue of the existence and uniqueness theorem by any initial data* $(x_0, y_1^0, \ldots, y_n^0)$ *lying inside domain* (6):

$$y_k = \varphi_k(x, x_0, y_1^0, \ldots, y_n^0) \quad (k = 1, 2, \ldots, n),$$

is continuous with respect to its arguments and possesses partial derivatives $\mathrm{d}\varphi_k/\mathrm{d}y_s^0$ *with respect to the initial data which are continuous functions of the arguments* $(x, x_0, y_1^0, \ldots, y_n^0)$.

To save breaking into the exposition, we shall postpone the proof of this theorem to a later section.

We return to Cauchy's problem. Suppose that the equation of the curve l is given in the parametric form:

$$x_0 = x_0(t); \; y_0 = y_0(t); \; u_0 = u_0(t) \;\; (t_0 \leqslant t \leqslant t_1) \tag{7}$$

and that the right-hand sides of equations (4) satisfy the conditions of the above theorem in some domain of the (x, y, u) space containing l inside it. On taking the coordinates of the points of l as the initial data with $s = 0$, we obtain the solutions of (4):

$$x = x(s, x_0, y_0, u_0); \quad y = y(s, x_0, y_0, u_0); \quad u = u(s, x_0, y_0, u_0)$$

when s is sufficiently close to zero, or, by (7):

$$x = x(s, t); \quad y = y(s, t); \quad u = u(s, t). \tag{8}$$

Assuming that the right-hand sides of equations (7) are continuously differentiable with respect to t, and using the above theorem, we can say that functions (8) have continuous derivatives with respect to t as well as s. Given any t of the interval $t_0 < t < t_1$, functions (8) are defined for all s sufficiently close to zero. We form the functional determinant of the first two of these functions with respect to s and t:

$$\Delta = x_s y_t - x_t y_s. \tag{9}$$

An essential point for what follows is whether or not this determinant is zero. We consider first the case when $\Delta \neq 0$ along the curve l, and second, the case when $\Delta = 0$ along the whole of l. We start with the first case:

$$\Delta \neq 0 \quad (\text{along } l), \tag{10}$$

i.e. $\Delta \neq 0$ for $s = 0$, and hence, in view of the continuity of the derivatives, $\Delta \neq 0$ also in some neighbourhood of the initial value $s = 0$ and of the value of t corresponding to some point M of l. The first two of equations (8) can now be solved for s and t for all x and y belonging to the neighbourhood of the coordinates (x_0, y_0) of the point M of l. This solution is unique, and the functions $s(x, y)$, $t(x, y)$ obtained have continuous first order derivatives [III, 19]. On substituting these functions $s(x, y)$ and $t(x, y)$ in the third of equations (8), we in fact get a function $u(x, y)$ in the neighbourhood in question, having continuous first order derivatives, where the surface $u = u(x, y)$ contains some part of the curve l in the neighbourhood of M. It follows at once from the geometrical discussion of the previous section that $u(x, y)$ satisfies equation (2). This will also be proved analytically below.

We remark that the solution $u(x, y)$ obtained is only valid in a neighbourhood of any given point M of l, which is usually expressed by saying that a local solution of the problem has been obtained. With certain conditions imposed on a, b, c and curve l, we can prove the possibility of constructing an integral surface in a neighbourhood of the whole of l, i.e. for all x and y sufficiently close to the curve $x = x_0(t)$, $y = y_0(t)$ on the (x, y) plane. We assume here that the derivatives $x_0'(t)$ and $y_0'(t)$ do not vanish simultaneously. A strict statement of the detailed results will be given in the next section.

The question of the existence of a solution of the equation in a previously assigned domain of the (x, y) plane presents greater difficulties. It is possible to construct a domain B of the plane and a function $b(x, y)$ in it with derivatives of all orders, so that the only solution of the equation

$$u_x + b(x, y)\, u_y = 0$$

having first order continuous derivatives and existing throughout B is $u = \text{const.}$

We now show that the function $u(x, y)$ obtained is in fact a solution of equation (2). By using the rule for differentiation of composite functions and equations (4), we can write:

$$\frac{du}{ds} = u_x a + u_y b.$$

This equation holds for all s and t lying in the neighbourhood of $s = 0$ and t corresponding to some point $M(x_0, y_0, z_0)$ of the curve l.

But $\mathrm{d}u/\mathrm{d}s = c$, so that $u(x, y)$ satisfies equation (2) for all (x, y) lying in the neighbourhood of (x_0, y_0).

The uniqueness is proved simply by showing that any smooth integral surface $u = u(x, y)$ through l can be formed by characteristics. We form the system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a[x, y, u(x, y)]; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = b[x, y, u(x, y)]. \tag{11}$$

In view of our assumptions, the right-hand sides are such that the existence and uniqueness theorem holds for all (x, y) in the neighbourhood of (x_0, y_0). It follows from the fact that the integral surface has an explicit equation $u = u(x, y)$ and passes through a piece of the curve l in the neighbourhood of $M(x_0, y_0, z_0)$ that the coordinates (x, y) of distinct points of l in the neighbourhood of M are distinct (l is assumed not to intersect itself). On taking these coordinates as the initial data when integrating system (11) and substituting the solutions obtained in the function $u = u(x, y)$, we get a family of curves on the integral surface. By (11), the first two of equations (4) are satisfied along these curves. It is easily shown that the third equation is also satisfied. For we have, by using (11):

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u_x a + u_y b.$$

But $u = u(x, y)$ is an integral surface, i.e. $u_x a + u_y b = c$, whence $\mathrm{d}u/\mathrm{d}s = c$. Thus the curves in question covering the surface $u = u(x, y)$ are in fact characteristics. Hence, *given condition* (10), *Cauchy's problem has a unique solution.* We shall return later to the question of the uniqueness when considering non-linear first order equations.

Now let

$$\Delta = x_s y_t - x_t y_s = 0 \tag{12}$$

along l, i.e. when $s = 0$.

We show that, if there exists in this case an integral surface $u = u(x, y)$ with continuous first order derivatives passing through l, this curve must be a characteristic. Here, as above, when we say that the surface passes through l, we mean this in a local sense, i.e. we only consider a piece of l.

We shall assume that a and b differ from zero along l. On taking into account the first two of equations (4), we can write condition (12) as

$$\frac{x_t}{a} = \frac{y_t}{b} = k \quad (s = 0), \tag{13}$$

where k has been used to denote the common value of the ratios. Let $u = u(x, y)$ be an integral surface passing through l. On substituting the expressions $x = x_0(t)$, $y = y_0(t)$ in $u(x, y)$, differentiating with respect to t and using (13), we get: $\mathrm{d}u/\mathrm{d}t = u_x ka + u_y kb$. On recalling that $u = u(x, y)$ is a solution of equation (2) and using this equation, we can further write $\mathrm{d}u/\mathrm{d}t = kc$, which leads us to the system:

$$\frac{x_t}{a} = \frac{y_t}{b} = \frac{u_t}{c} \quad (s = 0),$$

from which it follows that l is a characteristic. Thus, *if $\Delta = 0$, the necessary condition for the existence of an integral surface through l is that l be a characteristic.* Now, as we saw in the previous section, an infinite set of integral surfaces passes through l. It was obviously essential in the above proof that the integral surface $u = u(x, y)$ through l should have continuous derivatives at points of l; it may happen, as we shall see by an example, that l is not a characteristic, that $\Delta = 0$ along it, and yet an integral surface passes through l, though such that the partial derivatives of $u(x, y)$ cease to be continuous at points of l, i.e. in other words, l is a singular curve of the integral surface. If l is not a characteristic, but $\Delta = 0$ along it, this means that along l:

$$\frac{x_t}{a} = \frac{y_t}{b} \neq \frac{u_t}{c}.$$

Let us notice a special feature of system (4). The auxiliary parameter s does not appear in the right-hand side of the equations, and one of the arbitrary constants appears as a term added to s. This arbitrary constant does not play an essential part and amounts to an arbitrary choice of the initial value of s. We thus have two essential arbitrary constants from the integration of the system. This fact is immediately obvious if system (4) is written in the form (3).

We recall that the quasi-linear non-homogeneous equation (2) can be reduced to a purely linear homogeneous equation if the solution is sought in the implicit form [II, 21]:

$$\varphi(x, y, u) = C, \tag{14}$$

where C is an arbitrary constant. By the rule for differentiating implicit functions we have:

$$u_x = -\frac{\varphi_x}{\varphi_u}; \quad u_y = -\frac{\varphi_y}{\varphi_u},$$

and equation (2) reduces to the purely linear homogeneous equation:

$$a(x, y, u)\,\varphi_x + b(x, y, u)\,\varphi_y + c(x, y, u)\,\varphi_u = 0. \tag{15}$$

The corresponding system of ordinary differential equations will be system (3). If

$$\varphi_1(x, y, u) = C_1; \quad \varphi_2(x, y, u) = C_2$$

are two independent solutions of this system, then

$$\varphi = F(\varphi_1, \varphi_2),$$

where F is an arbitrary function of its arguments, will be a solution of equation (15). We have seen how the form of this function is determined from the conditions of the Cauchy problem [II, 23].

The foregoing discussion gives rise to the following question. We have sought the solution of equation (2) as a solution appearing in a class of solutions having the implicit equation (14), which contains an arbitrary constant C. It may readily be shown that no solution of our equation is lost by this means. Briefly, the point here is that, because of the arbitrariness of the initial data in Cauchy's problem, every solution of our equation can be regarded as belonging to a class of solutions containing an arbitrary constant. On solving for this arbitrary constant, it will be seen that every solution can in fact be obtained from a formula of the form (14). We could only have lost solutions (singular solutions) such as could not be obtained by the process described for solving Cauchy's problem. There can be no such solutions if the functions a, b and c satisfy certain general conditions. We shall not dwell on the details of the proof.

101. The case of any number of variables. Let us take a linear equation with any number of independent variables:

$$a_1(x_1, \ldots, x_n, u)\,p_1 + \ldots + a_n(x_1, \ldots, x_n, u)\,p_n = c(x_1, \ldots, x_n, u). \tag{16}$$

We shall always assume in future that the coefficients $a_1, a_2, \ldots, a_n$ do not vanish simultaneously for the values of the variables $x_1, x_2, \ldots, x_n$, u considered, i.e. $a_1^2 + a_2^2 + \ldots + a_n^2 > 0$. We shall use geometrical terminology when investigating equation (16), in analogy with the three-dimensional case. Here we have an $(n + 1)$-dimensional space with coordinates $(x_1, \ldots, x_n, u)$. We define a manifold of

dimensionality m (or m-manifold) in this space as a set of points whose coordinates are given in terms of m arbitrary parameters:

$$x_k = x_k(t_1, \ldots, t_m); \; u = u(t_1, \ldots, t_m) \;\; (k = 1, 2, \ldots, n),$$

it being assumed that any m of the equations written are soluble with respect to $t_1, \ldots, t_m$. When $m = n$ we have an n-manifold, which we shall term a surface. If we take $x_1, \ldots, x_n$ as parameters, we have the explicit equation of the surface: $u = u(x_1, \ldots, x_n)$. The equation of an integral surface of equation (16) must be of this form. When $m = 1$ the corresponding one-dimensional manifold is called a curve in $(n + 1)$-dimensional space.

We define the characteristics of equation (16) as follows:

$$\frac{dx_k}{ds} = a_k(x_1, \ldots, x_n, u); \qquad \frac{du}{ds} = c(x_1, \ldots, x_n, u), \tag{17}$$

where s is an auxiliary parameter. Every solution of this equation, apart from the solution in which all the x_k and u are constant, gives a curve in $(n + 1)$-dimensional space. A solution in which all the x_k and u are constant cannot exist since $a_1^2 + \ldots + a_n^2 > 0$. The coordinates of points of such a curve will be expressed in terms of the parameter s. In order to construct a surface from these curves, we have to take a family of the curves depending on $(n - 1)$ arbitrary parameters. In general a set of points is obtained, depending on n parameters. *If a smooth surface $u = u(x_1, \ldots, x_n)$ is formed by a family of characteristics which depend on $(n - 1)$ parameters, it is an integral surface of equation* (16). For, we obtain by differentiating $u(x_1, \ldots, x_n)$ with respect to s and using equations (17):

$$\frac{du}{ds} = \sum_{k=1}^{n} u_{x_k} a_k.$$

But, by the last of the equations, $du/ds = c$, whence equation (16) follows. Conversely, *every smooth integral surface $u = u(x, y)$ can be regarded as formed by a family of characteristics depending on $(n - 1)$ parameters.* For given the integral surface $u = u(x_1, \ldots, x_n)$, we can determine the x_k from the system of equations:

$$\frac{dx_k}{ds} = a_k[x_1, \ldots, x_n, u(x_1, \ldots, x_n)] \;\; (k = 1, 2, \ldots, n), \tag{18}$$

which gives us $(n - 1)$ arbitrary constants. An arbitrary constant, appearing additively in s, will not play an essential part. On sub-

stituting the solution of system (18) in the right-hand side of $u = u(x_1, \ldots, x_n)$, differentiating with respect to s and using equations (16) and (18), it will be seen that u satisfies the last of equations (17).

We assume as in [100] that $u(x_1, \ldots, x_n)$ and the right-hand sides of equations (17) have continuous first order derivatives.

Cauchy's problem for equation (16) consists in finding the integral surface containing a given $(n - 1)$-manifold:

$$x_k = x_k(t_1, \ldots, t_{n-1}); \; u = u(t_1, \ldots, t_{n-1}) \; (k = 1, 2, \ldots, n), \tag{19}$$

where the right-hand sides are continuous and have continuous first order partial derivatives inside some domain D of $(n - 1)$-dimensional space $(t_1, \ldots, t_{n-1})$.

It is assumed that the rank of the matrix made up of the derivatives $\partial x_k/\partial t_l$ is equal to $(n-1)$ and that different systems of values $(t_1, \ldots, t_{n-1})$ correspond to different points $(x_1, \ldots, x_n)$. Further, as mentioned above, the coefficients $a_k(x_1, \ldots, x_n, u)$ and $c(x_1, \ldots, x_n, u)$ are assumed to have continuous first order derivatives in some domain of space containing manifold (19) inside it.

In a particular case this condition in Cauchy's problem can consist in specifying, for a given value of one of the independent variables, the required function u as a function of the remainder:

$$u|_{x_1=x_1^{(0)}} = \varphi(x_2, \ldots, x_n). \tag{20}$$

The solution of the problem follows exactly the same lines as in the case of two independent variables. Expressions (19) are taken as the initial conditions when integrating system (17). We therefore obtain a solution of the form:

$$x_k = x_k(s, t_1, \ldots, t_{n-1}); \quad u = u(s, t_1 \ldots, t_{n-1}). \tag{21}$$

An essential role in what follows is played by the determinant:

$$\Delta = \begin{vmatrix} \dfrac{\partial x_1}{\partial s}, & \dfrac{\partial x_2}{\partial s}, & \ldots, & \dfrac{\partial x_n}{\partial s} \\ \dfrac{\partial x_1}{\partial t_1}, & \dfrac{\partial x_2}{\partial t_1}, & \ldots, & \dfrac{\partial x_n}{\partial t_1} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial x_1}{\partial t_{n-1}}, & \dfrac{\partial x_2}{\partial t_{n-1}}, & \ldots, & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix}, \tag{22}$$

which, by taking (17) into account, can be rewritten as

$$\Delta = \begin{vmatrix} a_1, & a_2, & \ldots, & a_n \\ \dfrac{\partial x_1}{\partial t_1}, & \dfrac{\partial x_2}{\partial t_1}, & \ldots, & \dfrac{\partial x_n}{\partial t_1} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial x_1}{\partial t_{n-1}}, & \dfrac{\partial x_2}{\partial t_{n-1}}, & \ldots, & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix}. \tag{23}$$

If this determinant is non-zero on the manifold (19), i.e. when $s = 0$, the first n of equations (21) can be solved for $s, t, \ldots, t_{n-1}$ and, on substituting this solution in the last of equations (21), we obtain the integral surface of equation (16). There can be no other solutions of the Cauchy problem in this case. All this is proved precisely as in the case of two independent variables. We consider the case when the initial condition has the form (20), and let the role of parameters ${}_1, \ldots, t_{n-1}$ be played by $x_2, \ldots, x_n$. We take a linear equation and assume that determinant (23) is non-zero on our manifold. On using the fact that $\partial x_p/\partial x_q = 0$ for $p \neq q$ and $\partial x_p/\partial x_p = 1$, we get $\Delta = a_1 \neq 0$. On dividing the equation by the coefficient a_1, we arrive at an equation of the form:

$$p_1 + a_2(x_1, \ldots, x_n)\, p_2 + \ldots + a_n(x_1, \ldots, x_n)\, p_n = b(x_1, \ldots, x_n)\, u + c(x_1, \ldots, x_n). \tag{24}$$

Suppose that a_k, b and c are continuous and have continuous first order partial derivatives with respect to $x_2, \ldots, x_n$ for $a \leqslant x_1 \leqslant \beta$ and arbitrary real $x_2, \ldots, x_n$. Suppose, in addition, that in these conditions the functions in question are bounded: $|a_k| \leqslant M$; $|b| \leqslant M$; $|c| \leqslant M$.

On choosing x_1 as the independent variable, system (17) can be written as:

$$\frac{\mathrm{d}x_k}{\mathrm{d}x_1} = a_k(x_1, \ldots, x_n) \quad (k = 2, \ldots, n), \tag{25}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x_1} = b(x_1, \ldots, x_n)\, u + c(x_1, \ldots, x_n). \tag{26}$$

Let $x_1^{(0)}$ be the initial value of x_1 from the interval $[a, \beta]$. We solve system (25) under certain initial conditions:

$$x_k |_{x_1 = x_1^{(0)}} = x_k^{(0)} \quad (k = 2, \ldots, n).$$

It follows from $|a_k| \leqslant M$ that the solutions x_k of system (25) have bounded derivatives $|\mathrm{d}x_k/\mathrm{d}x_1| \leqslant M$, and hence the x_k themselves re-

main bounded in absolute value: $| x_k - x_k^{(0)} | \leqslant M(\beta - a)$. On applying the method of successive approximations [II, 51], it is easily seen that these solutions:

$$x_k = \varphi_k(x_1, x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad (k = 2, \ldots, n) \tag{27}$$

exist throughout the interval $a \leqslant x_1 \leqslant \beta$, with arbitrary initial data $x_k^{(0)}$ $(k = 2, \ldots, n)$. We can say that the integral curve passing through the point $A_0(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ passes through $A(x_1, x_2, \ldots, x_n)$ the coordinates of which are given by formulae (27). On using the uniqueness theorem, we can say that, if the point A is taken as the initial point, the corresponding integral curve passes through A_0. Hence it follows that equations (27) are soluble with respect to, $x_2^{(0)}, \ldots, x_n^{(0)}$ for any x_k, the solution being of the form:

$$x_k^{(0)} = \varphi_k(x_1^{(0)}, x_1, x_2, \ldots, x_n) \quad (k = 2, \ldots, n). \tag{27_1}$$

Suppose we want to solve Cauchy's problem with initial data (20). In view of what has been said, we have to solve equations (25) and (26) under the initial conditions:

$$x_k|_{x_1 = x_1^{(0)}} = x_k^{(0)} \quad (k = 2, \ldots, n);$$

$$u|_{x_1 = x_1^{(0)}} = \varphi(x_2^{(0)}, \ldots, x_n^{(0)}),$$

where the arbitrary magnitudes $x_2^{(0)}, \ldots, x_n^{(0)}$ play the role of $t_1, \ldots, t_{n-1}$. On substituting (27) in (26) and integrating (26), we get:

$$u = e^{\omega}\left[\varphi(x_2^{(0)}, \ldots, x_n^{(0)}) + \int_{x_1^{(0)}}^{x_1} c(x_1, \varphi_2, \ldots, \varphi_n)\, e^{-\omega}\, dx_1\right], \tag{28}$$

where

$$\omega = \int_{x_1^{(0)}}^{x_1} b(x_1, \varphi_2, \ldots, \varphi_n)\, dx_1$$

and $\varphi_k(x_1, x_1^{(0)}, \ldots, x_n^{(0)})$ appear in the arguments of b and c. On substituting expressions (27_1) in the right-hand side of (28), we obtain the required solution $u(x_1, \ldots, x_n)$ of Cauchy's problem. It will exist throughout the interval $a \leqslant x_1 \leqslant \beta$ and for any $x_2, \ldots, x_n$. This is bound up with the linearity of the equation and the assumptions made regarding a_k, b and c.

Given certain assumptions regarding a_k and c, a domain of existence can be indicated for solutions of the quasi-linear equation (16).

Let $a_1 = 1$, and let a_k and c be continuous and bounded, with continuous derivatives provided that

$$|x_1 - x_1^{(0)}| \leqslant a; \tag{29}$$

$$b_k < x_k < c_k \quad (k = 2, \ldots, n) \tag{30}$$

for any real u, the absolute values of these derivatives being not greater than some constant A. Let $\varphi(x_2, \ldots, x_n)$ be continuous and bounded, given conditions (30), and have continuous first order derivatives whose absolute values do not exceed some constant B. Equation (16) $(a_1 = 1)$ under condition (20) now has a solution in the domain defined by

$$|x_1 - x_1^{(0)}| < a; \; |x - x_1^{(0} \;| < \frac{1}{nA} \log \left[1 + \frac{n}{(n-1)(B+1)}\right]$$

and by inequalities (30) (Kamke, *Differentialgleichungen reeller Funktionen;* p. 335).

We now consider the case when $\Delta = 0$ on manifold (19). We shall assume that one of the minors of Δ corresponding to an element of the first row is non-zero. The equation $\Delta = 0$ shows that the elements of the first row now consist of linear combinations of the corresponding elements of the remaining rows, i.e.

$$a_k = \sum_{j=1}^{n-1} \lambda_j \frac{\partial x_k}{\partial t_j}, \tag{31}$$

where λ_j are definite functions of parameters $(t_1, \ldots, t_{n-1})$. If the function c is also expressible as

$$c = \sum_{j=1}^{n-1} \lambda_j \frac{\partial u}{\partial t_j} \tag{32}$$

on manifold (19), (19) is here called a *characteristic manifold* of our equation. It can be shown that every characteristic manifold (19) of equation (16) can be formed by characteristics of the equation and that, if $\Delta = 0$ on manifold (19) and a smooth integral surface $u = u(x_1, x_2, \ldots, x_n)$ passes through this manifold, it is a characteristic manifold.

An infinite set of integral surfaces can pass through a characteristic manifold.

The quasi-linear non-homogeneous equation (16) can be reduced to a purely linear homogeneous equation precisely as in the case of two independent variables, by seeking the solution of (16) in the implicit form:

$$\varphi(x_1, \ldots, x_n, u) = C,$$

where C is an arbitrary constant. We obtain the equation for φ:

$$a_1 \varphi_{x_1} + \ldots + a_n \varphi_{x_n} + c\varphi_u = 0.$$

The corresponding system of ordinary differential equations is

$$\frac{dx_1}{a_1} = \ldots = \frac{dx_n}{a_n} = \frac{du}{c}\,. \tag{33}$$

If

$$\varphi_1(x_1, \ldots, x_n, u) = C_1;\ \ldots;\ \varphi_n(x_1, \ldots, x_n, u) = C_n \tag{34}$$

are independent solutions of it, the equation

$$F(\varphi_1, \ldots, \varphi_n) = 0,$$

where F is an arbitrary function of its arguments, gives the solution of (16) in the implicit form. We have written zero instead of an arbitrary constant on the right-hand side of the last equation because F is an arbitrary function of its arguments. To draw an integral surface containing a given manifold (19), we substitute expressions (19) in the left-hand sides of integrals (34). On eliminating from the n equations thus obtained the $(n-1)$ parameters $t_1, \ldots, t_{n-1}$, a relationship is obtained between the arbitrary constants:

$$F(C_1, \ldots, C_n) = 0.$$

The left-hand side of this relationship in fact defines the form of the function F. To put the matter more simply, substitution of functions $\varphi_k(x_1, \ldots, x_n, u)$ for the C_k in the left-hand side of the last equation gives us the equation of the required integral surface.

102. Examples. 1. Let us take the equation:

$$3(u-y)^2 p - q = 0. \tag{35}$$

System (4) has the form:

$$\frac{dx}{ds} = 3(u-y)^2; \quad \frac{dy}{ds} = -1; \quad \frac{du}{ds} = 0, \tag{36}$$

and its solution, expressed in terms of the initial values of the variables (x, y, u), becomes

$$x = (u_0 - y_0 + s)^3 + x_0 - (u_0 - y_0)^3; \quad y = -s + y_0; \quad u = u_0. \tag{37}$$

Suppose that equations (7) of the curve l through which the required integral surface must pass have the form:

$$x = 0,\ y = t; \quad u = t. \tag{38}$$

On substituting $x_0 = 0$, $y_0 = u_0 = t$ in (37), we get:

$$x = s^3; \quad y = -s + t; \quad u = t;$$

the determinant

$$\Delta = x_s y_t - x_t y_s = 3s^2$$

vanishes for $s = 0$, i.e. along l. The curve (38) is not a characteristic of equation (35), since, by the last of equations(36), u must be constant along a characteristic. There is nevertheless an integral surface of equation (35) through curve (38), viz.

$$u = \sqrt[3]{x} + y.$$

Here, $p = u_x = (1/3)\, x^{-2/3}$, and this partial derivative becomes infinite along curve (38).

2. We take the equation for a function u of three independent variables:

$$p_1 + p_2 + p_3 = u.$$

On forming system (17) and solving it, we obtain the following solution in terms of the initial values x_k^0 and u_0:

$$x_k = s + x_k^0; \quad u = u_0\, e^s \quad (k = 1, 2, 3). \tag{39}$$

Suppose we want to find the integral surface containing the manifold:

$$x_1 = t_1 + t_2; \quad x_2 = t_1 - t_2; \quad x_3 = 1; \quad u = t_1 t_2.$$

On substituting these expressions for the initial values in (39), we obtain:

$$x_1 = s + t_1 + t_2; \quad x_2 = s + t_1 - t_2; \quad x_3 = s + 1; \quad u = t_1 t_2\, e^s. \tag{40}$$

The first three equations are soluble for s, t_1 and t_2 (the case $\Delta \neq 0$):

$$s = x_3 - 1; \quad t_1 = \frac{1}{2}(x_1 + x_2 - 2x_3 + 2); \quad t_2 = \frac{1}{2}(x_1 - x_2);$$

on substituting these expressions in the last of equations (40), we obtain the equation of the required integral surface:

$$u = \frac{1}{4}(x_1 + x_2 - 2x_3 + 2)(x_1 - x_2)\, e^{x_3 - 1}.$$

3. Given the equation:

$$u_x - u_y = f(x + y),$$

let us seek the solution which is continuous together with its first order derivatives and satisfies the condition $u = 0$ for $x = 0$. We can carry out the change of variables:

$$x = x_1; \quad x + y = y_1,$$

and hence easily obtain the following answer:

$$u(x, y) = x f(x + y).$$

This formula in fact gives the solution of the problem provided $f(t)$ has a continuous derivative. If $f(t)$ does not have a continuous derivative, our problem has no solutions at all. It can be shown that continuous functions $f(t)$ exist which nowhere have a derivative. The present examples show the essential importance of the assumption that the derivatives of c in equation (2) exist and are continuous. (Perron, *Math. Zeitschr. Bd.* 27, Heft 4, 1928).

103. Auxiliary theorem. We shall prove in the present section the theorem stated in [100]. An auxiliary proposition must first be proved. Let the right-hand sides of the system of equations:

$$\frac{dy_k}{dx} = f_k(x, y_1, \ldots, y_n, \lambda) \quad (k = 1, 2, \ldots n) \tag{41}$$

contain the parameter λ. Further, let these right-hand sides be continuous functions, having continuous derivatives with respect to all the y_k for

$$|x-a| \leqslant A; \quad |y_k - b_k| \leqslant B \quad (k = 1, 2, \ldots, n) \tag{42}$$

where a and b_k are given numbers, and for λ varying in $\alpha \leqslant \lambda \leqslant \beta$.

Let M be the greatest of the absolute values

$$|f_k(x, y, \ldots, y_n, \lambda)| \quad (k = 1, \ldots, n)$$

for the values of the variables indicated. System (41) now has a unique solution satisfying the initial conditions:

$$y_k|_{x=a} = b_k \quad (k = 1, \ldots, n). \tag{43}$$

This solution exists in the interval $|x - a| \leqslant h$, where h is the lesser of the two numbers: A and B/M, and it can be obtained in this interval by the method of successive approximations [II, 51]. The successive approximations, obtained from the formulae of [II, 51], will be continuous functions of x and λ, and, by virtue of the uniform convergence of the successive approximations with respect to x and λ [II, 51], we can say that the functions yielding a solution of system (41) that satisfies initial conditions (43) will be continuous functions of the arguments x and λ. Of course we could have assumed that the right-hand sides of equations (41) contain several parameters instead of one.

We can thus regard the following lemma as proved:

LEMMA. *If the right-hand sides of equations* (41) *contain several parameters* $\lambda_1, \ldots, \lambda_k$ *and satisfy the conditions indicated above, the solution of the system satisfying initial conditions* (43), *where* a *and* b_k *are given numbers, consists of functions* $y_k = \psi_k(x, \lambda_1, \ldots, \lambda_k)$ *continuous with respect to* x *and* λ_s.

Note. Let x_0 and y_k^0 be values lying inside domain (42). The solutions satisfying the initial data $y_k(x_0) = y_k^0$ will be functions of these initial data:

$$y_k = \psi_k(x, x_0 y_1^0; \ldots, y_n^0), \tag{44}$$

these functions being defined in some neighbourhood of $x = x_0$. If we introduce the new independent variable $\xi = x - x_0$ and the new functions $\eta_k = y_k - y_k^0$, the system can be written as

$$\frac{d\eta_k}{dx} = f_k(\xi + x_0, \eta_1 + y_1^0, \eta_2 + y_2^0, \ldots, \eta_n + y_n^0, \lambda),$$

i.e. the initial values appear as parameters in the right-hand sides, whilst only definite numbers will appear in the initial conditions $\eta_k(0)=0$. By virtue of the above lemma, we can say that functions (44) are continuous in their arguments.

We now turn to the proof of the theorem stated in [100], where, for simplicity we shall first consider the case of a single equation:

$$\frac{dy}{dx} = f(x, y). \tag{45}$$

We suppose that the right-hand side is continuous and has a continuous derivative with respect to y for

$$|x-a| \leqslant A; \quad |y - b| \leqslant B. \tag{46}$$

Let us consider the solution of (45) satisfying the initial condition $y(x_0) = y_0$, where x_0 and y_0 lie inside domain (46). This solution will be a function of x_0 and y_0:

$$y = \varphi(x, x_0, y_0) \tag{47}$$

and will be defined for x sufficiently close to x_0. We vary slightly the initial value of the function and take the new solution:

$$y^+ = \varphi(x, x_0, y_0 + \Delta y_0). \tag{48}$$

If Δy_0 is sufficiently small in absolute value, solutions (47) and (48) exist in a definite neighbourhood of $x = x_0$.

It follows from equation (45) that

$$\frac{\mathrm{d}(y^+ - y)}{\mathrm{d}x} = f(x, y^+) - f(x, y),$$

and this equation can be rewritten as

$$\frac{\mathrm{d}(y^+ - y)}{\mathrm{d}x} = a(x, \Delta y_0)\,(y^+ - y), \tag{49}$$

where

$$a(x, \Delta y_0) = \frac{f(x, y^+) - f(x, y)}{y^+ - y}\,. \tag{50}$$

This ratio will be assumed to be a known function of x and Δy_0, since solutions (47) and (48) are assumed known. It is easily seen that the function $a(x, \Delta y_0)$ is continuous in its arguments. This is obvious for the values of x and Δy_0 for which $y^+ - y \neq 0$. If, as $x \to x'$ and $\Delta y_0 \to \alpha'$, the functions y^+ and y have the common limit y', it follows at once from the condition that a continuous derivative exist that:

$$\frac{f(x, y^+) - f(x, y)}{y^+ - y} = f_y\,[x, y + \theta\,(y^+ - y)] \to f_y\,(x, y'),$$

i.e. in this case also, function (50) is continuous. On dividing both sides of (49) by Δy_0, we get a differential equation for the ratio $(y^+ - y) : \Delta y_0$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^+ - y}{\Delta y_0}\right) = a\,(x, \Delta y_0)\cdot\frac{y^+ - y}{\Delta y_0}\,. \tag{51}$$

With $x = x_0$, we have: $y^+|_{x=x_0} = y_0 + \Delta y_0$ and $y|_{x=x_0} = y_0$, i.e.

$$\left.\frac{y^+ - y}{\Delta y_0}\right|_{x=x_0} = 1. \tag{52}$$

Thus the ratio $(y^+ - y) : \Delta y_0$ is a solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = a(x, \Delta y_0)\,u, \tag{53}$$

satisfying the initial condition:

$$u|_{x=x_0} = 1. \tag{54}$$

Since the right-hand side of (53) is a continuous function of the parameter Δy_0 for all Δy_0 sufficiently close to zero, the solution u, satisfying condition (54), is also a continuous function of Δy_0, and in particular, the ratio has a limit as $\Delta y_0 \to 0$, i.e. function (47) has a partial derivative $\varphi_{y_0}(x, x_0, y_0)$ with respect to y_0. This partial derivative must be a solution of equation (53) with $\Delta y_0 = 0$. But, by (50), $a(x, 0) = f_y[x, \varphi(x, x_0, y_0)]$, and we can thus say that the partial derivative $\varphi_{y_0}(x, x_0, y_0)$ is a solution of the equation:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f_y\,[x, \varphi(x, x_0, y_0)]\,u, \tag{55}$$

satisfying condition (54). Since the right-hand side of equation (55) is a continu. ous function of parameters x_0 and y_0, we can again use the lemma to assert that the partial derivative $\varphi_{y_0}(x, x_0, y_0)$ is also a continuous function of its arguments, and the theorem is thus proved.

Note 1. By giving x_0 the increment Δx_0 and repeating the above arguments, we could prove that the function (47) has a continuous partial derivative $\varphi_{x_0}(x, x_0, y_0)$. This partial derivative must also satisfy equation (55), but not now the initial condition (54); it satisfies instead the condition

$$u|_{x=x_0} = -f(x_0, y_0).$$

This condition is obtained at once if equation (45) is written with the initial condition $y(x_0) = y_0$ in the form of the integral equation [II, 51]:

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,\mathrm{d}x.$$

On differentiating both sides with respect to x_0, we in fact get the above initial condition for $u = \varphi_{x_0}(x, x_0, y_0)$.

Note 2. The above proof also retains its force for the system of equations (5). We shall have as the solution of this system:

$$y_k = \varphi_k\,(x, x_0, y_1^0, \ldots, y_n^0) \quad (k = 1, 2, \ldots, n). \tag{56}$$

On assigning y_i^0 the increment Δy_i^0, we obtain another solution:

$$y_k^+ = \varphi_k\,(x, x_0, y_1^0, \ldots, y_{i-1}^0, y_i^0 + \Delta y_i^0, y_{i+1}^0, \ldots, y_n^0).$$

If we write system (5) for y_k and y_k^+ and subtract the resulting equations term by term, a difference is obtained on the right-hand side which can be rewritten as:

$$\begin{aligned} f_k\,(x, y_1^+, \ldots, y_n^+) - f_k\,(x, y_1, \ldots, y_n) = \\ = [f_k\,(x, y_1^+, y_2^+, y_3^+, \ldots, y_n^+) - f_k\,(x, y_1, y_2^+, y_3^+, \ldots, y_n^+)] + \\ + [f_k\,(x, y_1, y_2^+ y_3^+, \ldots, y_n^+) - f_k\,(x, y_1, y_2, y_3^+, \ldots, y_n^+)] + \\ + \ldots + [f_k\,(x, y_1, y_2, \ldots, y_{n-1}, y_n^+) - f_k(x, y_1, y_2, \ldots, y_{n-1}, y_n)], \end{aligned}$$

We obtain a system of linear equations for the ratios $u_k = (y_k^+ - y_k) : \Delta y_i^0$:

$$\frac{\mathrm{d}u_k}{\mathrm{d}x} = \sum_{j=1}^{n} a_{kj}\,(x, \Delta y_i^0)\,u_j,$$

where

$$a_{kj} = \frac{f_k(x, y_1, \ldots, y_{j-1}, y_j^+, \ldots, y_n^+) - f_k(x, y_1, \ldots, y_{j-1}, y_j, y_{j+1}^+, \ldots, y_n^+)}{y_j^+ - y_j},$$

and the initial conditions are:

$$u_k|_{x=x_0} = 0\ (k \neq i); \quad u_i|_{x=x_0} = 1. \tag{57}$$

The rest of the proof follows the same lines, and functions (56) are shown to have continuous partial derivatives with respect to y_i^0. Instead of equation (55), a system of equations is obtained for these partial derivatives:

$$\frac{du_k}{dx} = \sum_{j=1}^{n} \frac{\partial f_k(x, y_1, \ldots, y_n)}{\partial y_j} u_j, \tag{58}$$

where we have to substitute functions (56) for the y_k in the coefficients of this linear system. As before, the initial conditions will be given by formulae (57). We remark that system (58) can be obtained directly by substituting functions (56) in equation (5) and differentiating both sides with respect to y_i^0. But we cannot assert the existence of the partial derivative with respect to $y_k^{(0)}$ without a preliminary proof, and we cannot strictly speaking change the order of differentiations with respect to x and $y_i^{(0)}$ in the left-hand side. We remark further that, in the case of a single equation, the linear homogeneous equation (55) can be solved in the closed form.

Note 3. If the right-hand sides f_k of equations (5) have, under condition (6), continuous partial derivatives with respect to y_s up to some order m, the functions $\varphi_k(x, x_0, y_1^{(0)}, \ldots, y_n^{(0)})$ also have continuous partial derivatives with respect to $y_s^{(0)}$ up to order m. If the f_k have continuous partial derivatives with respect to x, it follows from equation (5) itself that φ_k will have continuous derivatives up to the second order with respect to x.

104. Non-linear first order equations. We pass to a discussion of first order partial differential equations in the general case. As in the case of linear equations, we shall start by assuming that there are only two independent variables. A first order partial differential equation for a function of two independent variables has the form:

$$F(x, y, u, p, q) = 0. \tag{59}$$

Let us first explain the geometrical significance of this equation. At any fixed point (x, y, u), equation (59) represents a relationship between p and q, i.e. a relationship between the direction cosines of the normal to a surface. The normals satisfying this relationship form a conical surface with vertex (x, y, u). The planes passing through (x, y, u) and perpendicular to the generators of the cone represent all the possible positions of the tangent plane at the fixed point (x, y, u) to the required integral surface. This family of planes,

like the family of generators of the cone of normals, will depend on one parameter. The envelope of the family of planes will be a new cone, which we shall call the cone T. Equation (59) is thus equivalent to specifying a cone T at every point of space, whilst the required integral surface of (59) must have the property that the tangent plane at every point of it touches the cone T corresponding to this point.

Let us find the equations of the generators of the cone T at a given point (x, y, u). Let p and q be functions of a parameter a, which satisfy equation (59) at a fixed point (x, y, u). The cone T is the envelope of the family of planes:

$$p(a)(X - x) + q(a)(Y - y) - (U - u) = 0. \tag{60}$$

On differentiating with respect to the parameter a, we get the additional equation:

$$\frac{dp}{da}(X - x) + \frac{dq}{da}(Y - y) = 0. \tag{61}$$

On differentiating (59) with respect to a, we get:

$$P\frac{dp}{da} + Q\frac{dq}{da} = 0, \tag{62}$$

where

$$P = F_p; \quad Q = F_q. \tag{63}$$

We shall assume that F_p and F_q do not vanish simultaneously for the values in question of the variables, i.e. $F_p^2 + F_q^2 > 0$. The only exception will be the case of singular solutions of (59). On the assumption that dp/da and dq/da cannot vanish simultaneously, the homogeneous equations (61) and (62) give:

$$\frac{X - x}{P} = \frac{Y - y}{Q},$$

and finally, equation (60) gives us the equations of the generators of the cone:

$$\frac{X - x}{P} = \frac{Y - y}{Q} = \frac{U - u}{pP + qQ}. \tag{64}$$

To obtain different generators of a cone T, we have to substitute in the denominators different values of p and q satisfying (59) at the fixed point (x, y, u).

In the case of linear equation (2) we had one definite direction at every point, and the tangent plane to the required integral surface had to contain this direction. In the present case we have a cone T

at every point, instead of a single direction, and the tangent plane to the required integral surface must touch this cone. Hence we cannot construct the characteristic curves for non-linear equation (59) directly as we did for linear equation (2), where we had a definite direction field. Here we have a field of cones T instead of a direction field. But we shall soon show that, having obtained an integral surface $S : u = u(x, y)$ of equation (59), we can cover it with curves which are completely analogous to the characteristic curves of linear equation (2). In fact, the tangent plane at every point of the integral surface must touch the cone T corresponding to this point, and hence must contain a generator of the cone, this generator being the curve of contact. These generators of cones T at different points of the surface produce a direction field on the integral surface, and hence, by solving the first order differential equation corresponding to this direction field, the surface can be covered by a family of curves l' depending on a single parameter. The direction cosines of the direction field must be proportional to the denominators of equations (64), where p and q are given directly by the equations of the integral surface concerned. Hence, the following relationships must be satisfied along curves l' covering a given integral surface:

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}u}{pP + qQ}, \tag{65}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}s} = P; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = Q; \quad \frac{\mathrm{d}u}{\mathrm{d}s} = pP + qQ. \tag{66}$$

To find these curves on a given integral surface, we only need to integrate the first order equation

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}, \tag{67}$$

where the denominators contain only the variables x and y, since the function u and its partial derivatives p and q are known functions of x and y on the given surface. On integrating (67) and using the equation of the surface $u = u(x, y)$, we in fact obtain the above-mentioned curves l'. The right-hand sides of equations (66) have a definite meaning only with a definite choice of integral surface $u = u(x, y)$. Knowledge of the integral surface gives us p and q as functions of (x, y). We next complete the system of equations (66) by two further equations, containing the differentials $\mathrm{d}p$ and $\mathrm{d}q$, in order to obtain a system of differential equations which does not

depend on the choice of the integral surface of equation (59). We write r, σ and t for the second derivatives of the function u:

$$r = u_{xx}; \quad \sigma = u_{xy}; \quad t = u_{yy},$$

and X, Y and U for the derivatives of the left-hand side of (59) with respect to x, y and u:

$$X = F_x; \quad Y = F_y; \quad U = F_u.$$

Total differentiation of the left-hand side of (55) with respect to x and y gives us:

$$X + Up + Pr + Q\sigma = 0; \quad Y + Uq + P\sigma + Qt = 0.$$

On the other hand, we obviously have:

$$\frac{\mathrm{d}p}{\mathrm{d}s} = r\frac{\mathrm{d}x}{\mathrm{d}s} + \sigma\frac{\mathrm{d}y}{\mathrm{d}s} = Pr + Q\sigma$$

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \sigma\frac{\mathrm{d}x}{\mathrm{d}s} + t\frac{\mathrm{d}y}{\mathrm{d}s} = P\sigma + Qt.$$

It follows at once from these equations that:

$$\frac{\mathrm{d}p}{\mathrm{d}s} = -(X + Up); \qquad \frac{\mathrm{d}q}{\mathrm{d}s} = -(Y + Uq),$$

so that we can add the last two equations to equations (66) and thus obtain the following system of five differential equations with five functions of the auxiliary parameter s:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}s} = P; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = Q; \quad \frac{\mathrm{d}u}{\mathrm{d}s} = pP + qQ; \\ &\frac{\mathrm{d}p}{\mathrm{d}s} = -(X + Up); \qquad \frac{\mathrm{d}q}{\mathrm{d}s} = -(Y + Uq). \end{aligned} \tag{68}$$

We can therefore say that equations (68) must be satisfied along every curve l' as above constructed on any integral surface. The system (68) can be regarded independently of the integral surfaces of equation (59) as a system in its own right. It is called the *characteristic system of equation* (59).

We remark that, when deducing equations (68) we used the second order derivatives of the function u. In addition, an essential point when integrating (68) is that the right-hand sides must have continuous first order derivatives. On taking all this into account, the result obtained can be stated as follows. Let $u(x, y)$ be a solution of (59) having continuous derivatives up to the second order in the

neighbourhood of a point (x_0, y_0). We use the notation $u_0 = u(x_0, y_0)$, $p_0 = u_x(x_0, y_0)$; $q_0 = u_y(x_0, y_0)$. We assume that $F(x, y, u, p, q)$ is single-valued and continuous, and has continuous derivatives up to the second order in a neighbourhood of $(x_0, y_0, u_0, p_0, q_0)$. System (68) now has one definite solution

$$x_0(s),\ y_0(s),\ u_0(s),\ p_0(s),\ q_0(s)$$

under the initial conditions $(x_0, y_0, u_0, p_0, q_0)$ at $s = 0$. It follows from the above discussion that the integral surface $u = u(x, y)$ contains the above-mentioned solution of system (68) for all s sufficiently close to zero, i.e.

$$u_0(s) = u[x_0(s),\ y_0(s)]; \qquad p_0(s) = u_x[x_0(s),\ y_0(s)];$$
$$q_0(s) = u_y[x_0(s),\ y_0(s)].$$

System (68) can be considered on its own, as remarked above, independently of equation (59), as a first order system for the functions (x, y, u, p, q). It is easily shown that it has the solution

$$F(x, y, u, p, q) = C. \tag{69}$$

For, on differentiating the left-hand side of this last equation with respect to s and using equations (68), we obtain:

$$\frac{\mathrm{d}F}{\mathrm{d}s} = XP + YQ + U(pP + qQ) - P(X + Up) - Q(Y + Uq) \equiv 0.$$

105. Characteristic manifolds. Every solution of system (68) consists of five functions of the auxiliary parameter s:

$$x(s),\ y(s),\ u(s),\ p(s),\ q(s). \tag{70}$$

We have only noticed the solutions of the system which give the constant C a value equal to zero when substituted in equation (69). Such solutions of system (68) will be called characteristics strips of equation (59), i.e. *a characteristic strip of equation* (59) *is a system of functions* (70) *satisfying system* (68) *and the relationship:*

$$F(x, y, u, p, q) = 0. \tag{71}$$

The first three of functions (70) define a spatial curve, whilst the last two define a tangent plane along this curve. Every spatial curve forming part of a characteristic strip is usually called a *characteristic curve* of equation (59). We showed in the previous section that every integral surface can be covered by characteristic strips, and con-

sequently by the characteristic curves corresponding to these strips. If we take a point (x_0, y_0, u_0) on an integral surface and the values $p = p_0$, $q = q_0$ corresponding to this point, by the existence and uniqueness theorem, system (68) defines a unique characteristic strip satisfying the initial values $(x_0, y_0, u_0, p_0, q_0)$, and the strip must belong entirely to the integral surface in question, i.e. *if a characteristic strip has an element in common with the integral surface, it lies wholly on this integral surface.* A direct consequence of this assertion is that, if two integral surfaces touch at a point, i.e. have the same p and q at this point, the characteristic strip corresponding to these initial values must belong to both integral surfaces. In other words, *if two integral surfaces touch at a point, they must touch along the whole of the characteristic strip having the point of contact of the surfaces as initial element.* We are naturally assuming in all this that the integral surfaces and function F satisfy the conditions indicated in the previous section; and we are referring throughout to a neighbourhood of some point (x_0, y_0).

We remark further that a solution of system (68) can be shown to satisfy relationship (71), i.e. to be a characteristic strip, simply by showing that, in view of (69), this relationship is satisfied by the initial data $(x_0, y_0, u_0, p_0, q_0)$ of the solution in question, i.e.

$$F(x_0,\ y_0,\ u_0,\ p_0,\ q_0) = 0. \tag{72}$$

106. Cauchy's method. The connection has been indicated between system (68) and equation (59). In particular, it has been explained that every integral surface consists of a family of characteristic strips depending on one parameter. Suppose that we have been able to solve system (68) and thus find all the possible characteristic strips. We shall show how the integral surface of equation (59) can be obtained from these characteristic strips. We shall assume that the solution of system (68) is given in terms of the parameter s and the initial values of the functions of (68):

$$\begin{aligned} x &= x(s, x_0, y_0, u_0, p_0, q_0) \\ y &= y(s, x_0, y_0, u_0, p_0, q_0) \\ u &= u(\ldots\ldots\ldots\ldots\ldots) \\ p &= p(\ldots\ldots\ldots\ldots\ldots) \\ q &= q(\ldots\ldots\ldots\ldots\ldots) \end{aligned} \tag{73}$$

Since we want to find the family of characteristic strips, we assume that the initial data $(x_0, y_0, u_0, p_0, q_0)$ are assigned as functions of a parameter t:

$$x_0(t),\ y_0(t),\ u_0(t),\ p_0(t),\ q_0(t), \tag{74}$$

where these functions must satisfy relationship (72). We assume in addition that they have continuous derivatives for $t_0 < t < t_1$, and that the right-hand sides of equations (68) have continuous derivatives with respect to (x, y, u, p, q) in some domain containing manifold (74) inside it. As we saw in the previous section, (71) will be satisfied for any values of s.

On substituting functions (74) in the right-hand sides of (73), we have:

$$x = x(s, t); \quad y = y(s, t); \quad u = u(s, t), \tag{75}$$

$$p = p(s, t); \quad q = q(s, t). \tag{76}$$

Equations (75) define a surface parametrically. If the determinant

$$\Delta = x_s y_t - x_t y_s \tag{77}$$

is non-zero, as we shall assume in future, we can find the explicit equation $u = u(x, y)$ of this surface precisely as in the case of a linear equation. As we saw above, (71) will be satisfied, but the question remains open as to whether the functions p and q defined by (75) are partial derivatives of $u(x, y)$ with respect to x and y. If this is the case, differentiation of $u(x, y)$ with respect to s and t gives us:

$$\frac{\partial u}{\partial s} - p\frac{\partial x}{\partial s} - q\frac{\partial y}{\partial s} = 0, \quad \frac{\partial u}{\partial t} - p\frac{\partial x}{\partial t} - q\frac{\partial y}{\partial t} = 0. \tag{78}$$

Since the second order determinant formed from the coefficients of p and q is non-zero by hypothesis, we can say that, conversely, if p and q defined by (76) satisfy relationships (78), they must be the partial derivatives of $u(x, y)$ with respect to x and y. The first of relationships (78) follows directly from the first three equations of system (68). It only remains to explain the conditions under which the last of relationships (78) will be fulfilled. We assume that $F(x, y, u, p, q)$ has continuous derivatives up to the second order in the neighbourhood of $(x_0, y_0, u_0, p_0, q_0)$. The right-hand sides of equations (68) now have continuous first order derivatives, and it follows from these equations that x_s, y_s, u_s have continuous derivatives with respect to t, i.e. the continuous derivatives x_{st}, y_{st}, u_{st} exist. The continuous derivatives x_{ts}, y_{ys}, u_{ts} now also exist, and are equal to the derivatives just men-

tioned. This follows from the fact that, if the function $f(x, y)$ has a continuous derivative f_{xy} in some domain, the derivative f_{yx} also exists, and $f_{yx} = f_{xy}$. This theorem can be proved by a slight modification of the arguments of [I, 155] (see, e.g., G. M. Fikhtengol'ts, *Course of Differential and Integral Calculus, I* (Kurs differential'nogo i integral'nogo ischisleniya, t. I).) On writing L for the left-hand side of the second of equations (78) and differentiating with respect to s, we get:

$$\frac{\partial L}{\partial s} = \frac{\partial^2 u}{\partial t\,\partial s} - p\frac{\partial^2 x}{\partial t\,\partial s} - q\frac{\partial^2 y}{\partial t\,\partial s} - \frac{\partial p}{\partial s}\frac{\partial x}{\partial t} - \frac{\partial q}{\partial s}\frac{\partial y}{\partial t}.$$

On the other hand, on differentiating with respect to t the first of relationships (78), which is certainly fulfilled, as we have just seen, we have:

$$0 = \frac{\partial^2 u}{\partial s\,\partial t} - p\frac{\partial^2 x}{\partial s\,\partial t} - q\frac{\partial^2 y}{\partial s\,\partial t} - \frac{\partial p}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial q}{\partial t}\frac{\partial y}{\partial s}.$$

On subtracting the last equation term by term from the former, we can write $\partial L/\partial s$ as:

$$\frac{\partial L}{\partial s} = \frac{\partial p}{\partial t}\frac{\partial x}{\partial s} + \frac{\partial q}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial x}{\partial t} - \frac{\partial q}{\partial s}\frac{\partial y}{\partial t},$$

or, on using system (68):

$$\frac{\partial L}{\partial s} = P\frac{\partial p}{\partial t} - Q\frac{\partial q}{\partial t} + (X + Up)\frac{\partial x}{\partial t} + (Y + Uq)\frac{\partial y}{\partial t}.$$

On differentiating relationship (71), which is satisfied by functions (75) and (76), with respect to t we obtain:

$$0 = X\frac{\partial x}{\partial t} + Y\frac{\partial y}{\partial t} + U\frac{\partial u}{\partial t} + P\frac{\partial p}{\partial t} + Q\frac{\partial q}{\partial t}.$$

On subtracting this equation from the previous one, we can transform the expression for $\partial L/\partial s$ to:

$$\frac{\partial L}{\partial s} = U\left(p\frac{\partial x}{\partial t} + q\frac{\partial y}{\partial t} - \frac{\partial u}{\partial t}\right) \quad \text{or} \quad \frac{\partial L}{\partial s} = -UL,$$

whence it follows that:

$$L = L_0 e^{-\int_0^s U\,ds},$$

where L_0 is the value of the left-hand side of the second of equations (78) for $s = 0$:

$$L_0 = \frac{\partial u_0}{\partial t} - p_0\frac{\partial x_0}{\partial t} - q_0\frac{\partial y_0}{\partial t}.$$

It follows directly from the last expression that the equation $L = 0$ will be satisfied for any s if it is satisfied for $s = 0$, i.e. the necessary and sufficient condition for the second of equations (78) to be satisfied is that functions (74) satisfy

$$\frac{du_0}{dt} = p_0 \frac{dx_0}{dt} + q_0 \frac{dy_0}{dt} . \tag{79}$$

We can therefore assert that, if the determinant Δ is non-zero for $s = 0$ and $t = t'$ ($t_0 < t' < t_1$), and if functions (74) satisfy

$$F(x_0, y_0, u_0, p_0, q_0) = 0, \quad \frac{du_0}{dt} = p_0 \frac{dx_0}{dt} + q_0 \frac{dy_0}{dt}, \tag{80}$$

then equations (75) define an integral surface $u = u(x, y)$ of equation (59) for s and t close to $s = 0$ and $t = t'$. The first two of equations (75) yield continuously differentiable functions $s(x, y)$ and $t(x, y)$. On substituting these in $u(s, t)$, $p(s,t)$, $q(s,t)$, we obtain continuously differentiable functions of x and y, where $p = u_x$, $q = u_y$, whence it is clear that $u(x, y)$ has continuous second order derivatives. For the solution obtained, we have with $s = 0$: $u(t) = u[x(t), y(t)]$ for t close to t', i.e. the surface $u = u(x, y)$ contains a part of the curve $x = x(t)$, $y = y(t)$, $u = u(t)$ corresponding to a neighbourhood of $t = t'$. A propos of this, the solution of Cauchy's problem obtained in the next section has a local character as in [100].

107. Cauchy's problem. The Cauchy problem for equation (59) is formulated as in the case of a linear equation: *we require to find the surface passing through a given curve l.* We first take the particular case of the problem when the given curve lies in the $x = x_0$ plane parallel to the (y, u) plane, and has the explicit equation $u = \varphi(y)$ in this plane, i.e. we suppose that the integral surface to be found satisfies the following condition:

$$u|_{x=x_0} = \psi(y). \tag{81}$$

When considering a Cauchy problem, apart from proving the existence and uniqueness of the solution, we shall always prove the continuity (in some sense) of the dependence of the solution on the initial data. Let u_1 be a solution of the problem for which $\psi(y)$ in condition (81) is replaced by $\psi(y) + \delta(y)$. The above-mentioned continuous dependence here amounts to the following: in some finite domain of variation of (x, y), $| u - u_1 |$ can be made as small as desired if $| \delta(y) |$ is sufficiently small. This continuous dependence on the initial data is usually de-

scribed as implying *a correct statement of Cauchy's problem*. We shall assume that equation (59) is written explicitly with respect to p:

$$p = f(x, y, u, q). \tag{82}$$

It follows at once from Cauchy's condition (81) that we can take the variable y as a parameter, where the parametric equation of the curve l will have the form: $x = x_0$; $y = y$; $u = \psi(y)$. It still remains for us to define p and q along this curve as functions of the parameter y so as to satisfy the two conditions (80). These conditions can be rewritten in the present case as

$$p_0 = f(x_0, y, \psi(y), q_0); \quad \psi'(y) = q_0, \tag{83}$$

whence it is clear that p_0 and q_0 are defined uniquely along l, and the solution of the problem can be obtained by applying the method indicated in the previous section.

In order to fulfil the conditions indicated in the previous section, we have to make use of the existence of the continuous second order derivative of function $\psi(y)$. The conditions for f follow from the conditions for F indicated in [106].

We now take more general initial data, and try to find an integral surface passing through a curve with the equation

$$x = \varphi(y); \quad u = \psi(y). \tag{84}$$

This problem can be reduced to the previous one with the aid of a replacement of the independent variables: we replace (x, y) by new variables (x', y') in accordance with the formulae:

$$x = x' + \varphi(y'); \quad y = y',$$

and express the derivatives with respect to the new variables in terms of the previous derivatives:

$$u_{x'} = p; \quad u_{y'} = p\varphi'(y') + q,$$

whence

$$p = u_{x'}; \quad q = u_{y'} - u_{x'}\varphi'(y'),$$

and equation (59) in the new independent variables becomes

$$F[x' + \varphi(y'),\ y',\ u,\ u_{x'},\ u_{y'} - u_{x'}\varphi'(y')] = 0. \tag{85}$$

The curve (84) in the new variables may be written as

$$x' = 0; \quad u = \psi(y'),$$

i.e. we have Cauchy's problem of the type discussed above. The possibility of solving it depends on whether equations (85) can be solved with respect to the $u_{x'}$.

When the curve l is given in the parametric form:

$$x = x_0(t); \quad y = y_0(t); \quad u = u_0(t),$$

we must define functions $p_0(t)$ and $q_0(t)$ from the two equations:

$$\left.\begin{aligned} F[x_0(t),\ y_0(t),\ u_0(t),\ p_0(t),\ q_0(t)] = 0 \\ u_0'(t) - p_0(t)\,x_0'(t) - q_0(t)\,y_0'(t) = 0. \end{aligned}\right\} \tag{86}$$

The functional determinant of the left-hand sides with respect to p_0 and q_0:

$$\Delta_0 = y_0'(t)\,F_{p_0} - x_0'(t)\,F_{q_0} \tag{87}$$

is exactly the same as determinant (77) at $s = 0$, as follows directly from the first two equations of system (68). We assume that determinant (77) is non-zero along l and that system (86) gives completely determined values for p_0 and q_0 along l. The method of the previous section can now be used for constructing the solution; it must be noted here that determinant (87) will differ from zero not only for $s = 0$ but also for s close to zero. For the functions $p_0(t)$, $q_0(t)$ to have continuous first order derivatives, we have to require the existence of continuous second order derivatives of $x_0(t)$, $y_0(t)$, $u_0(t)$. This is clear from the second of equations (86).

108. Uniqueness of the solution. We constructed an integral surface with the aid of characteristic strips when solving Cauchy's problem. At first sight, the uniqueness of the solution follows directly from the fact that every integral surface can be covered by characteristic strips. But we used in the proof of this the existence of continuous second order derivatives of $u(x,y)$. Under the assumptions made above, we obtained in [107] a solution in which $u(x, y)$ has continuous second order derivatives. But this simple proof of uniqueness is inadequate if we assume only continuous first order derivatives for $u(x, y)$.

The uniqueness theorem can be easily proved even on the assumption that only first order derivatives exist. We shall do this for equation (82) with Cauchy's condition (81).

The proof is based on the following lemma:

LEMMA. *Let the function $u(x, y)$ be continuous in the closed triangle Δ, formed by the straight lines*

$$x = x_0;\; x - x_0 = \frac{1}{A}(y - y_1);\; x - x_0 = -\frac{1}{A}(y - y_2) \qquad (88)$$

$$(y_1 < y_2),$$

and be defined and have continuous first order derivatives for $x > x_0$ in the larger triangle formed by the straight lines

$$x = x_0;\;\; x - x_0 = \frac{1}{A}(y - y_3);\; x - x_0 = -\frac{1}{A}(y - y_4) \qquad (89)$$

$$(y_3 < y_1 < y_2 < y_4).$$

Further, let these derivatives satisfy

$$|u_x| \leqslant A\,|u_y| + B\,|u|, \qquad (90)$$

throughout the triangle Δ except on the base $x = x_0$, whilst on the base $x = x_0$ we have:

$$|u(x_0,\, y)| \leqslant M. \qquad (91)$$

In this case we have, throughout the triangle Δ,

$$|u(x, y)| \leqslant M\mathrm{e}^{B(x-x_0)}. \qquad (92)$$

We shall first prove the lemma for $A = B = 1$. We shall prove a contradiction. Let there be points in Δ at which $|\, u(x, y)\,| > M\mathrm{e}^{x-x_0}$. The function $u(x, y)\mathrm{e}^{x_0-x}$ must now attain its maximum absolute value off the base.

Since all the conditions contain only the absolute values of $u(x, y)$ and its derivatives, we can assume, by changing the sign of $u(x, y)$ if necessary, that the product $u(x, y)\mathrm{e}^{x_0-x}$ attains its greatest positive value off the base. We can now fix as small as desired a number λ such that the function

$$v(x, y) = u(x, y)\, \mathrm{e}^{-(1+\lambda)(x-x_0)} \qquad (93)$$

attains its greatest positive value off the base. This leads to a contradiction; for, if the statement is true inside Δ, we must have $v_x = v_y = 0$ at the corresponding point, whence it follows that $u_x - (1 + \lambda)u = 0$, $u_y = 0$ $(u > 0)$, which contradicts (90) with $A = B = 1$. If the statement is true on the side $x - x_0 = y - y_1$ (not at a vertex), we must have $v_y \leqslant 0$ at the corresponding point, and differentiation along this side gives $v_x + v_y = 0$. This leads to the expressions:

$$u_y \leqslant 0;\quad u_x = -\,u_y + (1 + \lambda)\,u \quad (u > 0), \qquad (94)$$

which again contradict (90) with $A = B = 1$. If the statement is true on the side $x - x_0 = -(y - y_2)$, we obtain similarly $v_y \geqslant 0, v_x - v_y = 0$, whence

$$u_y \geqslant 0; \quad u_x = u_y + (1 + \lambda)\, u \quad (u > 0), \tag{95}$$

which again contradicts (90) with $A = B = 1$. Suppose, finally, that function (93) attains its greatest positive value at the vertex of triangle Δ. We must always have at this point: $v_x \geqslant 0$, i.e. $v_x \geqslant (1 + \lambda)u$. If $u_y = 0$ here, we again arrive at a contradiction of (90). If $u_y < 0$ at the vertex, differentiation along the side $x - x_0 = y - y_1$ gives us $v_x + v_y \geqslant 0$, which leads to

$$u_y < 0, \quad u_x \geqslant - u_y + (1 + \lambda)\, u,$$

which contradicts (90) with $A = B = 1$. If $u_y > 0$ at the vertex, differentiation along the side $x - x_0 = -(y - y_2)$ leads as above to a contradiction. Thus the lemma is proved for $A = B = 1$. Let us extend it to the general case. We have conditions (90) and (91) in the triangle with sides (80). We introduce new independent variables $x' = Bx$, $y' = By/A$. The triangle Δ becomes a triangle Δ' with sides.

$$x' = Bx_0; \quad x' - Bx_0 = y' - y_1'; \quad x' - Bx_0 = -\,(y' - y_2')$$
$$\left(y_k' = \frac{B}{A}\, y_k\right),$$

and we have instead of (90):

$$|u_{x'}| \leqslant |u_{y'}| + |u|,$$

whilst condition (91) takes the form, as before: $|\, u(Bx_0, y')\, | \leqslant M$. By what has been proved above, we have $|\, u(x', y')\, | \leqslant Me^{(x' - Bx_0)}$ in Δ' or, on returning to the old variables, we get inequality (92) in Δ.

We pass to the proof of the uniqueness of the solution of (82) under condition (81) and the assumption made above. Let there be two solutions $u_1(x, y)$, $u_2(x, y)$ in the above-mentioned triangle Δ, both these solutions being situated in the domain for which f has continuous derivatives. We can now write the inequality:

$$|f(x, y, u_2, q_2) - f(x, y, u_1, q_1)| \leqslant B\,|u_2 - u_1\,| + A\,|q_2 - q_1|,$$

where A and B are constants. We write p_1, p_2, q_1, q_2 for the corresponding partial derivatives of u_1 and u_2. Thus:

$$|p_2 - p_1| \leqslant A|q_2 - q_1| + B\,|u_2 - u_1|.$$

On applying our lemma to the difference $u_2 - u_1$ and noting that this difference vanishes for $x = x_0$ (i.e. $M = 0$), we see from (92) that $u_2(x, y) - u_1(x, y) = 0$ throughout Δ, i.e. the uniqueness of the solution is proved. The general case of equation (59) with Cauchy data on any curve can be reduced to the case analysed with the aid of a change of variables and the solution of a differential equation with respect to one of the derivatives. We have already dealt with this in [107].

We now consider in triangle Δ two solutions $u_1(x, y)$ and $u_2(x, y)$ of equation (82) under different conditions:

$$u_1|_{x=x_0} = \psi_1(y); \qquad u_2|_{x=x_0} = \psi_2(y).$$

On applying the lemma to $(u_2 - u_1)$, we obtain the inequality in Δ :

$$|u_2(x, y) - u_1(x, y)| \leqslant \max_{y_1 \leqslant y \leqslant y_2} |\psi_2(y) - \psi_1(y)| \, e^{B(x-x_0)}.$$

This inequality proves the continuous dependence of the solution on the initial data $\psi(y)$ appearing in (81).

A further point may be mentioned in connection with the solution of Cauchy's problem. If the function $\psi(y)$ has no continuous second order derivative, application of Cauchy's method can lead to a surface $u = u(x, y)$ for which $u(x, y)$ has no derivative. It can be shown that the problem has in this case no solution with continuous derivatives (Haar, *Acta Szeged*, t. IV, fasc. II, 1928). The proof of the above lemma in a more general case and its application to the proof of the uniqueness theorem for partial differential equations can be found in A. Myshkis's article, The uniqueness of the solution of Cauchy's problem (*Uspekhi matematicheskikh nauk*, t. III, vyp. 2).

109. The singular case. Suppose we are given a strip satisfying the two equations (80) and such that determinant (87) vanishes along the strip:

$$\Delta_0 = x_s y_t - x_t y_s|_{s=0} = y_0'(t)\, F_{p_0} - x_0'(t)\, F_{q_0} = 0. \tag{96}$$

Suppose there is an integral surface $u = u(x, y)$ passing through the strip, where $u(x, y)$ has continuous derivatives up to the second order. If F_{p_0} and F_{q_0} do not vanish simultaneously, it follows from (96) and the second of equations (80) that our strip satisfies equations (66), the parameter s being denoted by the letter t. But the working of [104] now shows us that the strip satisfies all the equations (68), i.e. is a characteristic strip. Hence, *if, when conditions* (96) *are satisfied, there*

exists an integral surface containing a given strip, this must be a characteristic strip, on the assumption that F_{p_0} and F_{q_0} do not vanish simultaneously. Now, precisely as in the case of a linear equation, an infinite set of integral surfaces can pass through our strip. The proof of this statement is similar to that for a linear equation. We have to draw a strip ω which has a common point (x_0, y_0, u_0) with strip (74), and common p_0 and q_0 at this point, and such that determinant (87) differs from zero along the new strip ω. When certain conditions are satisfied, a definite integral surface, which contains all the characteristic strip (74) (since it contains its initial element), will pass through this strip. Since the strip ω has been chosen arbitrarily, we in fact obtain an infinite set of solutions of the problem.

If $\Delta_0 = 0$ along a given strip, which is not, however, a characteristic strip, no solution of the problem is possible if we are confined to functions $u(x, y)$ with continuous derivatives up to the second order. But it may happen that the corresponding curve is singular for the integral surface. We remark here that use was made of second order derivatives of $u(x, y)$ when we carried out Cauchy's method. If equation (96) is satisfied and the strip is not characteristic, only the first three equations of system (68) are satisfied along the strip.

The above arguments have a simple geometrical meaning. If we are given a curve l, the first of conditions (80) shows that, along this curve, the plane defined by $p_0(t)$ and $q_0(t)$ must touch the cone T, whilst the second condition is equivalent to the fact that this plane must contain the tangent to l. On recalling equations (64) for the generators of the cone, we see that the condition $\Delta_0 \neq 0$ is equivalent to the fact that, along l, the tangents to l do not coincide with the generators of the cone T. The possibility of solving equations (86) with respect to $p_0(t)$ and $q_0(t)$ amounts to the possibility of drawing planes which contain the tangent to l and touch the cone T. Suppose that we can draw planes through the tangents to l such that the planes touch T and vary continuously along l ($p_0(t)$ and $q_0(t)$ must have continuous derivatives).

We make up the curve l in this way into a strip, and obtain an integral surface by taking this strip as the initial values in solutions (73). If the tangents to l are generators of the cones T, we obtain the values of p_0 and q_0 along T by drawing the tangent plane to the cone along the corresponding generator. The strip thus obtained may be a characteristic strip. In this case the problem has an infinite set of solutions. It is sufficient to cut the curve l by another curve l_1, the tangent to which at the point of intersection lies in the same plane as the tangent

to l, but does not coincide with this tangent, and such that the tangents along l_1 do not coincide with the generators of the cones T. The integral surface drawn through l_1 will also contain l. Finally, it may happen that the tangents to the curve l coincide with the generators of the cones T, but that the curve is not characteristic, i.e. that making it up into a strip by the method indicated above does not lead to a characteristic strip. In this case we can nevertheless produce from every point of l a characteristic strip having the initial values $(x_0, y_0, u_0, p_0, q_0)$. If these characteristic strips form an integral surface with the explicit equation $u = u(x, y)$, the curve l is a singular curve on the integral surface. We are assuming in this discussion that there is a definite cone T at every point.

An important type of integral surface of equation (59) must be mentioned. We fix some point (x_0, y_0, u_0). The second of equations (80) will now be fulfilled for any values of p_0 and q_0, since all the derivatives appearing in this relationship now become identically zero. We obtain only the first of equations (80), which generally speaking gives us an infinite set of values for p_0 and q_0. These will be precisely the values of p_0, q_0 which define all the possible positions of the tangent plane at the fixed point (x_0, y_0, u_0). As above, we can regard $p_0(t)$ and $q_0(t)$ as functions of a parameter t. On substituting the fixed values (x_0, y_0, u_0) and the above-mentioned expressions for $p_0(t)$ and $q_0(t)$ in formula (73), we obtain an integral surface of equation (59) having the form of a conical surface with vertex at (x_0, y_0, u_0). This surface will in general have curvilinear generators, which touch the generators of the cone T at the vertex (x_0, y_0, u_0). This surface is usually called an *integral conoid of equation* (59) *with vertex* (x_0, y_0, u_0). It can be shown that the solution of Cauchy's problem can be reduced to the following construction. Draw the integral conoids having vertices on the given curve l, and take their envelope; this in fact leads to the solution of the problem. All these last statements naturally require strict analytic proof, but we shall not dwell on this.

110. Any number of independent variables. Let us take the first order equation in the case of any number of independent variables:

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0. \tag{97}$$

Cauchy's method of integrating this equation follows precisely the same lines as in the case of two independent variables, and we shall

confine ourselves to indicating the results. The characteristic system corresponding to equation (97) has the form:

$$\frac{dx_1}{P_1} = \ldots = \frac{dx_n}{P_n} = \frac{du}{p_1 P_1 + \ldots + p_n P_n} = \frac{dp_1}{-(X_1 + Up_1)} = \ldots =$$

$$= \frac{dp_n}{-(X_n + Up_n)} = ds \quad (X_k = F_{x_k};\ \ P_k = F_{p_k};\ \ U = F_u). \qquad (98)$$

Let us indicate the formal approach to deducing system (98). Let $u = u(x_1, \ldots, x_n)$ be a solution of equation (97) with continuous derivatives up to the second order. After substituting $u = u(x_1, \ldots, x_n)$ and $p_k = u_{x_k}(x_1, \ldots, x_n)$, X_k, P_k and U will be functions of $(x_1, \ldots, x_n)$.

We write the system of first order equations:

$$\frac{dx_k}{ds} = P_k \quad (k = 1, \ldots, n),$$

where s is an auxiliary variable. On substituting the solution of this system in the equation $u = u(x_1, \ldots, x_n)$, we get:

$$\frac{du}{ds} = \sum_{i=1}^{n} p_i \frac{dx_i}{ds} = \sum_{i=1}^{n} p_i P_i, \text{ and similarly: } \frac{dp_k}{ds} = \sum_{i=1}^{n} u_{x_k x_i} P_i.$$

But differentiation of (97) with respect to x_k gives:

$$X_k + Up_k + \sum_{i=1}^{n} P_i \frac{\partial p_i}{\partial x_k} = X_k + Up_k + \sum_{i=1}^{n} u_{x_k x_i} P_i,$$

whence

$$\frac{dp_k}{ds} = -(X_k + Up_k).$$

We have thus obtained all the equations of system (98). Let us consider (98) in more detail, as a system in functions x_k, u, p_k of the auxiliary variable s.

It admits of the obvious solution:

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = C.$$

Suppose we have succeeded in solving the system:

$$\left.\begin{aligned} x_k &= x_k(s,\ x_k^{(0)},\ u^{(0)},\ p_k^{(0)}) \\ u &= u(s,\ x_k^{(0)},\ u^{(0)},\ p_k^{(0)}) \\ p_k &= p_k(s,\ x_k^{(0)},\ u^{(0)},\ p_k^{(0)}), \end{aligned}\right\}, \qquad (99)$$

where $x_k^{(0)}$, $u^{(0)}$, $p_k^{(0)}$ are the initial values of the functions for $s = 0$. We shall regard these initial values as functions of $(n - 1)$ parameters:

$$x_k^{(0)}(t_1, \ldots, t_{n-1}),\ u^{(0)}(t_1, \ldots, t_{n-1}), p_k^{(0)}(t_1, \ldots, t_{n-1}). \tag{100}$$

On substituting these in (99), we get expressions for the variables x_k and u in terms of n parameters. Let us consider the functional determinant:

$$\Delta = \frac{D(x_1, \ldots, x_n)}{D(s, t_1, \ldots, t_{n-1})},$$

which, by virtue of the first equations of the system, can be written as:

$$\Delta = \begin{vmatrix} P_1, & \ldots, & P_n \\ \dfrac{\partial x_1}{\partial t_1}, & \ldots, & \dfrac{\partial x_n}{\partial t_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial t_{n-1}}, & \ldots, & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix}. \tag{101}$$

If this determinant is non-zero in the neighbourhood of the initial value $s = 0$, equations (100) give us a surface which can be written explicitly as $u = u(x_1, \ldots, x_n)$. The necessary and sufficient condition for this to be an integral surface of equation (97) is that functions (100) satisfy the following n relationships:

$$F(x_1^{(0)}, \ldots, x_n^{(0)}, u^{(0)}, p_1^{(0)}, \ldots, p_n^{(0)}) = 0, \tag{102}$$

$$\frac{\partial u^{(0)}}{\partial t_j} = \sum_{s=1}^{n} p_s^{(0)} \frac{\partial x_s^{(0)}}{\partial t_j} \quad (j = 1, 2, \ldots, n - 1). \tag{103}$$

Cauchy's problem consists in seeking the integral surface of equation (97) containing a given $(n - 1)$-manifold:

$$x_k^{(0)}(t_1, \ldots, t_{n-1}), u^{(0)}(t_1, \ldots, t_{n-1}).$$

This manifold will be assumed to be made up into manifold (100), so that (102) and (103) are satisfied. If determinant (101) now differs from zero along this manifold, our present method leads to a solution of Cauchy's problem, and this solution is unique.

Precisely as in the case of two independent variables, we can construct an integral conoid of equation (97) by fixing a point $(x_1^{(0)}, x_n^{(0)}, u^{(0)})$ and choosing $p_1^{(0)}, \ldots, p_n^{(0)}$ as functions of $(n - 1)$ parameters so that expressions (102) are satisfied.

If equation (97) is soluble with respect to p_1:

$$p_1 = f(x_1, \ldots, x_n, u, p_2, \ldots, p_n) \tag{104}$$

and if Cauchy's condition has the form:

$$u|_{x_1=x_1^{(0)}} = \psi(x_2, \ldots, x_n), \tag{105}$$

Cauchy's problem has a single definite solution.

We shall not quote here all the conditions of continuity and existence necessary for the derivatives of f and ψ. These are just the same as for $n = 2$. Given certain definite assumptions regarding f and ψ, we can define for equation (104) with initial data (105) the domain in which the integral surface exists. Suppose that f is continuous and has continuous derivatives f_{x_k}, f_{p_k}, and $f_u (k = 2, \ldots, n)$ for $|x_1 - x_1^{(0)}| \leqslant a$ and arbitrary x_k, p_k and u. Suppose, further, that these derivatives have continuous derivatives with respect to x, x_k, u and p_k, and that the derivatives $f_{x_1}, f_{x_k}, f_u, f_{p_k}, f_{x_k x_l}, f_{x_k u}, f_{x_k p_l}, f_{uu}, f_{up_k}, f_{p_k p_l}$ are bounded in absolute value by the number A for the values in question of the arguments. Suppose also that $\psi(x_2, \ldots, x_n)$ has continuous derivatives up to the second order and that the inequality holds:

$$|\psi_{x_k}| + \sum_{i=2}^{n} |\psi_{x_k x_i}| \leqslant B \quad (k = 2, \ldots, n).$$

There now exists a twice continuously differentiable solution of equation (104) under conditions (105) in the domain $|x_1 - x_1^{(0)}| < \alpha$ and arbitrary $x_k (k = 2, \ldots, n)$, where $\alpha \leqslant a$ and the further condition

$$\alpha < \frac{1}{A} \log \left[1 + \frac{\log 3}{2(n-1)(B+1)}\right]$$

must be satisfied (see Kamke, *Math. Zeitschr.* Bd. 49, Heft 3, 1943).

111. Complete, general and singular integrals. In this section and the next we shall give another method of solving the equation

$$F(x, y, u, p, q) = 0 \tag{106}$$

and in particular, of solving the Cauchy problem. It often leads readily to a solution of the problem in purely concrete examples. When discussing Cauchy's method, we explained the conditions in which this method, i.e. Cauchy's, can be applied, and in which a solution of Cauchy's problem exists and is unique. We shall for the present be chiefly concerned with the formal side of the question and shall make wide use of the theory of envelopes of a family of surfaces depending on one or two parameters.

To apply Cauchy's method for integrating equation (106) we have to be able to find the complete integral of the corresponding system of ordinary equations:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{pP + qQ} = \frac{dp}{-(X + Up)} = \frac{dq}{-(Y + Uq)}\,. \tag{107}$$

We now show that the problem of solving (106) only requires a knowledge of the solution of this equation depending on two arbitrary constants. Suppose we have such a solution:

$$u = \varphi(x, y, a, b), \tag{108}$$

where a and b are arbitrary constants. The partial derivatives p and q will be given by:

$$p = \varphi_x(x, y, a, b); \quad q = \varphi_y(x, y, a, b), \tag{109}$$

and we therefore have the following relationship:

$$F[x, y, \varphi(x, y, a, b), \varphi_x(x, y, a, b), \varphi_y(x, y, a, b)] = 0, \tag{110}$$

which must be satisfied identically not only with respect to (x, y) but also with respect to (a, b). We assume that a and b can be eliminated from the three equations (108) and (109) and that the elimination leads us to equation (106). In this case solution (108) of equation (106) will be called the *complete integral* of the equation. A further solution of the equation is easily obtained from the complete integral. Suppose that the constant b in (108) is some function of the constant a, i.e. $b = \omega(a)$. We arrive in this way at a family of integral surfaces, depending on one parameter:

$$u = \varphi[x, y, a, \omega(a)]. \tag{111}$$

The envelope of this family, which is obtained by eliminating a from equation (111) and the equation

$$\varphi_a[x, y, a, \omega(a)] + \varphi_b[x, y, a, \omega(a)]\,\omega'(a) = 0, \tag{112}$$

will have the same p and q along the curve of contact with the enveloped surface as has the enveloped surface itself, so that the envelope is also an integral surface of equation (106). Given any choice of differentiable function $\omega(a)$, the set of all these integral surfaces forms a *general solution* of equation (106). This solution, as may be seen, already contains an arbitrary function $\omega(a)$. We can further construct the envelope of the family of integral surfaces (108) depending on two para-

meters a and b. This is done by the elimination of a and b from equation (108) and the equations:

$$\varphi_a(x, y, a, b) = 0; \ \varphi_b(x, y, a, b) = 0. \tag{113}$$

The integral surface obtained does not contain any arbitrary elements and is usually called a *singular integral* of equation (106). We naturally assume here that all the above-mentioned eliminations are possible and lead to functions having continuous derivatives.

The general and singular integrals can be obtained by using the method of variation of the arbitrary constants instead of the above geometrical arguments. Let us seek the solution of (106) in the form (108), taking a and b as required functions of (x, y). The partial derivatives of the function u will no longer be evaluated from (109), but from the following formulae:

$$p = \varphi_x + \varphi_a a_x + \varphi_b b_x; \quad q = \varphi_y + \varphi_a a_y + \varphi_b b_y.$$

If we subject the required functions a and b to the two equations:

$$\varphi_a a_x + \varphi_b b_x = 0; \quad \varphi_a a_y + \varphi_b b_y = 0. \tag{114}$$

the expressions for the partial derivatives remain as before, and function (108) will yield an integral surface, as above. The problem reduces to a consideration of equations (114). These equations have the obvious solutions: $a =$ const. and $b =$ const, which lead us again to the complete integral. The second obvious solution is obtained if a and b satisfy the relationships:

$$\varphi_a = 0; \quad \varphi_b = 0.$$

This leads us to the singular integral. If at least one of these equations is not satisfied, the determinant of the homogeneous (in φ_a and φ_b) system (114) must vanish:

$$\begin{vmatrix} a_x, & b_x \\ a_y, & b_y \end{vmatrix} = 0.$$

We assume here that a and b are not constant simultaneously. On equating this functional determinant to zero, we get a relationship between a and b [III_1, 18]. Suppose that this relationship has the form $b = \omega(a)$. Equations (114) now reduce to one, which can be written in the form:

$$\varphi_a + \varphi_b \omega'(a) = 0,$$

and we obtain the general integral. It can be shown that, given certain conditions, all the solutions of equation (106) are exhausted by those found above. The nub of the matter is that Cauchy's problem can be solved once the complete integral is obtained.

112. The complete integral and Cauchy's problem. We now show just how the solution of Cauchy's problem follows from the complete integral. Let the integral surface be required, that passes through the curve:

$$x = x(t); \quad y = y(t); \quad z = z(t), \tag{115}$$

The problem amounts to finding a function $b = \omega(a)$ in the general integral defined by (111) and (112) such that the integral surface obtained passes through curve (115). Suppose we have a family of surfaces with one parameter:

$$\psi(x, y, u, a) = 0. \tag{116}$$

Suppose that a surface of family (116) passes through every point M of curve (115), the tangent at M to curve (115) being contained in the tangent plane to the surface at M. We show that in this case the envelope of family (116) contains the curve (115). In fact, we have by hypothesis:

$$\psi[x(t),\ y(t),\ u(t), a] = 0, \tag{117}$$

where different points M of curve (115) correspond to different values of the constant a, i.e. different surfaces of family (116). On differentiating this last identity with respect to t, we get:

$$\psi_x x_t + \psi_y y_t + \psi_z z_t + \psi_a a_t = 0.$$

On the other hand, the fact that the tangent plane to the surface contains the tangent to the curve leads to the identity:

$$\psi_x x_t + \psi_y y_t + \psi_u u_t = 0, \tag{118}$$

and these last two identities give $\psi_a a_t = 0$, or, since $a_t \neq 0$, we have $\psi_a = 0$. Thus functions (115) satisfy the equations $\psi = 0$ and $\psi_a = 0$ identically with respect to t, i.e. the envelope of family (116) in fact contains curve (115).

We now suppose that the complete integral of (106) is known, written in the implicit form:

$$\psi(x, y, u, a, b) = 0. \tag{119}$$

We have to find a function $b = \omega(a)$ such that (117) and (118) are satisfied. The left-hand side of (118) represents the derivative with respect to t of the left-hand side of (117). Let $\Psi(t, a, b)$ denote the result of substituting functions (115) in the left-hand side of (119). We must therefore write two equations:

$$\Psi(t, a, b) = 0; \quad \Psi'_t(t, a, b) = 0. \tag{120}$$

On eliminating t from these equations, we get a relationship between a and b, i.e. we find the required function: $b = \omega(a)$. Therefore to solve Cauchy's problem, given the complete integral, we have to substitute in the equation of the complete integral functions (115), differentiate the equation obtained with respect to t and eliminate t from the two equations thus obtained. This leads us to a relationship between the constants a and b. The general solution corresponding to this relationship will in fact pass through curve (115).

Another procedure can be adopted. We use (120) to express a and b in terms of t. On substituting in (119), a family of surfaces is obtained, depending on the single parameter t. The required integral surface through curve (115) is obtained by finding the envelope of this family.

Mention may be made of the connection between the general solution and the characteristic strips which were obtained as a result of solving system (107). The envelope of family (111) touches one of the enveloped surfaces along some curve l_a. On drawing along thiscurve the tangent plane common to the envelope and enveloped surface, a strip is obtained. This strip belongs to two integral surfaces, viz. the envelope and the enveloped surface, and is therefore a characteristic strip. We can therefore say that the formulae

$$\left.\begin{aligned} &u = \varphi(x, y, a, b); \quad \varphi_a(x, y, a, b) + \varphi_b(x, y, a, b)\,\omega'(a) = 0 \\ &p = \varphi_x(x, y, a, b); \quad q = \varphi_y(x, y, a, b) \end{aligned}\right\} \tag{121}$$

define, for any fixed a and any choice of $b = \omega(a)$, a solution of system (107) satisfying condition (106).

We can assume that formulae (121) define four of the quantities x, y, u, p, q as functions of the fifth and three arbitrary constants a, b and $c = \omega'(a)$. The general solution of system (107) contains four arbitrary constants. But, in view of the presence of relationships (106), the family of all the characteristic strips must depend only on three arbitrary constants, which have in fact been obtained in accordance with (121). In one of the following sections we shall prove by means

of direct evaluation the fact that equations (121) actually yield the solution of system (107) for the case of any number of independent variables.

Let us see how it is possible to find the singular integral directly from the differential equation without the aid of the complete integral. On differentiating the identity (110) with respect to a and b, we get

$$F_u \varphi_a + F_p \varphi_{xa} + F_q \varphi_{ya} = 0; \quad F_u \varphi_b + F_p \varphi_{xb} + F_q \varphi_{yb} = 0 .$$

Taking into account the definition of singular integral (113), we can say that the following two equations are fulfilled on the singular integral surface:

$$F_p \varphi_{xa} + F_q \varphi_{ya} = 0; \quad F_p \varphi_{xb} + F_q \varphi_{yb} = 0 .$$

We shall assume that the determinant of this system (homogeneous in F_p and F_q) does not vanish on the singular integral surface, which amounts in essence to assuming the possibility of solving equations (109) for a and b. The above system now gives us:

$$F_p = F_q = 0 . \tag{122}$$

Thus the singular integral can be obtained by eliminating p and q from the following three equations:

$$F(x, y, u, p, q) = 0; \; F_p(x, y, u, p, q) = 0; \; F_q(x, y, u, p, q) = 0. \tag{123}$$

Equations (122) indicate the impossibility of applying the implicit function theorem to equation (106) with respect to the variable p or the variable q. This proves the impossibility of obtaining the singular integral as a result of solving Cauchy's problem, as we did in [107], where the equation was assumed to be soluble with respect to p (or q). We can arrive at the same result by another method. No matter what curve we take on the singular integral surface, determinant (96) vanishes along this curve by virtue of conditions (122), which shows that no definite solution of Cauchy's problem exists, whatever the choice of curve on the singular integral surface.

113. Examples. 1. The equation

$$u = xp + yq + f(p, q) \tag{124}$$

is analogous to Clairaut's equation, which we discussed earlier [II, 8]. On replacing p and q by a and b, its complete integral is obtained, as may easily be verified:

$$u = ax + by + f(a, b) .$$

The equation

$$u = xp + yq + pq$$

has the complete integral:

$$u = ax + by + ab\,,$$

and application of the method described above gives the singular integral:

$$u = -xy\,.$$

If we take any curve on this surface:

$$x_0 = \varphi(t);\quad y_0 = \psi(t);\quad u_0 = -\varphi(t)\,\psi(t)\,, \tag{125}$$

then equations (86):

$$\varphi(t)\,p_0 + \psi(t)\,q_0 + p_0\,q_0 + \varphi(t)\,\psi(t) = 0$$
$$\varphi'(t)\,\psi(t) + \psi'(t)\,\varphi(t) + \varphi'(t)\,p_0 + \psi'(t)\,q_0 = 0$$

have the solutions $p_0 = -\psi(t)$, $q_0 = -\varphi(t)$, and we have along curve (125):

$$F_{p_0} = q + \varphi(t) = 0;\quad F_{q_0} = p + \psi(t) = 0\,.$$

The singular integral for the equation

$$u = xp + yq - \frac{1}{2}(p^2 + q^2) \tag{124$_1$}$$

will be

$$u = \frac{1}{2}(x^2 + y^2)\,. \tag{126}$$

If equation (124_1) is solved for p, we obtain:

$$F = p - x + \sqrt{x^2 + 2qy - 2u - q^2} = 0\,,$$

and the partial derivative of the left-hand side of the equation with respect to u becomes infinite along the surface (126).

2. Suppose we have an equation containing only p and q:

$$f(p, q) = 0\,.$$

This has the obvious solution:

$$u = ax + cy + b\,,$$

where constants a and c must satisfy the relationship: $f(a, c) = 0$. On solving this for c: $c = f_1(a)$, we obtain the complete integral of the equation in the form:

$$u = ax + f_1(a)\,y + b\,.$$

This equation gives a family of planes. The general integral will be the envelope of a one-parameter family of planes, i.e. a developable surface [II, 141].

Let us take as an example the equation:

$$p^2 + q^2 = k^2\,. \tag{127}$$

If we observe that the direction-cosine of the normal to the required surface with respect to the u axis is given by

$$\cos(n, u) = \pm \frac{1}{\sqrt{1 + p^2 + q^2}} = \pm \frac{1}{\sqrt{1 + k^2}},$$

we see that equation (127) reduces to the requirement that the normal to the required surface form a constant angle with the u axis. The complete integral represents the family of planes:

$$u = ax + \sqrt{k^2 - a^2}\, y + b\,.$$

System (68) becomes:

$$\frac{dx}{ds} = 2p; \quad \frac{dy}{ds} = 2q; \quad \frac{du}{ds} = 2(p^2 + q^2); \quad \frac{dp}{ds} = 0; \quad \frac{dq}{ds} = 0\,,$$

and its solution, expressed in terms of the initial data, is:

$$x = 2p_0\, s + x_0; \quad y = 2q_0\, s + y_0; \quad u = 2(p_0^2 + q_0^2)\, s + u_0; \quad p = p_0; \quad q = q_0. \tag{128}$$

The characteristic strips are obtained by subjecting p_0 and q_0 to the condition $p_0^2 + q_0^2 = k^2$. They will be straight lines, and p and q will remain constant along these lines.

Suppose we want to draw the integral surface through the circle:

$$x_0 = \cos t; \quad y_0 = 0; \quad u_0 = \sin t\,.$$

Equations (86) become in this case:

$$p_0^2 + q_0^2 = k^2; \quad \cos t = -p_0 \sin t,$$

whence

$$p_0 = -\cot t; \quad q_0 = \sqrt{k^2 - \cot^2 t}\,.$$

On substituting in the first three of equations (128), we get the parametric equation of the surface, in terms of parameters s and t:

$$x = -2s\cot t + \cos t; \quad y = 2s\sqrt{k^2 - \cot^2 t}; \quad u = 2k^2 s + \sin t\,.$$

3. The following first order equation is of a more general type:

$$f_1(x, p) = f_2(y, q)\,.$$

The complete integral is obtained by supposing that both sides of the equation are equal to the same constant a : $f_1(x, p) = a$ and $f_2(y, q) = a$. On solving these equations for p and q, we obtain: $p = \varphi_1(x, a)$ and $q = \varphi_2(y, a)$, and the complete integral becomes:

$$u = \int \varphi_1(x, a)\, dx + \int \varphi_2(y, a)\, dy + b\,,$$

where b is the second arbitrary constant. When applied to the equation

$$pq - xy = 0 \text{ or } \frac{p}{x} = \frac{y}{q} \tag{129}$$

this method gives:

$$u = \frac{1}{2} ax^2 + \frac{1}{2a} y^2 + b \,. \tag{130}$$

Suppose we want to draw the integral surface through the curve

$$x = t; \quad y = \frac{1}{t}; \quad u = 1 \,.$$

On substituting in (130) and differentiating with respect to t, we obtain:

$$1 = \frac{1}{2} at^2 + \frac{1}{2at^2} + b; \quad at - \frac{1}{at^3} = 0 \,.$$

Elimination of t gives $b = 0$, and we get the one-parameter family of integral surfaces:

$$u = \frac{1}{2} ax^2 + \frac{1}{2a} y^2 \,,$$

whilst the envelope of this family leads to the required integral surface:

$$u = xy \,.$$

If we took as the initial data the curve

$$x = t; \quad y = t; \quad u = t^2 \,, \tag{131}$$

the above method would lead to the equations:

$$\left(\frac{1}{2} a + \frac{1}{2a} - 1\right) t^2 + b = 0; \quad 2\left(\frac{1}{2} a + \frac{1}{2a} - 1\right) t = 0 \,,$$

from which we have $a = 1, b = 0$, so that no integral surface can be found through curve (131). It is easily seen that curve (131) can be made up into a characteristic strip by putting $p = t$ and $q = t$. In fact, the functions

$$x = t; \quad y = t; \quad u = t^2; \quad p = t; \quad q = t$$

satisfy equation (129) and system (107).

4. If the equation does not contain the independent variables:

$$F(u, p, q) = 0 \,,$$

the complete integral can be obtained by seeking the solution of an equation of the form:

$$u = \varphi(x + ay) \,, \tag{132}$$

where a is an arbitrary constant. Let us take as an example:

$$pq - u = 0 \,. \tag{133}$$

On carrying out substitution (132) and setting $\xi = x + ay$, we obtain

$$a[\varphi'(\xi)]^2 - \varphi(\xi) = 0 \,,$$

and integration of this ordinary differential equation gives for the complete integral of equation (133):

$$u = \frac{(x + ay + b)^2}{4a} \,.$$

System (68) for equation (133) has the form:

$$\frac{dx}{ds} = q; \quad \frac{dy}{ds} = p; \quad \frac{du}{ds} = 2\,pq; \quad \frac{dp}{ds} = p; \quad \frac{dq}{ds} = q\,,$$

and integration of it gives:

$$x = q_0\,e^s + (x_0 - q_0); \quad y = p_0\,e^s + (y_0 - p_0); \quad u = p_0\,q_0\,e^{2s} + (u_0 - p_0\,q_0)$$
$$p = p_0\,e^s; \quad q = q_0\,e^s\,. \tag{133$_1$}$$

Suppose we want the integral surface through the straight line:

$$x_0 = t; \quad y_0 = 1; \quad u_0 = t\,.$$

We have the equation for p_0 and q_0:

$$p_0\,q_0 = t; \quad p_0 = 1\,,$$

whence $p_0 = 1$ and $q_0 = t$. On substituting in the first three of equations (133$_1$) and setting $e^s = v$, we get the parametric equation of the required surface in terms of parameters v and t:

$$x = tv; \quad y = v; \quad u = tv^2\,,$$

or, in the explicit form: $u = xy$.

114. The case of any number of variables. The *complete integral* of the equation

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0 \tag{134}$$

is defined as the solution:

$$u = \varphi(x_1, \ldots, x_n, a_1, \ldots, a_n)\,, \tag{135}$$

containing n arbitrary constants a_s and such that elimination of the a from the equations:

$$p_k = \varphi_{x_k}(x, \ldots, x_n, a_1, \ldots, a_n) \qquad (k = 1, 2, \ldots, n), \tag{135$_1$}$$

and equation (135) leads to equation (134). We shall assume that the a_k are functions of $(n - 1)$ parameters:

$$a_k = a_k(t_1, \ldots, t_{n-1}) \qquad (k = 1, 2, \ldots, n)\,. \tag{136}$$

On substituting these expressions in (135) and eliminating the $(n - 1)$ parameters from the n equations:

$$\begin{gathered} u = \varphi(x_1, \ldots, x_n, a_1, \ldots, a_n) \\ \varphi_{t_j}(x_1, \ldots, x_n, a_1, \ldots, a_n) = 0 \qquad (j = 1, \ldots, n - 1)\,, \end{gathered} \tag{137}$$

we obtain the general solution of equation (134). It depends on the choice of the n functions (136). Let us turn to the solution of Cauchy's

problem. Suppose we want to find the integral surface of equation (134) containing a given $(n-1)$-manifold:

$$u = u(t_1, \ldots, t_{n-1}); \quad x_k = x_k(t_1, \ldots, t_{n-1}) \quad (k = 1, 2, \ldots, n). \quad (138)$$

This problem is solved precisely as in the case of two independent variables. Substitution of expression (138) in (135) leads to an equation of the form

$$\psi(t_1, \ldots, t_{n-1}, a_1, \ldots, a_n) = 0 . \quad (139)$$

On associating with this equation the $(n-1)$ further equations obtained by differentiating the last equation with respect to $t_1, \ldots, t_{n-1}$:

$$\psi_{t_1} = 0; \quad \psi_{t_2} = 0; \quad \ldots; \ \psi_{t_{n-1}} = 0 , \quad (140)$$

we obtain n equations, from which the a_k $(k = 1, 2, \ldots, n)$ can be found as functions of the parameters $t_1, \ldots, t_{n-1}$, i.e. functions (136) are defined from these n equations. We substitute the functions obtained in (137), and on eliminating $t_1, \ldots, t_{n-1}$ from the n equations (137), obtain an integral surface containing manifold (138). We remark that the number of independent parameters in (139) may be less than $(n-1)$. Instead of (140), we now have to differentiate with respect to our independent parameters.

If we fix the values of parameters t_j, the n equations (137) with $n+1$ variables $(u, x_1, \ldots, x_n)$ define a curve in $(n+1)$-dimensional space. This curve can be made up into a first order strip by adding the equations (135_1). This strip belongs to two integral surfaces — the envelope obtained by eliminating the parameters t_j from (137), and one of the enveloped surfaces. This strip must therefore be a characteristic strip, i.e. must satisfy Cauchy's system (98). This enables us to construct from our knowledge of the complete integral (135), a solution of system (98) depending on $(2n-1)$ arbitrary constants. We shall assume for simplicity that a_n is a function of $(a_1, \ldots, a_{n-1})$, where these last play the role of the parameters $t_1, \ldots, t_{n-1}$. Formulae (137) and (135_1) take the form:

$$\begin{aligned} u &= \varphi(x_1, \ldots, x_n, a_1, \ldots, a_n) \\ \varphi_{a_j} + \varphi_{a_n} b_j &= 0 \qquad (j = 1, 2, \ldots, n-1) \\ p_k &= \varphi_{x_k}(x_1, \ldots, x_n, a_1, \ldots, a_n) \qquad (k = 1, 2, \ldots, n) , \end{aligned} \quad (141)$$

where we have written b_j for the derivative of a_n with respect to a_j. Expressions (141) define the above-mentioned first order strip, where $b_1, \ldots, b_{n-1}$, as well as $a_1, \ldots, a_n$, can be assumed arbitrary, since the

choice of the function $a_n(a_1, \ldots, a_{n-1})$ is arbitrary. We shall prove formally that the strip defined by (141) satisfies system (98).

On substituting for u and p_k from (141) in (134), we must obtain an identity in the x_k and a_k $(k = 1, \ldots, n)$. Differentiation of this identity with respect to the a_s gives:

$$U\varphi_{a_j} + \sum_{k=1}^{n} P_k \varphi_{x_k a_j} = 0 \qquad (j = 1, 2, \ldots, n-1)$$

$$U\varphi_{a_n} + \sum_{k=1}^{n} P_k \varphi_{x_k a_n} = 0 .$$

On multiplying the last equation by b_j, adding to the previous equation and using (141), we get the following $(n - 1)$ equations:

$$\sum_{k=1}^{n} P_k(\varphi_{a_j x_k} + \varphi_{a_n x_k} b_j) = 0 . \tag{142}$$

On the other hand, on taking the total differentials of the left-hand sides of the second set of equations (141), we get the $(n-1)$ equations:

$$\sum_{k=1}^{n} (\varphi_{a_j x_k} + \varphi_{a_n x_k} b_j)\, \mathrm{d}x_k = 0 \qquad (j = 1, 2, \ldots, n-1) . \tag{143}$$

Assuming that at least one of the determinants of order $(n - 1)$ formed from the coefficients of system (142) or (143) differs from zero, we can say that the $\mathrm{d}x_k$ must be proportional to the P_k, i.e. we have:

$$\frac{\mathrm{d}x_1}{P_1} = \ldots = \frac{\mathrm{d}x_n}{P_n} .$$

Further, it follows from (141) that $\mathrm{d}u = \sum_{k=1}^{n} p_k\, \mathrm{d}x_k$, and we can complete the equations written above:

$$\frac{\mathrm{d}x_1}{P_1} = \ldots = \frac{\mathrm{d}x_n}{P_n} = \frac{\mathrm{d}u}{p_1 P_1 + \ldots + p_n P_n} . \tag{144}$$

On returning once more to the identity obtained by substituting for u and p_k from (141) in (134), differentiation of this identity with respect to the x_k gives:

$$X_k + U\varphi_{x_k} + \sum_{i=1}^{n} P_i \varphi_{x_i x_k} = 0 .$$

On multiplying by $\mathrm{d}x_k$ and substituting, by (144): $P_i \mathrm{d}x_k = P_k \mathrm{d}x_i$, we obtain:

$$(X_k + Up_k)\,\mathrm{d}x_k + P_k \sum_{i=1}^{n} \varphi_{x_i x_k}\,\mathrm{d}x_i = 0 .$$

But

$$\sum_{i=1}^{n} \varphi_{x_i x_k}\,\mathrm{d}x_i = \mathrm{d}p_k ,$$

so that:

$$(X_k + Up_k)\,\mathrm{d}x_k + P_k\,\mathrm{d}p_k = 0, \text{ i.e. } \frac{\mathrm{d}x_k}{P_k} = \frac{\mathrm{d}p_k}{-(X_k + Up_k)} ,$$

and we therefore finally obtain the system:

$$\frac{\mathrm{d}x_1}{P_1} = \ldots = \frac{\mathrm{d}x_n}{P_n} = \frac{\mathrm{d}u}{p_1 P_1 + \ldots + p_n P_n} =$$
$$= \frac{\mathrm{d}p_1}{-(X_1 + Up_1)} = \ldots = \frac{\mathrm{d}p_n}{-(X_n + Up_n)} . \tag{145}$$

115. Jacobi's theorem. We now consider the particular case of equation (134) when it does not contain the required function u and is soluble with respect to one of the derivatives. For the sake of symmetrical writing, we denote the independent variables by $t, x_1, \ldots, x_n$, and let the equation be soluble with respect to $p_0 = u_t$, i.e. it has the form:

$$p_0 + H(t, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 . \tag{146}$$

System (145) corresponding to this equation may be written as:

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x_1}{H_{p_1}} = \ldots = \frac{\mathrm{d}x_n}{H_{p_n}} = \frac{\mathrm{d}p_1}{-H_{x_1}} = \ldots = \frac{\mathrm{d}p_n}{-H_{x_n}}$$
$$= \frac{\mathrm{d}u}{p_0 + p_1 H_{p_1} + \ldots + p_n H_{p_n}} . \tag{147}$$

None of these relationships, except the last, contains p_0 and u, and we obtain the so-called canonical system:

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = H_{p_k}; \quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -H_{x_k} \qquad (k = 1, 2, \ldots, n) ,$$

where x_k and p_k are assumed to be functions of t. If we can succeed in integrating this system, p_0 is found from (146), whilst u is found with the aid of quadratures from the equation:

$$\mathrm{d}u = (p_0 + p_1 H_{p_1} + \ldots + p_n H_{p_n})\mathrm{d}t.$$

Since equation (146) does not contain u, we can add an arbitrary constant to every solution of this equation. Suppose that we have the

complete integral of equation (146), which must contain $(n+1)$ arbitrary constants, one of these being in the form of an added term:

$$u = \psi(t, x_1, \ldots, x_n, a_1, \ldots, a_n) - a_0 .$$

We apply equation (141) to the present case, the role of a_n being now played by the constant a_0. Since we now have $\theta_{a_0} = -1$, the general solution of the canonical system is obtained in the form

$$\psi_{a_k} = b_k; \quad p_k = \psi_{x_k} \qquad (k = 1, \ldots, n) .$$

This is the essence of the familiar Jacobi Theorem, already mentioned above in [82].

We remark that, if equation (134) does not contain the required function u, but is not soluble with respect to any of the p_k, i.e. has the form:

$$F(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 ,$$

system (145) corresponding to this equation becomes:

$$\frac{dx_1}{F_{p_1}} = \ldots = \frac{dx_n}{F_{p_n}} = \frac{dp_1}{-F_{x_1}} = \ldots = \frac{dp_n}{-F_{x_n}} = \frac{du}{p_1 F_{p_1} + \ldots + p_n F_{p_n}} = ds,$$

and we again obtain a canonical system:

$$\frac{dx_k}{ds} = F_{p_k}; \quad \frac{dp_k}{ds} = -F_{x_k} \qquad (k = 1, \ldots, n) ,$$

in which the role of independent variable is played by the auxiliary parameter s. If we are able to integrate this system, u can be found with the aid of quadratures.

The Jacobi theorem given above shows us how the corresponding canonical system can be integrated, given a knowledge of the complete integral of equation (146). The Cauchy method that we explained in [110], shows that, conversely, if we can integrate system (147), we can find the solutions of (146) satisfying any initial Cauchy conditions, and by using this fact it is easily shown that, in particular, the complete integral of (146) can be constructed.

116. Systems of two first order equations. We have given a number of examples where the complete integral can be found with the aid of purely elementary methods. The question arises as to the possibility of developing a general method for discovering the complete integral for any first order equation. Before describing such a method, we first

need to consider the problem of finding the solution of two first order equations with one required function:

$$F(x, y, u, p, q) = 0; \quad \Phi(x, y, u, p, q) = 0 .$$

We shall assume that these equations are soluble for p and q, so that we have equations of the following form:

$$p = f(x, y, u); \quad q = g(x, y, u) . \tag{148}$$

We shall describe this system as *completely integrable* if it has a solution depending on an arbitrary constant; we shall find the necessary and sufficient condition for this, then give a method of finding the solution if this condition is fulfilled. On differentiating the first of equations (148) with respect to y, and the second with respect to x, we obviously obtain:

$$f_y + f_u q = g_x + g_u p ,$$

or, by (148):

$$f_y + f_u g = g_x + g_u f . \tag{149}$$

If this relationship between variables (x, y, u) is not fulfilled identically, it defines u as a function of x and y, and this function, which does not contain an arbitrary constant, can alone be a solution of system (148). Hence the necessary condition for system (148) to be completely integrable is that (148) can be satisfied identically. Let us show that this condition is also sufficient, whilst at the same time giving a method for finding the solutions of system (148). We can regard the first of equations (148) as an equation with one independent variable x, since y appears in this equation as a parameter. On integrating this first order equation, we obtain u as a function of the independent variable x, the parameter y and the arbitrary constant $c(y)$: which we can regard as a function of y:

$$u = \varphi[x, y, C(y)] . \tag{150}$$

This function must satisfy the second of equations (148), i.e. the equation must be satisfied:

$$\varphi_y + \varphi_C \frac{dC}{dy} = g(x, y, u) ,$$

or

$$\frac{dC}{dy} = \frac{g(x, y, u) - \varphi_y}{\varphi_C} , \tag{151}$$

where u has to be replaced on the right-hand side by its expression (150). We now show that, if (149) is satisfied identically, the right-

hand side of (151) does not contain x. In fact, on equating to zero the derivative of the right-hand side of (151) with respect to x, we get:

$$(g_x + g_u \varphi_x - \varphi_{yx}) \varphi_C - \varphi_{Cx}(g - \varphi_y) = 0 . \qquad (152)$$

But, since function (150) satisfies the first of equations (148), we have the following obvious relationships:

$$\varphi_x = f; \quad \varphi_{yx} = f_y + f_u \varphi_y; \quad \varphi_{Cx} = f_u \varphi_C ,$$

and with the aid of these relationships condition (152) can be written as:

$$(g_x + g_u f - f_y - f_u \varphi_y) \varphi_C - f_u \varphi_C(g - \varphi_y) = 0 ,$$

and it is obviously satisfied, since (149) is assumed to be satisfied identically. Therefore equation (151) now represents a first order equation in $C(y)$, solution of which gives us C in terms of y and an arbitrary constant b. On substituting this expression in (150), we get a solution of system (148) containing one arbitrary constant. Hence *the necessary and sufficient condition for complete integrability of system* (148) *is that* (149) *be satisfied as an identity*. If this condition is fulfilled, integration of system (148) amounts to integration of two ordinary differential equations of the first order, and the general solution of system (148) contains one arbitrary parameter.

The problem of solving the exact differential equation

$$P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}u = 0 , \qquad (153)$$

where P, Q and R are given functions of (x, y, u), is directly connected with the previous problem. Equation (153) reduces immediately to (148) if we put:

$$f = -\frac{P}{R} ; \quad g = -\frac{Q}{R} ,$$

and the condition for integrability (149) leads in this case to the relationship between the coefficients:

$$P(R_y - Q_u) + Q(P_u - R_x) + R(Q_x - P_y) = 0 .$$

This relationship has already been mentioned as the necessary and sufficient condition for complete integrability of equation (153) [II, 76].

117. The Lagrange–Charpit method. This method gives a general means of constructing the complete integral of a first order partial differential equation in two independent variables:

$$F(x, y, u, p, q) = 0 . \qquad (154)$$

We try to find a second equation of the form:

$$\Phi(x, y, u, p, q) = a\,, \tag{155}$$

where a is an arbitrary constant, such that (154) and (155) are soluble with respect to p and q and such that, after solution, a system of the form (148) is obtained which is completely integrable. If we can succeed in doing this, integration of the system obtained introduces a further arbitrary constant b and we in fact obtain the complete integral of (154). Condition (149) for complete integrability can be written as:

$$p_y + p_u q = q_x + q_u p\,. \tag{156}$$

We have to work out all the partial derivatives appearing in this identity by applying the rule for differentiation of implicit functions p and q of variables (x, y, u), defined by equations (154) and (155). Differentiation of (154) and (155) with respect to u gives us:

$$F_u + F_p p_u + F_q q_u = 0; \quad \Phi_u + \Phi_p p_u + \Phi_q q_u = 0\,,$$

whence

$$p_u = -\frac{\begin{vmatrix} F_u, & F_q \\ \Phi_u, & \Phi_q \end{vmatrix}}{\begin{vmatrix} F_p, & F_q \\ \Phi_p, & \Phi_q \end{vmatrix}}\,; \quad q_u = -\frac{\begin{vmatrix} F_p, & F_u \\ \Phi_p, & \Phi_u \end{vmatrix}}{\begin{vmatrix} F_p, & F_q \\ \Phi_p, & \Phi_q \end{vmatrix}}\,,$$

Similarly, differentiation with respect to x and y gives:

$$q_x = -\frac{\begin{vmatrix} F_p, & F_x \\ \Phi_p, & \Phi_x \end{vmatrix}}{\begin{vmatrix} F_p, & F_q \\ \Phi_p, & \Phi_q \end{vmatrix}}\,; \quad p_y = -\frac{\begin{vmatrix} F_y, & F_q \\ \Phi_y, & \Phi_q \end{vmatrix}}{\begin{vmatrix} F_p, & F_q \\ \Phi\ , & \Phi_q \end{vmatrix}}\,,$$

and condition (156) for integrability becomes:

$$-\begin{vmatrix} F_y, & F_q \\ \Phi_y, & \Phi_q \end{vmatrix} - \begin{vmatrix} F_u, & F_q \\ \Phi_u, & \Phi_q \end{vmatrix} q + \begin{vmatrix} F_p, & F_x \\ \Phi_p, & \Phi_x \end{vmatrix} + \begin{vmatrix} F_p, & F_u \\ \Phi_p, & \Phi_u \end{vmatrix} p = 0\,,$$

or

$$\begin{vmatrix} F_p, & F_x + F_u p \\ \Phi_p, & \Phi_x + \Phi_u p \end{vmatrix} + \begin{vmatrix} F_q, & F_y + F_u q \\ \Phi_q, & \Phi_y + \Phi_u q \end{vmatrix} = 0\,. \tag{157}$$

On expanding the determinants and using the notation of [104], we arrive at the following partial differential equation for the required function Φ:

$$P\Phi_x + Q\Phi_y + (pP + qQ)\,\Phi_u - (X + Up)\,\Phi_p - (Y + Uq)\,\Phi_q = 0. \tag{158}$$

Strictly speaking, this equation must be satisfied if the p and q in it are replaced by their expressions from (154) and (155). But we shall

require more than this, viz. that it be satisfied as an identity. The system of ordinary differential equations corresponding to the linear homogeneous equation (158) is precisely Cauchy's system (107):

$$\frac{\mathrm{d}x}{P}=\frac{\mathrm{d}y}{Q}=\frac{\mathrm{d}u}{pP+qQ}=\frac{\mathrm{d}p}{-(X+Up)}=\frac{\mathrm{d}q}{-(Y+Uq)}\,. \tag{159}$$

It is sufficient for us to find any one solution of this system, such that equations (154) and (155) be soluble with respect to p and q.

We know that system (159) has the obvious solution $F=C$. The existence of this solution can simplify the discovery of another solution of the system. Here we can use not only the solution mentioned, but also the simple relationship $F=0$.

If equation (154) does not contain the required function u, i.e. has the form

$$F(x, y, p, q)=0\,,$$

we can seek the solution as also independent of u:

$$\Phi(x, y, p, q)=a\,.$$

Condition (157) now becomes:

$$\begin{vmatrix} F_p, & F_x \\ \Phi_p, & \Phi_x \end{vmatrix}+\begin{vmatrix} F_q, & F_y \\ \Phi_q, & \Phi_y \end{vmatrix}=0\,, \tag{160}$$

or, in the expanded form:

$$(F_p\Phi_x-F_x\Phi_p)+(F_q\Phi_y-F_y\Phi_q)=0\,.$$

This leads us to seek the solution of the system:

$$\frac{\mathrm{d}x}{P}=\frac{\mathrm{d}y}{Q}=\frac{\mathrm{d}p}{-X}=\frac{\mathrm{d}q}{-Y}\,. \tag{161}$$

The expression on the left-hand side of (160) is usually called the *Poisson bracket* of functions F and Φ and is denoted by the symbol (F, Φ). The left-hand side of (157) is called the *Mayer bracket* of functions F and Φ and is denoted by the symbol $[F, \Phi]$. If we introduce conditional notation for any function ω depending on variables (x, y, u, p, q), and put

$$\frac{\mathrm{d}\omega}{\mathrm{d}x}=\omega_x+\omega_u\,p;\quad \frac{\mathrm{d}\omega}{\mathrm{d}y}=\omega_y+\omega_u\,q\,,$$

the Mayer bracket can be written as

$$[F, \Phi] = \begin{vmatrix} F_p, & \dfrac{\mathrm{d}F}{\mathrm{d}x} \\ \Phi_p, & \dfrac{\mathrm{d}\Phi}{\mathrm{d}x} \end{vmatrix} + \begin{vmatrix} F_q, & \dfrac{\mathrm{d}F}{\mathrm{d}y} \\ \Phi_q, & \dfrac{\mathrm{d}\Phi}{\mathrm{d}y} \end{vmatrix}, \tag{162}$$

The two functions F and Φ are said to be *in involution* if they cause the Poisson or Mayer bracket to vanish. In the first case the functions must be functions of the variables (x, y, p, q), and in the second the arguments further include u. The essence of the Lagrange–Charpit method consists therefore in trying to find a solution of system (159) or (161) which is in involution with F.

Let us mention a further fact that strikes the attention when comparing the Cauchy and Lagrange methods. When using the Cauchy method we have to find all the solutions of system (159), whilst in the Lagrange–Charpit method we have to find only one solution of this system. On the other hand, we find the complete integral of equation (154) when using the Lagrange–Charpit method, and this enables us to integrate system (159) completely.

118. Systems of linear equations. In order to generalize the Lagrange–Charpit method to the case of any number of independent variables, we must first consider the question of solving a sytem of linear homogeneous equations with one required function. We take the system consisting of m equations:

$$\left.\begin{aligned} X_1(u) &= a_{11}\,p_1 + a_{12}\,p_2 + \ldots + a_{1n}\,p_n = 0 \\ X_2(u) &= a_{21}\,p_1 + a_{22}\,p_2 + \ldots + a_{2n}\,p_n = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_m(u) &= a_{m1}\,p_1 + a_{m2}\,p_2 + \ldots + a_{mn}\,p_n = 0\,, \end{aligned}\right\} \tag{163}$$

where $p_k = u_{x_k}$, the coefficients a_{ik} are regarded as depending only on the independent variables x_s, and $x_k(u)$ denotes for brevity the left-hand side of the kth equation. The question arises of seeking a function u which will satisfy simultaneously all the equations of system (163). When speaking of a solution of the system, we exclude the obvious solution $u = \text{const}$, which is of no interest to us. We assume that equations (163) are linearly independent, i.e. there exist no factors λ_k, not all zero, which may be functions of x_s, such that

$$\sum_{k=1}^{m} \lambda_k X_k(u) = 0\,,$$

this being satisfied as an identity in x_s and p_s. If such factors were to exist and at least one were non-zero, the left-hand side of one of equations (163) could be expressed linearly in terms of the left-hand sides of the remaining equations. This equation would now be a consequence of the rest, and we could strike it out. We suppose that $m \geqslant n$ and consider the first n equations of

the system. Since these equations are linearly independent, the determinant formed from their coefficients must be non-zero. But in this case the system, which is homogeneous in the p_s, only has a zero solution $p_1 = \ldots = p_n = 0$, whence it follows that $u =$ const., i.e. *when $m \geqslant n$ the system has no non-trivial solution*. We shall therefore suppose in future that $m < n$.

We can form new linear homogeneous equations which are consequences of equations (163) yet may turn out to be linearly independent of equations (163). Let us first establish a number of elementary identities. If u_1 and u_2 are any two functions of the independent variables $x_1, \ldots, x_n$, we have the following two obvious identities:

$$X_k(u_1 + u_2) = X_k(u_1) + X_k(u_2); \quad X_k(u_1 u_2) = u_1 X_k(u_2) + u_2 X_k(u_1). \quad (164)$$

We replace the function u in $X_i(u)$ by the left-hand side of the kth equation, i.e. by $X_k(u)$. On taking (164) into account, we obtain

$$X_i(X_k(u)) = \sum_{s=1}^{n} X_i(a_{ks}) u_{x_s} + \sum_{s=1}^{n} a_{ks} X_i(u_{x_s}),$$

and similarly:

$$X_k(X_i(u)) = \sum_{s=1}^{n} X_k(a_{is}) u_{x_s} + \sum_{s=1}^{n} a_{is} X_k(u_{x_s}).$$

Further, we can obviously write:

$$\sum_{s=1}^{n} a_{ks} X_i(u_{x_s}) = \sum_{s=1}^{n} a_{ks} \sum_{t=1}^{n} a_{it} u_{x_s x_t} = \sum_{s,t=1}^{n} a_{it} a_{ks} u_{x_s x_t},$$

and the last expression is unchanged on interchanging subscripts i and k, i.e.

$$\sum_{s=1}^{n} a_{is} X_k(u_{x_s}) = \sum_{=1}^{n} a_{ks} X_i(u_{x_s}),$$

so that we have the following expression:

$$X_i(X_k(u)) - X_k(X_i(u)) = \sum_{s=1}^{n} [X_i(a_{ks}) - X_k(a_{is})] u_{x_s}, \quad (165)$$

the right-hand side of which is a linear homogeneous function of $p_s = u_{x_s}$ with coefficients depending on x_k. Let us extend the concept of the Poisson bracket to the case of any number of independent variables. If φ and ψ are any functions of variables $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$, the Poisson bracket of these two functions can be defined by analogy with the previous definition by the expression:

$$(\varphi, \psi) = \sum_{j=1}^{n} (\varphi_{pj} \psi_{xj} - \varphi_{xj} \psi_{pj}). \quad (166)$$

Let us put $\varphi = X_i(u)$ and $\psi = X_k(u)$ in this formula. Now:

$$\varphi_{pj} = a_{ij}; \quad \psi_{xj} = \sum_{s=1}^{n} \frac{\partial a_{ks}}{\partial x_j} p_s; \quad \varphi_{xj} = \sum_{s=1}^{n} \frac{\partial a_{is}}{\partial x_j} p_s; \quad \psi_{pj} = a_{kj} \, .$$

On substituting these in the right-hand side of (166), we get:

$$(X_i(u), X_k(u)) = \sum_{s=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \frac{\partial a_{ks}}{\partial x_j} - \sum_{j=1}^{n} a_{kj} \frac{\partial a_{is}}{\partial x_j} \right) p_s \, ,$$

or

$$(X_i(u), X_k(u)) = \sum_{s=1}^{n} [X_i(a_{ks}) - X_k(a_{is})] \, p_s \, .$$

On comparison with the right-hand side of (165), we arrive at the important identity:

$$X_i(X_k(u)) - X_k(X_i(u)) = (X_i(u), X_k(u)) \, . \tag{167}$$

If u satisfies all the equations of system (163), i.e.

$$X_l(u) = 0 \qquad (l = 1, \ldots, m) \, ,$$

this function must satisfy also the linear homogeneous equation

$$(X_i(u), X_k(u)) = 0 \tag{168}$$

for any choice of subscripts i and k. On assigning all possible values to the subscripts, we can thus form $(1/2)\, m(m-1)$ new linear homogeneous equations, which are, in the sense indicated, a consequence of system (163). Certain of these new equations may reduce to identities, i.e. all the coefficients of the p_k may vanish. We associate in some definite order the new equations which do not reduce to identities with the equations of system (163), whilst investigating in each case whether or not the added equation is a linear combination of the equations already present. If it is a linear combination, we naturally discard it. Having carried out this process, we may arrive at a new system in which the number of equations is greater than m. We again form the Poisson brackets of the left-hand sides of the new system — without, of course, repeating the Poisson brackets which have already been formed for the original system. As above, we add to the system the new equations obtained. On proceeding in this way, two cases are possible. It may happen that we arrive at a system in which the number of equations is equal to n. Such a system has only the trivial solution $u = \text{const}$, so that the original system has also only a trivial solution. The second possibility is that we arrive at a system in which the number of equations is less than n, and for which all the new equations obtained with the aid of the Poisson bracket prove to be linear combinations of the equations of the system itself. Such a system is said to be *complete*. Hence it follows from the above discussion that *our original system either has only a trivial solution, or it is equivalent to a complete system;* and hence we arrive at the problem of integrating a complete system. We shall assume that our original system (163)

is already complete, i.e. all the possible Poisson brackets $(X_i(u), X_k(u))$ are linear combinations of the left-hand sides of the equations:

$$(X_i(u), X_k(u)) = \sum_{l=1}^{m} \beta_l^{(i,k)} X_l(u)\,, \tag{169}$$

where the coefficients $\beta_l^{(i,k)}$ are functions of the x_k, or else the brackets are dentically zero.

119. Complete and Jacobian systems. Let us consider some fundamental properties of complete systems. Instead of the x_k, we introduce the new independent variables

$$y_k = \varphi_k(x_1, \ldots, x_n) \qquad (k = 1, 2, \ldots, n)\,,$$

these equations being assumed soluble for the x_k. System (163) becomes in the new independent variables:

$$Y_j(u) = b_{j1}\frac{\partial u}{\partial y_1} + \ldots + b_{jn}\frac{\partial u}{\partial y_n} = 0 \qquad (j = 1, 2, \ldots, m)\,,$$

where, by the rule for differentiation of functions of a function:

$$b_{jl} = \sum_{s=1}^{n} a_{js}\frac{\partial \varphi_l}{\partial x_s} = X_j(y_l)\,. \tag{170}$$

Whatever the choice of function u, we have $Y_j(u) = X_j(u)$, where the right-hand side is expressed in terms of the independent variables x_k, and the left-hand side in terms of the independent variables y_k. Consequently, for any subscripts i and k:

$$X_i(X_k(u)) = Y_i(Y_k(u))$$

and

$$X_i(X_k(u)) - X_k(X_i(u)) = Y_i(Y_k(u)) - Y_k(Y_i(u))\,.$$

On taking (167) and (169) into account, we can write:

$$Y_i(Y_k(u)) - Y_k(Y_i(u)) = \sum_{l=1}^{m} \gamma_l^{(i,k)} Y_l(u)\,,$$

where the coefficients $\gamma_1^{(i,k)}$ are obtained from the $\beta_1^{(i,k)}$ simply by passage to the new independent variables. It is thus seen that, *if the original system is complete, the new system obtained as the result of any change of the independent variables is also complete.*

We now consider a second property of complete systems. We form m linear combinations of the left-hand sides of equations (163):

$$Z_j(u) = d_{j1}X_1(u) + \ldots + d_{jm}X_m(u) \qquad (j = 1, \ldots, m)\,,$$

where the coefficients d_{jl} are assumed dependent on the x_k, and the determinant formed from these coefficients is assumed non-zero. With these assumptions, the system of equations

$$Z_j(u) = 0 \qquad (j = 1, 2, \ldots, m) \tag{171}$$

will obviously be equivalent to system (163). We shall in fact describe the new system as *equivalent* to system (163). We show that, *if the original system is complete, any system equivalent to it is also complete.* In fact, the Poisson bracket $(Z_i(u), Z_k(u))$ consists of a sum of expressions of the form:

$$d_{ip}\, X_p(d_{kq}\, X_q(u)) - d_{kq}\, X_q(d_{ip}\, X_p(u))\,,$$

or, by (164), a sum of expressions of the form:

$$d_{ip}[X_p(d_{kq})\, X_q(u) + d_{kq}\, X_p(X_q(u))] - d_{kq}[X_q(d_{ip})\, X_p(u) + d_{ip}\, X_q(X_p(u))]$$
$$= d_{ip}\, X_p(d_{kq})\, X_q(u) - d_{kq}\, X_q(d_{ip})\, X_p(u) + d_{ip}\, d_{kq}[X_p(X_p(u)) - X_q(X_p(u))]\,.$$

On observing that all the expressions $X_p(X_q(u) - X_q(X_p(u))$ are linear combinations of $X_j(u)$, we see that the Poisson bracket $(Z_i(u), Z_k(u))$ is expressible linearly in terms of the $X_j(u)$, and consequently in terms of the $Z_j(u)$ also, which proves the completeness of system (171). We now introduce a new concept, which is a particular case of the concept of completeness: we say that system (163) is a *Jacobian system* if all the Poisson brackets $(X_i(u), X_k(u))$ vanish identically for it, i.e. if all the coefficients of the p_s in these brackets are identically zero. It is easy to transform a complete to a Jacobian system with the aid of elementary algebraic operations. In fact, let us take the original system (163), which we shall assume complete. Since the equations of this system are linearly independent, the matrix of its coefficients has rank m, and we can solve the equations for m of the p_s. We can assume without loss of generality that the equations of the system are soluble for $p_1, \ldots, p_m$, i.e. instead of system (163) we can write the equivalent system:

$$\left.\begin{aligned} p_1 + c_{1,m+1}p_{m+1} + \cdots + c_{1,n}p_n &= 0 \\ p_2 + c_{2,m+1}p_{m+1} + \cdots + c_{2,n}p_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\ p_m + c_{m,m+1}p_{m+1} + \cdots + c_{m,n}p_n &= 0\,. \end{aligned}\right\} \tag{172}$$

This system, by what has been proved above, must be complete. We show that it is in fact Jacobian. As before, we write $X_i(u)$ for the left-hand sides of the equations of the system. We have to show that all the coefficients $\beta_1^{(i,k)}$ in formula (169) vanish identically. It follows at once from the form of system (172) and the definition of Poisson bracket that the expression on the left-hand side of (169) does not contain p_s for $s \leqslant m$, whilst on the right-hand side the coefficient of $p_s\,(s \leqslant m)$ is obviously equal to $\beta_s^{(i,k)}$. Hence it follows at once that all the coefficients $\beta_s^{(i,k)}$ must vanish, i.e. system (172) is in fact Jacobian. We remark that not every Jacobian system is necessarily of form (172), but, by what has been proved, *on reduction of a complete system to form* (172), *the latter proves to be Jacobian.*

120. Integration of complete systems. Instead of integrating the complete system (163), we can integrate the equivalent Jacobian system (172).

We take the first equation of this system and the system of ordinary differential equations corresponding to it:

$$\frac{dx_1}{1} = \frac{dx_2}{0} = \ldots = \frac{dx_m}{0} = \frac{dx_{m+1}}{c_{1,\,m+1}} = \ldots = \frac{dx_n}{c_{1,\,n}}\,.$$

This system must have $(n-1)$ independent solutions

$$\varphi_2(x_1, \ldots, x_n) = C_2; \quad \ldots; \quad \varphi_n(x_1, \ldots, x_n) = C_n,$$

where the left-hand sides of these equations must be solutions of the first of equations (172). We remark that $(m-1)$ solutions can be written down at once, viz.

$$x_2 = \text{const}; \quad \ldots; \quad x_m = \text{const}.$$

We introduce $(n-1)$ new variables.

$$y_s = \varphi_s(x_1, \ldots, x_n) \qquad (s = 2, \ldots, n). \tag{173}$$

By virtue of the independence of the solutions, the above equations must be soluble with respect to $(n-1)$ of the variables x_k, and we can choose the function $\varphi_1(x, \ldots, x_n)$ such that the complete change of variables

$$y_s = \varphi_s(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n)$$

is soluble for all the x_k. If, for instance, equations (173) are soluble for $x_1, \ldots, x_{n-1}$, it is sufficient for us to take $\varphi_1 = x_n$. We transform system (172) to the new independent variables. Using (170), and the fact that $\varphi_2, \ldots, \varphi_n$ are solutions of the first of equations (172), we see that the first of the equations written reduces to the form: $\partial u/\partial y_1 = 0$. Using this equation, we can strike out all the terms containing $\partial u/\partial y_1$ in the remaining $(m-1)$ equations, and, by virtue of the linear independence of the equations, we can solve these equations for $(m-1)$ of the derivatives $\partial u/\partial y_s$. We can assume without loss of generality that the remaining equations can be solved for $\partial u/\partial y_2, \ldots, \partial u/\partial y_m$. Thus, the transformed system has the form

$$\left.\begin{aligned} Y_1(u) &= \frac{\partial u}{\partial y_1} = 0 \\ Y_2(u) &= \frac{\partial u}{\partial y_2} + h_{2,\,m+1}\frac{\partial u}{\partial y_{m+1}} + \ldots + h_{2n}\frac{\partial u}{\partial y_n} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_m(u) &= \frac{\partial u}{\partial y_m} + h_{m,\,m+1}\frac{\partial u}{\partial y_{m+1}} + \ldots + h_{mn}\frac{\partial u}{\partial y_n} = 0. \end{aligned}\right\} \tag{174}$$

The original system was Jacobian, and therefore complete, so that the transformed system must also be complete. But since it is soluble for the derivatives, it must also be Jacobian. We remark that it follows at once from the discussion of [119] that the transformation of a Jacobian system to new independent variables leads to a system which is also Jacobian.

The first of equations (174) shows that the function u must be independent of y_1. We show that the coefficients in the remaining equations of system (174) do not contain y_1. In fact, every expression:

$$Y_1(Y_i(u)) - Y_i(Y_1(u)) = \frac{\partial h_{i\,m+1}}{\partial y_1}\,\frac{\partial u}{\partial y_{m+1}} + \ldots + \frac{\partial h_{in}}{\partial y_1}\,\frac{\partial u}{\partial y_n}$$

must vanish identically by virtue of the fact that system (174) is Jacobian, which in fact proves our assertion. We can therefore throw away the first equation in system (174) and integrate the remainder on the assumption that

u does not depend on y_1. We thus arrive at a closed system of $(m - 1)$ equations with $(n - 1)$ independent variables. On carrying out the above-mentioned operation with this system, we arrive at a closed system of $(m - 2)$ equations with $(n - 2)$ independent variables and so on. We finally arrive at a single equation for the function u of $(n - m + 1)$ independent variables. On again writing $y_1, \ldots, y_{n-m+1}$ for these variables, we arrive at an equation of the form

$$\frac{\partial u}{\partial y_1} + g_2 \frac{\partial u}{\partial y_2} + \ldots + g_{n-m+1} \frac{\partial u}{\partial y_{n-m+1}} = 0,$$

where the independent variables y_j are functions of the original independent variables $x_1, \ldots, x_n$. The system of ordinary differential equations corresponding to this last equation has $(n - m)$ independent solutions:

$$\psi_1(y_1, \ldots, y_{n-m+1}) = C_1; \quad \ldots; \quad \psi_{n-m}(y_1, \ldots, y_{n-m+1}) = C_{n-m},$$

and the general solution of this equation takes the form

$$u = \Psi(\psi_1, \ldots, \psi_{n-m}),$$

where Ψ is an arbitrary function. This formula in fact gives the general solution of the original system (163).

121. Poisson brackets. We shall use the above results for developing a method for finding the complete integral of a non-linear first order equation in the case of any number of independent variables. As in the case of two independent variables, an auxiliary problem must be considered as a preliminary. Let the function $u(x_1, \ldots, x_n)$ be required, given its partial derivatives as functions of the independent variables x_k:

$$p_k = p_k(x_1, \ldots, x_n) \qquad (k = 1, 2, \ldots, n). \tag{175}$$

Since the result of differentiation is not dependent on the order, we see that functions (175) must satisfy the following $(1/2)\, n(n - 1)$ relationships:

$$\frac{\partial p_i(x_1, \ldots, x_n)}{\partial x_k} = \frac{\partial p_k(x_1, \ldots, x_n)}{\partial x_i}. \tag{176}$$

These relationships are sufficient as well as necessary for determining the function u. We have already proved this for the cases $n = 2$ and $n = 3$ [II, 73]. On generalizing Stokes' formula to the case of n-dimensional space, we discover, as in the case $n = 3$, that, when conditions (176) are satisfied, the line integral

$$u(x_1, \ldots, x_n) = \int^{(x_1, \ldots, x_n)} \sum_{s=1}^{n} p_s(x_1, \ldots, x_n)\, dx_s$$

does not depend on the path and gives a function u having the partial derivatives (175).

The sufficiency of conditions (176) can be proved in the general case by applying the method of complete induction. We shall assume that the sufficiency has been proved in the case of $(n - 1)$ independent variables, and we show that the assertion of sufficiency is now true as regards n variables. Suppose,

then, that functions (175) satisfy relationships (176). In view of our assumption that the conditions are sufficient for $(n-1)$ variables, we can use the first $(n-1)$ of functions (175) to construct a function u of independent variables $(x_1, \ldots, x_{n-1})$, having the partial derivatives: $u_{x_k} = p_k(x_1, \ldots, x_n)$ $(k = 1, 2, \ldots, n-1)$. This function will contain x_n as a parameter, since x_n appears in the p_k. In addition, we can add to the function u an arbitrary constant, which can be regarded as a function of the parameter x_n.

We thus obtain the function:

$$u(x_1, \ldots, x_{n-1}, x_n) + c(x_n),$$

which satisfies the $(n-1)$ conditions:

$$u_{x_k} = p_k(x_1, \ldots, x_n) \qquad (k = 1, 2, \ldots, n-1).$$

It further remains to choose $c(x_n)$ such that the condition $u_{x_n} = p_n(x_1, \ldots, x_n)$ is satisfied, which leads us to the equation:

$$\frac{\mathrm{d}c(x_n)}{\mathrm{d}x_n} = p_n(x_1, \ldots, x_n) - u_{x_n},$$

and it remains for us to verify that the right-hand side of this equation contains x_n only. On differentiating with respect to x_k for $k < n$ and taking (176) into account, together with the fact that $u_{x_k} = p_k$, we obtain:

$$\frac{\partial p_n}{\partial x_k} - \frac{\partial^2 u}{\partial x_n \partial x_k} = \frac{\partial p_n}{\partial x_k} - \frac{\partial}{\partial x_n}\left(\frac{\partial u}{\partial x_k}\right) = \frac{\partial p_n}{\partial x_k} - \frac{\partial p_k}{\partial x_n} \equiv 0,$$

which is what we had to prove.

We now suppose that the partial derivatives p_k are defined implicitly with the aid of the n equations:

$$F_s(x_1, \ldots, x_n, p_1, \ldots, p_n) = a_s \qquad (s = 1, 2, \ldots, n), \tag{177}$$

which we assume soluble with respect to the p_k. We show that the necessary and sufficient condition for the p_k defined by equations (177) to satisfy relationships (176) is that all the Poisson brackets of the left-hand sides of equations (177) vanish identically, i.e. we must have the following $(1/2)\, n(n-1)$ identities in x_i and p_i:

$$(F_i, F_k) = \sum_{s=1}^{n}\left(\frac{\partial F_i}{\partial p_s}\frac{\partial F_k}{\partial x_s} - \frac{\partial F_i}{\partial x_s}\frac{\partial F_k}{\partial p_s}\right) \equiv 0. \tag{178}$$

We assume here that the right-hand sides of equations (177) are arbitrary constants.

We take two of equations (177) and differentiate them with respect to the independent variable x_s:

$$\frac{\partial F_i}{\partial x_s} + \sum_{j=1}^{n}\frac{\partial F_i}{\partial p_j}\frac{\partial p_j}{\partial x_s} = 0; \qquad \frac{\partial F_k}{\partial x_s} + \sum_{j=1}^{n}\frac{\partial F_k}{\partial p_j}\frac{\partial p_j}{\partial x} = 0.$$

On multiplying the first of these equations by $\partial F_k/\partial p_s$, the second by $\partial F_i/\partial p_s$, subtracting the first from the second and summing over s, we get:

$$(F_i, F_k) + \sum_{j=1}^{n} \sum_{s=1}^{n} \frac{\partial F_k}{\partial p_j} \frac{\partial F_i}{\partial p_s} \frac{\partial p_j}{\partial x_s} - \sum_{j=1}^{n} \sum_{s=1}^{n} \frac{\partial F_i}{\partial p_j} \frac{\partial F_k}{\partial p_s} \frac{\partial p_j}{\partial x_s} = 0.$$

On changing the notation for the variables of summation in the second sum, we can rewrite the last formula as:

$$(F_i, F_k) + \sum_{j=1}^{n} \sum_{s=1}^{n} \frac{\partial F_k}{\partial p_j} \frac{\partial F_i}{\partial p_s} \left(\frac{\partial p_j}{\partial x_s} - \frac{\partial p_s}{\partial x_j} \right) = 0. \tag{179}$$

If p_k satisfies relationships (176), it follows at once from the last formulae that identities (178) must be fulfilled for any subscripts. We now suppose conversely that identities (178) are satisfied, and show that the p_k defined by (177) must satisfy relationship (176). If identities (178) are fulfilled, formula (179) can be rewritten as:

$$\sum_{=1}^{n} \sum_{s=1}^{n} \frac{\partial F_k}{\partial p_j} \frac{\partial F_i}{\partial p_s} \left(\frac{\partial p_j}{\partial x_s} - \frac{\partial p_s}{\partial x_j} \right) = 0,$$

where we can assign any values to the subscripts i and k. On assigning to the subscript k the values $k = 1, 2, \ldots, n$, we get n equations which can be regarded as n homogeneous equations in the n quantities:

$$\sum_{s=1}^{n} \frac{\partial F_i}{\partial p_s} \left(\frac{\partial p_j}{\partial x_s} - \frac{\partial p_s}{\partial x_j} \right) \qquad (j = 1, 2, \ldots, n). \tag{180}$$

The determinant of this homogeneous system is a functional determinant of functions F_s with respect to variables p_s, and we assume it to be different from zero [system (177) is soluble with respect to the p_s]. Consequently, we can say that quantities (180) must vanish. On fixing subscript j and assigning to i the values $i = 1, 2, \ldots, n$, we again obtain a homogeneous system in the quantities:

$$\frac{\partial p_j}{\partial x_s} - \frac{\partial p_s}{\partial x_j} \qquad (s = 1, 2, \ldots, n), \tag{181}$$

the determinant of which is again a functional determinant of the F_s with respect to the p_s. It follows at once from this that all the quantities (181) must vanish, which is what we wanted to prove. Thus, *the necessary and sufficient condition for system* (177) *to define p_k which are the partial derivatives of some function u is that the functions F_i are in involution in pairs.* We have assumed that the right-hand sides of equations (177) are arbitrary constants, and in connection with this it has been necessary to require that relationships (178) be satisfied identically. If we fix the values of certain of these constants, it is sufficient to require that relations (178) be satisfied by virtue of the equations thus obtained.

We shall mention further some elementary properties of the Poisson bracket. If φ and ψ are any two functions of variables x_k and p_k, and a and b are numbers,

it follows at once from the definition of Poisson bracket that the relationships hold:

$$(\varphi, \varphi) = 0; \quad (\psi, \varphi) = -(\varphi, \psi); \quad (0, \varphi) = 0; \qquad (a\varphi, b\psi) = ab(\varphi, \psi).$$

Let ω be a further function of the above-mentioned variables. The following identity holds:

$$((\varphi, \psi), \omega) + ((\psi, \omega), \varphi) + ((\omega, \varphi), \psi) = 0\,, \tag{1 82}$$

which is usually known as Poisson's identity. This identity contains doublePoisson brackets. To form the first term in the formula, we must form the Poisson bracket (φ, ψ) and then use the function thus obtained to form the bracket $((\varphi, \psi), \omega)$. To prove identity (182), we remark first of all that each term of the identity contains first order derivatives. In view of the symmetry of the identity with respect to all three functions, and also with respect to variables x_k and p_k, all we need do to prove the identity is to verify that all the terms containing $\partial\varphi/\partial p_k$ vanish on the left-hand side. By using the definition of the Poisson bracket, it can be seen that the coefficient of $\partial\varphi/\partial p_k$ on the left-hand side of the identity is:

$$\sum_{s=1}^{n}\left[\frac{\partial}{\partial p_s}\left(\frac{\partial\psi}{\partial x_k}\right)\frac{\partial\omega}{\partial x_s} - \frac{\partial}{\partial x_s}\left(\frac{\partial\psi}{\partial x_k}\right)\frac{\partial\omega}{\partial p_s}\right] -$$

$$-\frac{\partial}{\partial x_k}\sum_{s=1}^{n}\left(\frac{\partial\psi}{\partial p_s}\frac{\partial\omega}{\partial x_s} - \frac{\partial\psi}{\partial x_s}\frac{\partial\omega}{\partial p_s}\right) - \sum_{s=1}^{n}\left[\frac{\partial}{\partial p_s}\left(\frac{\partial\omega}{\partial x_k}\right)\frac{\partial\psi}{\partial x_s} - \frac{\partial}{\partial x_s}\left(\frac{\partial\omega}{\partial x_k}\right)\frac{\partial\psi}{\partial p_s}\right].$$

By carrying out the differentiations, we easily see that this coefficient in fact vanishes.

122. Jacobi's method. We now turn to the generalization of the Lagrange–Charpy method, viz., to the solution of the problem of seeking the complete integral of a first order equation with any number of independent variables; it will be assumed here that the equation does not contain the required function, i.e. that it has the form:

$$F_1(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0. \tag{183}$$

If we can succeed in choosing a further $(n-1)$ functions F_k such that the n functions obtained are in involution in pairs and such that they are soluble for the p_k, we can find p_k satisfying conditions (176) by taking system (177) in which we have substituted $a_1 = 0$. By this means, we can find u. System (177) gives us $(n-1)$ arbitrary constants, then a further arbitrary constant is obtained on finding u from its partial derivatives p_k. The functions f_k can be found in stages. Suppose that the first m functions $F_1, F_2, \ldots, F_m$ have already been found so that they are in involution in pairs and can be solved for m of the p_k. To obtain the next function F_{m+1} we have to form the m equations:

$$(F_1, u) = 0; \quad (F_2, u) = 0; \quad \ldots; \quad (F_m, u) = 0. \tag{184}$$

These are linear homogeneous equations for the required function F_{m+1} of $2n$ independent variables x_k and p_k.

We write the system for F_{m+1} in the expanded form:

$$\sum_{k=1}^{n}\left(\frac{\partial F_j}{\partial p_k}\frac{\partial u}{\partial x_k}-\frac{\partial F_j}{\partial x_k}\frac{\partial u}{\partial p_k}\right)=0 \qquad (j=1,\ldots,m). \tag{185}$$

Since the F_j are assumed soluble for m of the p_k, we must assume that some determinant of order m of the functions F_j in the variables p_k differs from zero, so that the rank of the matrix of coefficients of the derivatives in system (185) is equal to m, i.e. equations (185) are certainly linearly independent. We show that the system is complete. This is done by forming differences (165) for system (184):

$$(F_p,(F_q,u))-(F_q,(F_p,u)).$$

We have to show that these vanish identically. On applying identity (182), the differences can be transformed to:

$$-((F_q,u),F_p)-((u,F_p),F_q)=((F_p,F_q),u).$$

But functions F_p and F_q are in involution, whence it follows at once that the differences in question vanish. Therefore, by what was said in [120], system (185) has $2n-m$ independent solutions. We have the obvious solutions of this system:

$$u=F_1;\quad u=F_2;\quad \ldots;\quad u=F_m. \tag{186}$$

Consequently, there must exist in addition to these a further $2n-2m$ solutions, which must be soluble simultaneously with solutions (186) for $(2n-m)$ of the variables x_k and p_k. Thus there must certainly be a solution $u=F_{m+1}$ of system (185) such that the equations $F_1=0$, $F_2=a_2$; $\ldots$; $F_{m+1}=a_{m+1}$ are soluble for $(m+1)$ of the p_k. To find the next function F_{m+2}, we construct the system:

$$(F_1,u)=0;\quad \ldots;\quad (F_{m+1},u)=0,$$

for which we can perform the same process as for system (184). We can construct in this way all n functions such that they are in involution in pairs, and system (177) (with $a_1=0$) is soluble for all the p_k. As seen above, this in fact leads us to the complete integral of equation (183).

We have assumed that the equation does not contain the required function. If we have an equation containing this function:

$$F(x_1,\ldots,x_n,u,p_1,\ldots,p_n)=0,$$

by increasing the number of independent variables by one, we can arrive at an equation which does not contain the required function. For this, it is sufficient to seek the solution of the equation in the implicit form:

$$v(x_1,\ldots,x_n,u)=C,$$

where C is an arbitrary constant. On applying the implicit function differentiation rule, an equation is obtained as usual for v which does not contain the function itself.

123. Canonical systems. Let us establish the connection of the previous discussion with the Cauchy system. We shall take the case when the equation does not contain the required function and is soluble for one of the derivatives. For symmetry, we introduce $(n + 1)$ independent variables and denote one of these variables by t, whilst the derivative with respect to t is written as $p_0 = u_t$. The equation will have the form:

$$p_0 + H(t, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0, \tag{187}$$

whilst the corresponding Cauchy system will be the canonical system [115]:

$$\frac{dx_k}{dt} = H_{p_k}, \quad \frac{dp_k}{dt} = -H_{x_k}. \tag{188}$$

Let

$$\varphi(t, x_1, \ldots, x_n, p_1, \ldots, p_n) = C$$

be a solution of this system, i.e.

$$\varphi_t + \sum_{k=1}^{n} \left(\varphi_{x_k} \frac{dx_k}{dt} + \varphi_{p_k} \frac{dp_k}{dt} \right) = 0$$

by virtue of system (188). We can write the last equation in the alternative form

$$\varphi_t + (H, \varphi) = 0. \tag{189}$$

Consequently, the necessary and sufficient condition for the function φ to give a solution of the system is that it satisfy equation (189). Suppose that φ and ψ give two solution of the system. We show that their Poisson bracket (φ, ψ) is also a solution of the system (or becomes a constant). It follows at once from the definition of Poisson bracket that

$$\frac{\partial}{\partial t}(\varphi, \psi) = (\varphi_t, \psi) + (\varphi, \psi_t).$$

On substituting the function $\omega = (\varphi, \psi)$ instead of φ in (189), we get:

$$(\varphi_t, \psi) + (\varphi, \psi_t) + (H, (\varphi, \psi)) = 0.$$

But φ and ψ are solutions, so that we can substitute in the last equation:

$$\varphi_t = -(H, \varphi); \quad \psi_t = -(H, \psi),$$

and hence arrive at the relationship:

$$-((H, \varphi), \psi) - (\varphi, (H, \psi)) + (H, (\varphi, \psi)) = 0,$$

which is satisfied as an identity by virtue of (182). Thus, *the Poisson bracket of two solution of a canonical system is also a solution of the system, or else is constant.*

We now suppose that the n solutions of system (188) are given:

$$\varphi_s(t, x_1, \ldots, x_n, p_1, \ldots, p_n) = a_s \qquad (s = 1, 2, \ldots, n), \tag{190}$$

these being in involution in pairs and soluble for the p_k. We associate the differential equation (187) itself with equations (190) and show that the $(n + 1)$ functions obtained are in involution in pairs, if we take into account the inde-

pendent variables $t, x_1, \ldots, x_n$ and the corresponding derivatives $p_0, p_1, \ldots, p_n$. Functions (190) are obviously in involution in pairs even after adding the independent variable t, since none of them contain p_0. It is sufficient here to show that each of functions (190) is in involution with the left-hand side of equation (187). On equating to zero the corresponding Poisson bracket, we arrive at the equation:

$$\frac{\partial \varphi_s}{\partial t} + (H, \varphi_s) = 0\,,$$

which is certainly satisfied, since functions (190) are solutions of system (188). On taking into account the results of [122], we can say that, if equations (190) are solved for the p_k $(k = 1, \ldots, n)$ and equation (187) for p_0, and if the expressions obtained for the p_k are substituted in function H, the sum

$$p_1\,dx_1 + p_2\,dx_2 + \ldots + p_n\,dx_n - H\,dt$$

will be the total differential of some function $v(t, x_1, \ldots, x_n, a_1, \ldots, a_n)$. It will obviously yield the complete integral of equation (187). By using Jacobi's theorem, we can say that the remaining n solutions of canonical system (190) can be obtained by simple differentiation, viz., they are defined by the equations $v_{a_k} = b_k$ $(k = 1, \ldots, n)$.

124. Examples. 1. Let us take the system of two linear homogeneous equations:

$$\left.\begin{aligned} X_1 &= p_1 + (x_2 + x_4 - 3x_1)\,p_3 + (x_3 + x_1 x_2 + x_1 x_4)\,p_4 = 0 \\ X_2 &= p_2 + (x_3 x_4 - x_2)\,p_3 + (x_1 x_3 x_4 + x_2 - x_1 x_2)\,p_4 = 0\,. \end{aligned}\right\} \tag{191}$$

On forming the Poisson bracket of the left-hand sides, we obtain the further equation:

$$X_3 = p_3 + x_1 p_4 = 0\,.$$

The Poisson brackets (X_1, X_3) and (X_2, X_3) differ only by a factor from the left-hand side of the last equation. We thus have a complete system consisting of three equations. On solving it for p_1, p_2, p_3, we get the Jacobian system:

$$p_1 + (x_3 + 3x_1^2)\,p_4 = 0; \quad p_2 + x_2 p_4 = 0; \quad p_3 + x_1 p_4 = 0\,.$$

The last equation has the solutions: x_1, x_2 and $x_4 - x_1 x_3$. We introduce the independent variables: x_1, x_2, x_3 and $t = x_4 - x_1 x_3$. The system can be rewritten as:

$$\frac{\partial u}{\partial x_1} + 3x_1^2 \frac{\partial u}{\partial t} = 0\,; \quad \frac{\partial u}{\partial x_2} + x_2 \frac{\partial u}{\partial t} = 0, \quad \frac{\partial u}{\partial x_3} = 0\,.$$

The first two equations give a Jacobian system with independent variables: x_1, x_2, t. The second of them has the solution: x_1 and $t - (1/2)\,x_2^2$. We introduce the new independent variables x_1, x_2 and $\tau = t - (1/2)\,x_2^2$. The equations mentioned can be rewritten as:

$$\frac{\partial u}{\partial x_1} + 3x_1^2 \frac{\partial u}{\partial \tau} = 0\,; \quad \frac{\partial u}{\partial x_2} = 0\,.$$

The first of these equations has the solution:

$$u = \tau - x_1^3 \quad \text{or} \quad u = x_4 - x_1 x_3 - \frac{x_2^2}{2} - x_1^3,$$

and an arbitrary function of this argument is a solution of system (191).

2. Let us find the complete integral of the equation:

$$F_1 = (p_1 + x_2)^2 + (x_1 + p_2)^2 - x_3 p_3 = 0. \tag{192}$$

The equation $(F_1, u) = 0$ has the form:

$$2(p_1 + x_2)\frac{\partial u}{\partial x_1} + 2(x_1 + p_2)\frac{\partial u}{\partial x_2} - x_3\frac{\partial u}{\partial x_3} - 2(x_1 + p_2)\frac{\partial u}{\partial p_1} -$$
$$- 2(p_1 + x_2)\frac{\partial u}{\partial p_2} + p_3\frac{\partial u}{\partial p_3} = 0. \tag{193}$$

This equation has the obvious solution $u = x_1 + p_2$. We put:

$$F_2 = x_1 + p_2 = a_2. \tag{192$_1$}$$

We associate with equation (193) the equation $(F_2, u) = 0$, i.e.

$$\frac{\partial u}{\partial p_1} - \frac{\partial u}{\partial x_2} = 0.$$

This equation and (193) have the solution: $u = x_3 p_3$, i.e.

$$F_3 = x_3 p_3 = a_3. \tag{192$_2$}$$

We solve (192), (192$_1$) and (192$_2$) for p_1, p_2, p_3:

$$p_1 = -x_2 + \sqrt{a_3 - a_2^2}; \quad p_2 = a_2 - x_1; \quad p_3 = \frac{a_3}{x_3}.$$

On establishing the function u from its partial derivatives, we obtain the complete integral of equation (192):

$$u = -x_1 x_2 + \sqrt{a_3 - a_2^2}\, x_1 + a_2 x_2 + a_3 \log x_3 + a.$$

125. The method of majorant series. In our investigation of Cauchy's problem we have assumed that the given and required functions are real functions of real independent variables. The whole of the theory can be worked out on the assumption that all the functions are analytic in their arguments. In this case the arguments can take both real and complex values. We shall start from this stand-point in our proof of the existence and uniqueness theorem for the solution of Cauchy's problem. As a preliminary, certain supplementary propositions must be discussed.

Suppose we have a power series in m variables:

$$\varphi(z_1, \ldots, z_m) = \sum_{p_1, \ldots, p_m = 0}^{\infty} a_{p_1 \ldots p_m} z_1^{p_1}, \ldots, z_m^{p_m}, \tag{194}$$

convergent under the conditions:

$$|z_1| \leqslant R_1; \quad \ldots; \quad |z_m| \leqslant R_m. \tag{195}$$

We shall assume that the radius of convergence of series (194) is in fact somewhat greater than the numbers R_k. Let M be the greatest value of the modulus of function (194) when conditions (195) are satisfied. We have seen that the power series obtained by expansion of the function [III_2, 83]:

$$\frac{M}{\left(1-\frac{z_1}{R_1}\right)\left(1-\frac{z_2}{R_2}\right)\cdots\left(1-\frac{z_m}{R_m}\right)} \tag{196}$$

has positive coefficients throughout, which are not less than the moduli of the coefficients of series (194). In other words, the latter series is a *majorant* of series (194).

In general, a *majorant* of a series

$$\sum_{p_1, \ldots, p_m=0}^{\infty} c_{p_1 \ldots p_m} z_1^{p_1}, \ldots, z_m^{p_m} \tag{197}$$

is defined as a series of the same form in which the coefficients are non-negative (i.e. greater than or equal to zero) and are not less than the moduli of the corresponding coefficients of series (197). As we know, every power series is absolutely convergent inside its circle of convergence [III_2, 83]. If a majorant series of series (197) is convergent for $|z_k| < \varrho_k$ ($k = 1, 2, \ldots, m$), we can obviously assert that series (197) is convergent inside the circles $|z_k| < \varrho_k$. On taking all the numbers R_k as the same in expression (196) (all the R_k can be replaced by the greatest), we can consider the two functions:

$$\frac{M}{\left(1-\frac{z_1}{R}\right)\left(1-\frac{z_2}{R}\right)\cdots\left(1-\frac{z_m}{R}\right)} \tag{198}$$

and

$$\frac{M}{1-\frac{z_1+z_2+\ldots+z_m}{R}} \tag{199}$$

The expansions of these functions in power series have the forms respectively:

$$\sum_{p_1 \ldots, p_m=0}^{\infty} M \frac{z_1^{p_1} \ldots z_m^{p_m}}{R^{p_1+\ldots+p_m}} \text{ and } \sum_{p=0}^{\infty} M \frac{(z_1+\ldots+z_m)^p}{R^p},$$

and it may be seen by expanding $(z_1 + \ldots + z_m)^p$ that the coefficients in the expansion of function (199) are not less than the corresponding coefficients in expansion (198), i.e. function (199) (or the corresponding power series) will also be a majorant for function (197) (i.e. for the corresponding power series).

The method of majorant power series is used for proving the existence of solutions of differential equations in the case of analytic functions. We shall first give the proof for an ordinary first order differential equation. Suppose we have the equation:

$$\frac{dy}{dx} = f(x, y),$$

the right-hand side of which is a power series in x and y, convergent in the neighbourhood of $x = y = 0$, i.e.

$$\frac{dy}{dx} = \sum_{p,q=0}^{\infty} a_{pq} x^p y^q. \tag{200}$$

Let us seek a solution of this equation, regular at the point $x = 0$ and satisfying the initial condition:

$$y|_{x=0} = 0. \tag{201}$$

To obtain the required solution, it is sufficient to form its Maclaurin series, i.e. to evaluate the derivatives at $x = 0$. The constant term of the Maclaurin series is given by initial condition (201) and vanishes. The value of the first derivative at $x = 0$ is given by the differential equation, and we have $y_0' = a_{00}$. To find the second derivatives, we differentiate both sides of the equation with respect to x:

$$y'' = \sum_{p,q=0}^{\infty} p a_{pq} x^{p-1} y^q + \sum_{p,q=0}^{\infty} q a_{pq} x^p y^{q-1} y'$$

and substitute $x = 0$, $y = 0$, $y' = a_{00}$ in its right-hand side. The value of the second derivative at $x = 0$ is thus obtained as

$$y_0'' = a_{10} + a_{01} a_{00}.$$

On proceeding in this way, we can find the derivatives of all orders at $x = 0$ and form the Maclaurin series:

$$y_0 + \frac{y_0'}{1!} x + \frac{y_0''}{2!} x^2 + \ldots \tag{202}$$

It follows from the above working that there can exist only one solution which is regular and satisfies the initial condition. But in order to

assert that such a solution in fact exists, we have to show that series (202) has a radius of convergence greater than zero. We remark here that all the above operations performed on the series are valid by virtue of the fundamental properties of a power series inside its circle of convergence. If series (202) proves to be convergent, it follows at once from the actual procedure for obtaining its coefficients that its sum satisfies equation (200).

A direct consequence of the above working is that the coefficients of series (202) are polynomials in a_{pq} with non-negative numerical coefficients. For, during the successive differentiations of the equation and the substitution in its right-hand side of the initial values of the derivatives already obtained, we do no more than perform on the coefficients the operations of addition and multiplication. Hence, if we replace the series on the right-hand side of equation (200) by a majorant series, series (200) will also be replaced by a majorant series. If this majorant series proves to be convergent for x sufficiently close to zero, then series (202) for equation (200) will be all the more convergent. The vital point in the following proof is that the substitution of a majorant series for the series on the right-hand side of (200) leads us to an equation which can be integrated in a closed form. Suppose that the series on the right of (200) is absolutely and uniformly convergent for $|x| \leqslant R$ and $|y| \leqslant R$, and let M be the greatest value of the sum of this series under these conditions. On passing to a majorant series, we obtain the differential equation

$$\frac{dy}{dx} = \frac{M}{\left(1 - \frac{x}{R}\right)\left(1 - \frac{y}{R}\right)}, \tag{203}$$

in which the variables are separated:

$$\left(1 - \frac{y}{R}\right) dy = \frac{M}{\left(1 - \frac{x}{R}\right)} dx .$$

On integrating and taking (201) into account, we obtain:

$$y - \frac{y^2}{2R} = -MR \log\left(1 - \frac{x}{R}\right),$$

whence

$$y = R - R\sqrt{1 + 2M \log\left(1 - \frac{x}{R}\right)}, \tag{204}$$

where the value of the radical must be taken equal to unity for $x = 0$, i.e. such that initial condition (201) is satisfied. Function (204) is regular at the point $x = 0$, and can therefore be expanded in a power series.

The coefficients of this series obviously coincide with the coefficients that are obtained by the process described above of term by term differentiation of equation (203). Therefore, series (202) for the majorant equation is convergent in the neighbourhood of $x = 0$. It will be all the more convergent for the original equation, as we saw above. This proves not only the uniqueness, but also the existence of the regular solution of equation (200) satisfying the initial condition (201).

126. Kovalevskaya's theorem. We shall use the above method of majorant series or functions to prove the existence and uniqueness of the solution of Cauchy's problem for a partial differential equation. We shall here always consider equations of an explicit type. Suppose we have the first order equation:

$$p_1 = f(x_1, \ldots, x_n, u, p_2, \ldots, p_n), \tag{205}$$

where f is a regular function at the point:

$$x_1 = \ldots = x_n = 0; \quad u = u^{(0)},\ p_2 = p_2^{(0)}, \ldots; \quad p_n = p_n^{(0)}, \tag{206}$$

and the initial values of the independent variables can be taken as zero without loss of generality. We seek the solution satisfying the following Cauchy condition:

$$u\,|_{x_1=0} = \varphi(x_2, \ldots, x_n). \tag{207}$$

The function $\varphi(x_2, \ldots, x_n)$ is here assumed regular when its arguments become zero, and in addition,

$$(\varphi)_0 = u^{(0)}; \quad (\varphi_{x_k})_0 = p_k^{(0)} \qquad (k = 2, \ldots, n), \tag{208}$$

whilst the zero subscript always indicates that all the arguments are to be replaced by zero. Before solving this problem we shall simplify the Cauchy condition by means of an elementary change of the required function: instead of u we introduce the new required function u' in accordance with the formula:

$$u = u' + \varphi(x_2, \ldots, x_n) + Ax_1,$$

where the constant A is the value of the right-hand side of (205) with the initial values (206) of the arguments, i.e. in simpler language, A is the constant term in the expansion of the right-hand side of (205) in a power series:

$$A = f(0, \ldots, 0, (\varphi)_0, (\varphi_{x_2})_0, \ldots (\varphi_{x_n})_0) .$$

The new required function must satisfy the equation:

$$\begin{aligned} u'_{x_1} = f(x_1, \ldots, x_n, u' + \varphi + Ax_1, \ \varphi_{x_2} + u'_{x_2}, \ldots, \ \varphi_{x_n} + u'_{x_n}) - \\ - f(0, \ldots, 0, (\varphi)_0, (\varphi_{x_2})_0, \ldots, (\varphi_{x_n})_0) , \end{aligned} \tag{209}$$

and instead of (207) we have the initial condition:

$$u' |_{x_1=0} = 0 .$$

Let us consider the arguments of the function on the right-hand side of (209). The argument $u' + \varphi + Ax_1$ becomes equal to $(\varphi)_0$ if we put all the $x_s = 0$ and $u' = 0$. Similarly, each of the arguments $\varphi'_{x_k} + u'_{x_k}$ becomes equal to $(\varphi_{x_k})_0$ if we again put all the $x_s = 0$ and $u_{x_k} = 0$. Therefore the arguments of this function for zero values of x_s, u', u'_{x_k} coincide precisely with the initial values (208), for which function f is regular. We can therefore say that the right-hand side of (209) is a regular function at the point:

$$x_1 = \ldots = x_n = u' = u'_{x_2} = \ldots = u'_{x_n} = 0 . \tag{210}$$

In addition, on taking into account the subtracted term on the right of (209), we can say that this side vanishes for the initial values (210) of the arguments. We have thus reduced the initial Cauchy value and all the initial values of the arguments of the function on the right of the equation to zero. If we retain our previous notation, we now obtain the following problem: given the differential equation:

$$p_1 = f(x_1, \ldots, x_n, u, p_2, \ldots, p_n) , \tag{211}$$

where f is a regular function at the point:

$$x_1 = \ldots = x_n = u = p_2 = \ldots = p_n = 0 ,$$

and vanishes at this point, we require the solution satisfying the condition:

$$u |_{x_1=0} = 0 . \tag{212}$$

We remark that the right-hand side of (211) must be expandable in a series of the form:

$$f = \sum_{s_1, \ldots, s_n, t_1, \ldots, t_n = 0}^{\infty} a_{s_1 \ldots s_n t_1 \ldots t_n} x_1^{s_1} \ldots x_n^{s_n} u^{t_1} p_2^{t_2} \ldots p_n^{t_n}$$

$$(a_{0 \ldots 00 \ldots 0} = 0)\,, \tag{213}$$

convergent for all values of the arguments sufficiently close to zero.

Precisely as in the case of an ordinary differential equation, we can use equation (211) and the initial condition (212) to evaluate the coefficients of the Maclaurin series of the required function u, i.e. the values of all the partial derivatives for zero values of the arguments. When differentiating with respect to any argument except x_1, we can first put $x_1 = 0$.

Hence, the initial condition (212) shows us that

$$\left(\frac{\partial^{\alpha_2 + \ldots + \alpha_n} u}{\partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \right)_0 = 0\,, \tag{214}$$

where α_k are any desired non-negative integers. We shall now work out the initial values of the derivatives in which differentiation with respect to x_1 plays a part. It follows from (211) that:

$$\left(\frac{\partial u}{\partial x_1} \right)_0 = 0\,. \tag{215}$$

On differentiating both sides of (211) any number of times with respect to the variables $x_2, \ldots, x_n$, then introducing zero values of the arguments, we obtain on the right-hand side the values already calculated for derivatives (214) and (215), and hence find:

$$\left(\frac{\partial^{1 + \alpha_2 + \ldots + \alpha_n} u}{\partial x_1 \, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \right)_0$$

for any non-negative values $\alpha_2, \ldots, \alpha_n$. We now take the equation which is obtained from equation (211) by differentiation with respect to x_1, and follow the same procedure with this as with the original equation. This gives us completely defined values for the derivatives:

$$\left(\frac{\partial^{2 + \alpha_2 + \ldots + \alpha_n} u}{\partial x_1^2 \, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \right)_0 .$$

On continuing in this way, we can work out any partial derivative of the required function for the initial values of the arguments, and form

the Maclaurin series:

$$\sum_{a_1, \ldots, a_n=0}^{\infty} \frac{1}{a_1! \ldots a_n!} \left(\frac{\partial^{a_1+\ldots+a_n} u}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}\right)_0 x_1^{a_1} \ldots x_n^{a_n}. \tag{216}$$

The foregoing arguments, like those in the case of an ordinary differential equation, prove the uniqueness of the regular solution of our Cauchy problem. To prove the existence, we have to show that series (216) is convergent in certain circles with centre at the origin when the initial values obtained for the derivatives are substituted in it. We can say, precisely as in the previous section, that, if we replace series (213) by a majorant series, and if series (216), formed in the way described above for the majorant equation thus obtained, is convergent, it will be all the more convergent for the original equation. Suppose that series (213) is absolutely and uniformly convergent under the condition:

$$|x_1| \leqslant \varrho; \ \ldots; \ |x_n| \leqslant \varrho; \ |u| \leqslant \varrho; \ |p_2| \leqslant R; \ \ldots; \ |P_n| \leqslant R,$$

and let M be the greatest value of the modulus of the sum of this series under these conditions. The function

$$\frac{M}{\left(1-\frac{x_1}{\varrho}\right)\ldots\left(1-\frac{x_n}{\varrho}\right)\left(1-\frac{u}{\varrho}\right)\left(1-\frac{p_2}{R}\right)\ldots\left(1-\frac{p_n}{R}\right)} - M$$

will be majorant for (213), the number M being subtracted from the right-hand side of the previous formula in order to get rid of the constant term which is in fact absent in series (213). The function:

$$\frac{M}{\left(1-\frac{x_1+\ldots+x_n+u}{\varrho}\right)\left(1-\frac{p_2+\ldots+p_n}{R}\right)} - M$$

will be all the more majorant for series (213). If we divide the variable x_1 by some number a satisfying the condition $0 < a < 1$, the various powers of this number will appear in the denominators of the coefficients of the terms containing powers of x_1, and the function

$$\frac{M}{\left(1-\frac{\frac{x_1}{a}+x_2\ldots+x_n+u}{\varrho}\right)\left(1-\frac{p_2+\ldots+p_n}{R}\right)} - M$$

will still more be majorant for (213). We thus have the majorant equation:

$$p_1 = \frac{M}{\left(1 - \dfrac{\dfrac{x_1}{a} + x_2 + \ldots + x_n + u}{\varrho}\right)\left(1 - \dfrac{p_2 + \ldots + p_n}{R}\right)} - M \quad (217)$$

By evaluating the Maclaurin coefficients for the solution of this equation satisfying the initial condition (212), we obtain a power series which vanishes for $x_s = 0$, and which is majorant for series (216) formed for equation (211). If this power series proves to be convergent, series (216), formed for equation (211), will be all the more convergent. We next find the solution of equation (217) which satisfies

$$u\,|_{x_1=0} = \psi(x_2, \ldots, x_n)\,, \quad (218)$$

instead of the zero initial condition, ψ being a power series with non-negative coefficients. Successive evaluation of the Maclaurin coefficients for this solution can be performed precisely as above, except that the initial condition (218) leads to the appearance of certain non-negative numbers instead of zeros on the right-hand side of (214) for all non-negative values of the a_k. The evaluation of the further coefficients is carried out as above, and amounts to the operations of addition and multiplication on the non-negative coefficients already obtained and on the positive coefficients in the expansion of the right-hand side of equation (218). Therefore, if we replace the zero initial condition (212) by initial condition (218) for equation (217), ψ being expanded as a series with real non-negative coefficients, series (216) for equation (217) with initial condition (218) will be majorant with respect to series (216) for equation (217) with the zero initial condition (212), and all the more majorant with respect to series (216) for equation (211) with initial condition (212). Therefore, it is all a question of proving that series (216) for equation (217) with any initial condition of the form (218), where ψ possesses the above-mentioned property, is convergent inside certain circles with centre at the origin.

In other words, it is all a question of finding the solution of equation (217) satisfying a condition of the form (218), and of proving that this solution can be expanded as a Maclaurin series if x_k is suffi-

ciently close to zero. We shall seek such a solution as a function of the single argument $z = x_1 + a(x_2 + \ldots + x_n)$.

Here,

$$u_{x_1} = \frac{du}{dz}; \quad u_{x_k} = a\frac{du}{dz} \qquad (k = 2, \ldots, n),$$

so that equation (217) has the form:

$$\frac{du}{dz} = \frac{M}{\left(1 - \dfrac{\dfrac{z}{a} + u}{\varrho}\right)\left(1 - \dfrac{(n-1)\,a}{R}\,\dfrac{du}{dz}\right)} - M,$$

or

$$\left(1 - \frac{(n-1)\,Ma}{R}\right)\frac{du}{dz} - \frac{(n-1)\,a}{R}\left(\frac{du}{dz}\right)^2 = \frac{M}{1 - \dfrac{\dfrac{z}{a} + u}{\varrho}} - M.$$

We shall assume that the number a has been taken so close to zero that the coefficient of du/dz is positive. We obtain on the right-hand side of the equation, on expanding by the progression formula, a power series with no constant term and with positive coefficients. Our last equation can be written as

$$\left(\frac{du}{dz}\right)^2 - 2h\frac{du}{dz} + \varphi(z, u) = 0,$$

where $h > 0$ and $\varphi(z, u)$ is a power series with no constant term and with positive coefficients. On solving for du/dz, we get a first order equation:

$$\frac{du}{dz} = h - h\sqrt{1 - \frac{1}{h^2}\,\varphi(z, u)}, \tag{219}$$

where the radical has to be taken equal to unity for $z = u = 0$. On expanding by the binomial theorem, we get:

$$-h\sqrt{1 - \frac{1}{h^2}\,\varphi(z, u)} = -h + \frac{\frac{1}{2}}{1!}\,\frac{\varphi}{h} - \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\,\frac{\varphi^2}{h^3} + \\ + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}\,\frac{\varphi^3}{h^5} + \ldots,$$

and all the coefficients of powers of $\varphi(z, u)$ are positive. On further expanding in powers of z and u, we obtain on the right-hand side of

equation (219) a power series $\varphi_1(z, u)$ with positive coefficients and with no constant term, and we arrive at the first order equation:

$$\frac{du}{dz} = \varphi_1(z, u) .$$

A theorem has already been proved on the existence of a regular solution of this equation satisfying the initial condition $u_{z=0} = 0$. This solution will be represented by the series:

$$u = \sum_{k=1}^{\infty} c_k z^k ,$$

all the coefficients of which are positive. If we substitute $z = x_1 + a(x_2, + \ldots + x_n)$ in this expansion, we get a solution of equation (217) written as a power series with positive coefficients. This solution will satisfy, for $x_1 = 0$, some initial condition (218), where $\psi(x_2, \ldots, x_n)$ is a power series with positive coefficients. From what has been said above, the construction of such a solution of equation (216) amounts in the long run to a proof of the existence of the solution of the Cauchy problem. This present proof is due to Goursat. The theorem itself is generally known as Kovalevskaya's theorem, since she was the first to offer a finished proof.

It follows from the proof given above that the radii of the circles for the variables x_k, inside which the convergence of series (216) is established, depends only on the radius of convergence of the right-hand side of equation (213) and the maximum modulus M of this right-hand side, and does not depend on the concrete form of the function f. We recall that (216) gives the solution of problem (211), (212). As regards (205), (207), there is added the further dependence on the radii of convergence and maximum modulus of the function $\varphi(x_2, \ldots, x_n)$ appearing in condition (207). A similar remark applies to the results of the next section.

127. Equations of higher order. The method described above can be applied virtually without modification to the case of higher order equations. Let us take as an example a second order equation with two independent variables, solved for the second order derivative with respect to x:

$$r = f(x, y, u, p, q, s, t) \tag{220}$$

$$(p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy}) .$$

The Cauchy initial data in the present case consist in specifying u and p for the initial value of x:

$$u\,|_{x=0} = \varphi(y); \quad p\,|_{x=0} = \psi(y)\,. \tag{221}$$

Let $\varphi(y)$ and $\psi(y)$ be functions regular at the point $y = 0$. We use the notation:

$$\varphi(0) = u_0; \quad \psi(0) = p_0; \quad \varphi'(0) = q_0; \quad \psi'(0) = s_0; \quad \varphi''(0) = t_0$$

and suppose that the right-hand side of (220) is a regular function at the point:

$$x = y = 0; \quad u = u_0; \quad p = p_0; \quad q = q_0; \quad s = s_0; \quad t = t_0\,.$$

Equation (220) now has a unique regular solution satisfying the Cauchy conditions (221). We shall not give the proof of this assertion, which is similar to the above proof, and shall confine ourselves to indicating the possibility of a single-valued determination of the Maclaurin coefficients of the required solution. Initial conditions (221) give us immediately the values of the derivatives

$$\left(\frac{\partial^a u}{\partial y^a}\right)_0, \quad \left(\frac{\partial^{1+a} u}{\partial x\, \partial y^a}\right)_0,$$

for any non-negative values of a, i.e. the initial conditions give us the initial value of the function itself and of such derivatives of it as contain differentiation with respect to x not more than once. The equation itself gives us next:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_0.$$

On differentiating both sides of (220) several times with respect to y, we obtain the values:

$$\left(\frac{\partial^{2+a} u}{\partial x^2\, \partial y^a}\right)_0.$$

On differentiating both sides of (220) with respect to x and using the equations obtained precisely as was done just now regarding the original equation (220), we have the values:

$$\left(\frac{\partial^{3+a} u}{\partial x^3\, \partial y^a}\right)_0.$$

On proceeding further in this manner, all the Maclaurin coefficients of the required function can be uniquely determined.

We now state Kovalevskaya's theorem in the most general case for a system of equations of any order. Suppose we have a system of m equations in the required functions $u_1, \ldots, u_m$ of the independent variables $x_1, \ldots, x_n$:

$$\frac{\partial^{r_k} u_k}{\partial x_1^{r_k}} = f_k\left(x_i, u_i, \frac{\partial^l u_l}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}}\right) \qquad (k = 1, \ldots, m). \tag{222}$$

The right-hand sides of these equations contain the independent variables x_s, the functions u_k and their derivatives up to order r_k: the derivatives $\partial^{r_k} u_k / \partial x_1^{r_k}$, with respect to which the system is solved, must not appear in the right-hand sides. The Cauchy initial data now have the form:

$$\left.\begin{aligned} &u_k|_{x_1=0} = \varphi_k(x_2, \ldots, x_n); \quad \left.\frac{\partial u_k}{\partial x_1}\right|_{x_1=0} = \varphi_k^{(1)}(x_2, \ldots, x_n); \quad \ldots; \\ &\left.\frac{\partial^{r_k-1} u_k}{\partial x_1^{r_k-1}}\right|_{x_1=0} = \varphi_k^{(r_k-1)}(x_2, \ldots, x_n) \qquad (k = 1, \ldots, m). \end{aligned}\right\} \tag{223}$$

The functions on the right-hand sides of these last equations are assumed regular for the zero values of the arguments. We define the initial values of these functions and of their derivatives so that the common order of the derivative of function u_k does not exceed r_k. We assume that the right-hand sides of (222) are regular functions of their arguments, for values of the arguments equal to the initial values of the functions which we have just obtained by the method indicated of differentiation of functions (223).

A theorem now holds for the existence and uniqueness of the regular solution of system (222) with initial conditions (223).

We remark that a complete theory of partial differential equations can be constructed by confining ourselves to analytic functions only. The inadequacy of this stand-point will be indicated below when considering higher order equations.

§ 2. Equations of higher orders

128. Types of second order equation. We shall start our description of the general theory of higher order equations with an investigation of linear second order equations. Suppose we have a linear second order equation for the function u of independent variables $x_1, \ldots, x_n$:

$$\sum_{i,k=1}^{n} a_{ik}(x_1, \ldots, x_n)\, u_{x_i x_k} + \ldots = 0. \tag{1}$$

The coefficients a_{ik} are taken to be given functions of the independent variables x_s, and it can obviously be assumed that $a_{ki} = a_{ik}$, in view of the independence of the result of differentiation on the order. The terms not written do not contain second order derivatives. They can even be non-linearly dependent on the required function and its first order derivatives, so that, strictly speaking, our discussion is of equations which are only linear as regards the higher order derivatives. All the functions and independent variables are assumed real.

The general theory is based on a division of the equations into types. Equations of different types are associated with quite different fundamental problems, require different methods of solution, and are satisfied by functions with quite different analytic properties. Equations of the form (1) will be divided into types in the present section. For this purpose, we form the quadratic form of the auxiliary variables ξ_s:

$$\sum_{i,\,k=1}^{n} a_{ik}\,\xi_i\,\xi_k\,. \tag{2}$$

On assigning definite values $x_s^{(0)}$ to the variables x_s, we obtain a quadratic form with numerical coefficients. If this form is positive definite or negative definite [III, 35], equation (1) is said to belong to the *elliptic type* at the point $x_s = x_s^{(0)}$. Further, we shall say that the equation belongs to the elliptic type in some domain D of space $(x_1, \ldots, x_n)$, if it is of the elliptic type at all points of this domain. The property of being of the elliptic type is characterized by all the coefficients of quadratic form (2), after reduction to a sum of squares, being of the same sign at every point of domain D; also, no coefficient must vanish. Similarly, we say that equation (1) is of the *hyperbolic type* in domain D, or as it is sometimes said, is of the *normal hyperbolic type*, if the quadratic form (2), after reduction to a sum of squares, has coefficients which are all of a given sign apart from one, this remaining coefficient being of the opposite sign. If quadratic form (2), after reduction to a sum of squares, has at least one zero coefficient at every point of D, we say that (1) is of the *parabolic type* in D. If there is no zero coefficient, yet we have neither an elliptic nor a hyperbolic type, the equation is sometimes said to be of the *ultra-hyperbolic type*. If the coefficients a_{ik} are constant, the type of the equation is independent of the independent variables. The simplest equation of the elliptic type is Laplace's equation; the wave equation

is of the hyperbolic type, and finally, the heat conduction equation is of the parabolic type. Let us consider this last equation:

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz}) .$$

We can write in the notation of (1):

$$u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} - \frac{1}{a^2} u_{x_4} = 0 ,$$

and the quadratic form (2) becomes:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 .$$

This is already reduced to a sum of squares, the coefficient of ξ_4^2 being zero.

If the coefficients a_{ik} of equation (1) contain the function u and its partial derivatives u_{x_i}, we can only speak of the type of the equation after fixing some solution $u^{(0)}(x_1, \ldots, x_n)$ of it. On substituting $u = u^{(0)}$ and $u_{x_i} = u^{(0)}_{x_i}$ in the coefficients a_{ik}, we obtain a function of x_s only, which enables us to state the type of the equation for the given solution $u^{(0)}$. In certain cases the type of the equation can be determined for any u.

If the equation is non-linear:

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots, u_{x_n x_n}) = 0 ,$$

the type of the equation is determined for a given solution $u^{(0)}$ by constructing coefficients a_{ik} in accordance with the formulae:

$$a_{ik} = \frac{\partial F}{\partial u_{x_i x_k}} \quad \text{for} \quad u = u^{(0)} ,$$

then determining the type of the linear equation:

$$\sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} = 0 .$$

129. Equations with constant coefficients. Let us consider equation (1) with constant coefficients a_{ik} and write down the corresponding quadratic form. By means of a linear transformation of the independent variables we try to reduce the set of terms containing the second derivatives in equation (1) to the simplest form. In fact, we introduce new independent variables y_s instead of x_s with the aid of the linear transformation:

$$y_k = c_{k1} x_1 + \ldots + c_{kn} x_n \qquad (k = 1, 2, \ldots, n) ,$$

on the natural assumption that the determinant formed from the coefficients of the transformation is non-zero. The derivatives with respect to the old variables are given in terms of the derivatives with respect to the new by the expressions:

$$u_{x_i} = \sum_{s=1}^{n} c_{si} u_{y_s}, \qquad u_{x_i x_k} = \sum_{s,t=1}^{n} c_{si} c_{tk} u_{y_s y_t}.$$

On substituting in equation (1), we obtain a transformed equation of the form:

$$\sum_{i,k=1}^{n} a'_{ik} u_{y_i y_k} + \ldots = 0,$$

where the new coefficients a'_{ik} are given in terms of the old by

$$a'_{ik} = \sum_{s,t=1}^{n} c_{is} c_{kt} a_{st}. \tag{3}$$

On the other hand, if we introduce into quadratic form (2) new variables η_s in place of the ξ_s with the aid of a matrix which is the transpose of matrix c_{ik}, which amounts to expressing the old variables in terms of the new with the aid of this matrix, i.e. if we put

$$\xi_k = c_{1k}\eta_1 + \ldots + c_{nk}\eta_n \qquad (k = 1, \ldots, n),$$

it is easily seen that the transformed quadratic form will have precisely the coefficients a'_{ik} given by (3), i.e.

$$\sum_{i,k=1}^{n} a_{ik} \xi_i \xi_k = \sum_{i,k=1}^{n} a'_{ik} \eta_i \eta_k.$$

But we know that the coefficients c_{ik} can always be chosen so that quadratic form (2) reduces to a sum of squares, i.e.:

$$\sum_{i,k=1}^{n} a_{ik} \xi_i \xi_k = \sum_{i=1}^{n} \lambda_i \eta_i^2,$$

or, in other words, $a'_{ik} = 0$ for $i \neq k$ and $a'_{ii} = \lambda_i$. The signs of the coefficients λ_i in fact decide the type of the equation. On retaining the previous notation for the independent variables, we obtain a transformed equation of the form:

$$\sum_{i=1}^{n} \lambda_i u_{x_i x_i} + \ldots = 0.$$

If the equation is linear and has constant coefficients, not only with respect to the second order derivatives, the transformed equation takes the form:

$$\sum_{i=1}^{n} \lambda_i u_{x_i x_i} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f(x_1, \ldots, x_n). \tag{4}$$

By adding a suitably chosen numerical factor to the independent variables x_s, we can always arrange for the non-zero coefficients λ_i to be equal to $(+1)$ or (-1). Suppose that all the λ_i are non-zero, and let us show that in this case the function u can be freed of terms containing the first derivatives with the aid of an elementary transformation: in fact, we introduce instead of u the new required function v given by:

$$u = v e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{b_i}{\lambda_i} x_i}. \tag{5}$$

It is easily shown that substitution in equation (4) gives us an equation of the form:

$$\sum_{i=1}^{n} \lambda_i v_{x_i x_i} + c_1 v = f_1(x_1, \ldots, x_n).$$

All the λ_i have the same sign for an equation of the elliptic type, and we can assume, by multiplying both sides of the equation by (-1) if necessary, that all the λ_i are positive. On replacing the x_i by new independent variables $x_i = \sqrt{\lambda_i}\, x_i'$, we get rid of the coefficients λ_i, and we can say, retaining the previous notation, that every linear equation of the elliptic type with constant coefficients can be reduced to the form:

$$\sum_{i=1}^{n} u_{x_i x_i} + c_1 u = f_1(x_1, \ldots, x_n). \tag{6}$$

In the case of the hyperbolic type, we shall assume that there are $(n + 1)$ independent variables, and we shall write t for one of them. We can obviously assume that n of the λ_i are negative and the other one is positive, and we use t to denote the independent variable for which λ_i is positive. In the end, every linear equation with constant coefficients of the hyperbolic type can be reduced to the form:

$$u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} + cu = f_1(t, x_1, \ldots, x_n).$$

130. Normal forms with two independent variables. We proved in [129] that, in the case of constant coefficients, the set of terms of the equation which depend on the second order derivatives can be reduced with the aid of a linear transformation to some normal form. In the case of variable coefficients, depending on the x_s, we obviously cannot rely on carrying out the transformation to the normal form with the aid of a linear transformation of the variables, and we have to use more general transformations; though even in this case, the problem can only be solved for the case of two independent variables. We therefore take a second order equation with two independent variables, linear with respect to the second order derivatives:

$$a(x, y)\, u_{xx} + 2b(x, y)\, u_{xy} + c(x, y)\, u_{yy} + \ldots = 0\,. \tag{7}$$

We introduce instead of (x, y) the new independent variables (ξ, η):

$$\xi = \varphi(x, y); \quad \eta = \psi(x, y)\,. \tag{8}$$

The derivatives with respect to the old variables are given in terms of the derivatives with respect to the new by the formulae:

$$\begin{aligned}
u_x &= u_\xi \varphi_x + u_\eta \psi_x; \quad u_y = u_\xi \varphi_y + u_\eta \psi_y \\
u_{xx} &= u_{\xi\xi} \varphi_x^2 + 2u_{\xi\eta} \varphi_x \psi_x + u_{\eta\eta} \psi_x^2 + u_\xi \varphi_{xx} + u_\eta \psi_{xx} \\
u_{yy} &= u_{\xi\xi} \varphi_y^2 + 2u_{\xi\eta} \varphi_y \psi_y + u_{\eta\eta} \psi_y^2 + u_\xi \varphi_{yy} + u_\eta \psi_{yy} \\
u_{xy} &= u_{\xi\xi} \varphi_x \varphi_y + u_{\xi\eta}(\varphi_x \psi_y + \varphi_y \psi_x) + u_{\eta\eta} \psi_x \psi_y + u_\xi \varphi_{xy} + u_\eta \psi_{xy}\,.
\end{aligned}$$

On substituting in equation (7), we have the transformed equation:

$$a'(\xi, \eta)\, u_{\xi\xi} + 2b'(\xi, \eta)\, u_{\xi\eta} + c'(\xi, \eta)\, u_{\eta\eta} + \ldots = 0\,,$$

where

$$\left.\begin{aligned}
a'(\xi, \eta) &= a\varphi_x^2 + 2b\varphi_x \varphi_y + c\varphi_y^2 \\
c'(\xi, \eta) &= a\psi_x^2 + 2b\psi_x \psi_y + c\psi_y^2 \\
b'(\xi, \eta) &= a\varphi_x \psi_x + b(\varphi_x \psi_y + \varphi_y \psi_x) + c\varphi_y \varphi_y\,.
\end{aligned}\right\} \tag{9}$$

The following identity may be verified by direct substitution:

$$a' c' - b'^2 = (ac - b^2)\,(\varphi_x \psi_y - \varphi_y \psi_x)^2\,. \tag{10}$$

It is easily seen that the sign of $ac - b^2$ determines the type of equation (7). If $ac - b^2 < 0$, the equation is of the hyperbolic type, with $ac - b^2 > 0$ it is of the elliptic type, and with $ac - b^2 = 0$ it is parabolic. By (10), transformation of the variables does not change the sign of $ac - b^2$, i.e. of course, it does not change the type of the equation.

For equations of the hyperbolic type with constant coefficients, the simplest form in two independent variables is

$$u_{xx} - u_{yy} + \ldots = 0\,. \tag{11}$$

On introducing new independent variables (ξ, η) instead of (x, y):

$$\xi = \frac{x+y}{2}; \quad \eta = \frac{x-y}{2}, \tag{12}$$

we arrive at another elementary form of the hyperbolic type:

$$u_{\xi\eta} + \ldots = 0\,. \tag{13}$$

It can be seen that either (11) or (13) can be taken as the elementary form of the hyperbolic type in two independent variables. Either of these equations is easily transformed into the other.

We return to equation (7) and suppose it to be of the hyperbolic type in some domain D of the (x, y) plane. This means that, for (x, y) in D, the quadratic equation

$$a(x, y)\,\tau^2 + 2b(x, y)\,\tau + c(x, y) = 0 \tag{14}$$

has real distinct roots. Here, we take either $a \neq 0$ or $c \neq 0$. If $c = a = 0$, equation (7) is already of the elementary form (13). Obviously, we can assume without loss of generality that $a \neq 0$. Let us consider the first order partial differential equation:

$$a(x, y)\,u_x^2 + 2b(x, y)\,u_x u_y + c(x, y)\,u_y^2 = 0\,. \tag{15}$$

Writing $f_1(x, y)$ and $f_2(x, y)$ for the roots of (14), equation (15) will be seen to split into two:

$$u_x = f_1(x, y)\,u_y \tag{16$_1$}$$

and

$$u_x = f_2(x, y)\,u_y\,. \tag{16$_2$}$$

If the coefficients a, b and c, and hence also functions f_1 and f_2, are sufficiently smooth, the last equations have solutions with continuous derivatives up to the second order in some part of domain D [cf. 100]. Let us take as $\varphi(x, y)$, $\psi(x, y)$ of transformation (8) solutions of (16_1) and (16_2) respectively. These solutions can be chosen so that the determinant $\varphi_x\psi_y - \varphi_y\psi_x$ is non-zero in the part of D in question. It may be observed that

$$\varphi_x = f_1\,\varphi_y; \quad \psi_x = f_2\,\psi_y\,,$$

whence

$$\varphi_x \psi_y - \varphi_y \psi_x = (f_1 - f_2)\, \varphi_y \psi_y\,. \tag{17}$$

It follows from these last equations that, if the determinant vanishes at some point, both first order partial derivatives of φ or ψ vanish at this point. We must therefore find solutions of (16_1) and (16_2) such that both their first order partial derivatives do not vanish simultaneously.

Functions φ and ψ satisfy (15), and by (9), we have $a' = c' = 0$, whilst it follows from (10) that $b' \neq 0$, so that (7) reduces to form (13).

As we saw in [100], the solution of (16_1) and (16_2) is of a local nature, i.e. we can construct solutions of these equations differing from constants in a domain which, in general, will only be a part of the domain in which $f_k(x, y)$ are continuously differentiable, and the reduction of (7) to the normal form will only hold in this sub-domain. The same remark concerning the local nature of the reduction of (7) to the normal form applies for the discussion that follows.

Let us turn to equations of the elliptic type. Here, $ac - b^2 > 0$, and the roots of (14) are imaginary conjugates. We can write down (15) as before. Let us write one of equations (16):

$$u_x = \frac{-b + \sqrt{ac - b^2}\, i}{a}\, u_y\,,$$

where the root is taken as positive. On regarding the coefficients a, b and c as analytic functions of x and y, and $a \neq 0$, we can find the solution of this equation as an analytic function [126]: $u = \varphi(x, y) + \psi(x, y)\, i$, where

$$\varphi_x = -\frac{b}{a}\varphi_y - \frac{\sqrt{ac - b^2}}{a}\psi_y, \quad \psi_x = -\frac{b}{a}\psi_y + \frac{\sqrt{ac - b^2}}{a}\varphi_y\,.$$

We now carry out the change of variables (8). Using the system for φ and ψ, together with (9), we obtain:

$$b' = 0; \quad a' = c' = (ac - b^2)\,(\varphi_y^2 + \psi_y^2)\,,$$

and the equation becomes, after division by a':

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \ldots = 0\,. \tag{18}$$

Instead of (17), we have:

$$\varphi_x \psi_y - \varphi_y \psi_x = -\frac{2\sqrt{ac - b^2}}{a}\, \varphi_y \psi_y\,.$$

The problem is therefore solved for the elliptic type as well. A solution of this general problem, where a, b, c satisfy certain conditions but are not assumed to be analytic, may be found in I. N. Vekua's article in *DAN SSSR*, vol. 100, no. 2, 1955. We still have to consider equations of the parabolic type. Here, equation (14) has equal roots, and (15) reduces to a single equation, i.e. (16_1) and (16_2) coincide. We take the solution of this equation as $\varphi(x, y)$ and any function as $\psi(x, y)$, provided only that the functional determinant of φ and ψ is non-zero. By virtue of the choice of φ, we have $a' = 0$ in the transformed equation. In addition, since the equation is parabolic, $ac - b^2 = 0$, and (19) shows that $b' = 0$. We thus have $a' = b' = 0$ as a result of the transformation. The function c' cannot be identically zero, since if it were, our equation would be of the first order, and the inverse transformation from (ξ, η) to (x, y) could not give us the second order equation (7). We thus have the following canonical form in the parabolic case:

$$u_{\eta\eta} + \ldots = 0\,, \tag{19}$$

where the terms not written do not contain second order derivatives.

131. Cauchy's problem. We saw above [127] that, given the second order equation:

$$F(x, y, u, p, q, r, s, t) = 0\,, \tag{20}$$

the Cauchy initial data can consist in a particular case in specifying the function u and its derivative $u_x = p$ for the initial value of $x = x_0$:

$$u\,|_{x=x_0} = \varphi(y); \quad p\,|_{x=x_0} = \psi(y)\,. \tag{21}$$

Such data will be described as *special Cauchy data*. These initial conditions imply that the values of the required function u and its partial derivative p are given along the curve $x = x_0$ of the (x, y) plane. We remark that the value of the second partial derivative $q\,|_{x=x_0} = \varphi'(y)$ is obtained here directly from the first of conditions (21). Hence we know from the initial data the function itself and both its first order partial derivatives along $x = x_0$. It is easy to conceive of more general Cauchy data. Let λ be a non-self-intersecting curve on the (x, y) plane, and let the values of the required function u be given along this curve. We must automatically know the derivative of u along λ with respect to the tangential direction to λ. To find the first order derivative with respect to any direction, we must be given one further value along λ, i.e. the value of the derivative of u

with respect to any direction other than the tangential must be specified along λ. Knowing the derivatives with respect to two directions on the (x, y) plane along λ, we can find the derivative with respect to any direction on the plane along λ. Therefore, the value of the function and of its derivative with respect to any non-tangential direction must be specified along λ in our present case. The specification of u along a curve λ in the (x, y) plane leads us to a curve l in three-dimensional space (x, y, u). Furthermore, the partial derivatives p and q are known along λ. The Cauchy data therefore finally reduce to specifying a curve l in three-dimensional space (x, y, u) and to specifying the positions of the tangent plane along this curve. These general Cauchy data can be described as follows in parametric terms: we are given five functions of one parameter:

$$x(t),\ y(t),\ u(t),\ p(t),\ q(t)\,, \tag{22}$$

which must satisfy the relationship

$$\mathrm{d}u = p\,\mathrm{d}x + q\,\mathrm{d}y\,. \tag{23}$$

This last relationship amounts to the requirement that the specification of both derivatives p and q along λ does not contradict the specification of u itself along λ i.e. that the derivative with respect to the tangent to λ, worked out from the given p and q, has in fact the values obtaining along λ by virtue of the specification of u. The five functions (22) satisfying (23) define a *strip* in three-dimensional space (x, y, u) and the Cauchy problem amounts to seeking the integral surface of equation (20) which contains the strip.

The Cauchy problem is similarly posed in the general case for functions of any number of independent variables. Let us take say the second order equation in three independent variables:

$$F(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots) = 0\,. \tag{24}$$

The initial Cauchy data amount here to specifying the function u and its first order partial derivatives on a surface S of three-dimensional space (x_1, x_2, x_3). Since the values of u itself are given on S, to specify all the first order partial derivatives on S we only need to specify the derivative with respect to any direction not in the tangent plane to S. If the surface S supporting the initial Cauchy data is a plane $x_1 = x_1^{(0)}$, we have the special form of the initial Cauchy data:

$$u\,|_{x_1=x_1^{(0)}} = \varphi(x_2, x_3); \quad u_{x_1}\,|_{x_1=x_1^{(0)}} = \psi(x_2, x_3)\,. \tag{25}$$

The Cauchy problem in the parametric form amounts to specifying seven functions of two parameters:

$$\left.\begin{array}{l} x_1(t_1, t_2),\ x_2(t_1, t_2),\ x_3(t_1, t_2),\ u(t_1, t_2), \\ u_{x_1}(t_1, t_2),\ u_{x_2}(t_1, t_2),\ u_{x_3}(t_1, t_2), \end{array}\right\} \tag{26}$$

where the condition:

$$\mathrm{d}u = u_{x_1}\,\mathrm{d}x_1 + u_{x_2}\,\mathrm{d}x_2 + u_{x_3}\,\mathrm{d}x_3\,. \tag{27}$$

must be satisfied. The specification of functions x_1, x_2, x_3 amounts to specifying a surface, and the remaining data to specifying a function u and its first order partial derivatives on the surface. Data (26), satisfying condition (27), are usually described as a *strip*, or more precisely as a *first order strip* in four-dimensional space (x_1, x_2, x_3, u), and the Cauchy problem consists in defining the integral surface of equation (24) containing the given strip. In the case of a function u of n independent variables ($x_1, \ldots, x_n$) the strip is given in the form of $(2n+1)$ functions of $(n-1)$ parameters:

$$x_k(t_1, \ldots, t_{n-1}), u(t_1, \ldots, t_{n-1}), u_{x_k}(t_1, \ldots, t_{n-1}) \quad (k = 1, 2, \ldots, n)\,,$$

where the functions must satisfy the relationship

$$\mathrm{d}u = \sum_{k=1}^{n} u_{x_k}\,\mathrm{d}x_k\,.$$

If one of the independent variables is time t, and the surface supporting the initial Cauchy data is the plane $t = 0$, we get the usual problem of mathematical physics of integrating an equation with given initial conditions [II, 163].

The initial Cauchy data define the function u and all its first order partial derivatives on the curve or surface that supports the initial data. If we associate the differential equation itself with the initial data, as we saw in [127], in the case of the special Cauchy data we can also uniquely determine on our curve or surface all the second order derivatives of the required function. We shall describe the strip as a *characteristic strip* if the strip plus the differential equation itself does not lead to unique determination of the second order derivatives. We shall discuss this question in detail in the next section, for a quasilinear equation in two independent variables.

132. Characteristic strips. We shall consider an equation of the form

$$ar + 2bs + ct + h = 0\,, \tag{28}$$

in which the coefficients and the term h are given functions of (x, y, u, p, q). We want to find the integral surface of this equation containing the given strip:

$$x(t),\ y(t),\ u(t),\ p(t),\ q(t) \qquad (\mathrm{d}u = p\,\mathrm{d}x + q\,\mathrm{d}y)\,. \tag{29}$$

We obviously have:

$$\mathrm{d}p = r\,\mathrm{d}x + s\,\mathrm{d}y; \quad \mathrm{d}q = s\,\mathrm{d}x + t\,\mathrm{d}y\,,$$

and on adding the equation itself, we have three first degree equations for the second order derivatives of the required function on the base curve $\lambda : x(t), y(t)$, supporting the initial Cauchy data:

$$\left.\begin{aligned} \mathrm{d}x\cdot r + \mathrm{d}y\cdot s &= \mathrm{d}p \\ \mathrm{d}x\cdot s + \mathrm{d}y\cdot t &= \mathrm{d}q \\ ar + 2bs + ct &= -h\,. \end{aligned}\right\} \tag{30}$$

In this system r, s, t are the unknowns, whilst the remaining quantities, by (29), are known functions of the parameter t. If the determinant of the above system is non-zero, we obtain definite values for the second order derivatives. Hence, the necessary and sufficient condition for incompatibility or indeterminacy of the problem of seeking the second order derivatives is

$$\Delta = \begin{vmatrix} \mathrm{d}x, & \mathrm{d}y, & 0 \\ 0, & \mathrm{d}x, & \mathrm{d}y \\ a, & 2b, & c \end{vmatrix} = 0\,, \tag{31}$$

or, in the expanded form:

$$a\,\mathrm{d}y^2 - 2b\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}x^2 = 0\,. \tag{32}$$

Let us find a second condition guaranteeing that the problem is indeterminate, i.e. that system (31) has an infinite set of solutions. We shall assume that one of the second order minors of determinant (31) differs from zero. We assume for definiteness that

$$\begin{vmatrix} 0, & \mathrm{d}y \\ a, & c \end{vmatrix} = -a\,\mathrm{d}y \neq 0\,.$$

In this case system (30) will have one characteristic determinant [III, 9], and the necessary and sufficient condition (in addition to (31))

for it to be indeterminate is that this characteristic determinant vanishes [III_1, 9]:

$$\begin{vmatrix} \mathrm{d}x, & \mathrm{d}p, & 0 \\ 0, & \mathrm{d}q, & \mathrm{d}y \\ a, & -h, & c \end{vmatrix} = 0\,,$$

or, in the expanded form:

$$a\,\mathrm{d}p\,\mathrm{d}y + h\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}x\,\mathrm{d}q = 0\,. \tag{33}$$

On also recalling (23), we finally get the following three equations which completely define a characteristic strip as a strip along which the search for the second order derivatives of system (30) leads to an infinite set of answers:

$$\left.\begin{aligned} &a\,\mathrm{d}y^2 - 2b\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}x^2 = 0 \\ &a\,\mathrm{d}p\,\mathrm{d}y + h\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}x\,\mathrm{d}q = 0 \\ &\mathrm{d}u = p\,\mathrm{d}x + q\,\mathrm{d}y\,. \end{aligned}\right\} \tag{34}$$

Let us analyse separately the case of special Cauchy data:

$$u\,|_{x=x_0} = \varphi(y); \quad p\,|_{x=x_0} = \psi(y)\,. \tag{35}$$

Here, the role of parameter t in expressions (29) is played by the variable y, and the variable x retains the constant value $x = x_0$. Condition (32) leads to $a = 0$. We observe that this condition should not be satisfied as an identity after substitution of initial data (35) in the function a. System (30) now becomes

$$s\,\mathrm{d}y = \mathrm{d}p; \quad t\,\mathrm{d}y = \mathrm{d}q; \quad 2bs + ct = -h$$

and the necessary and sufficient condition for it to be indeterminate is that the third equation be a consequence of the first two. On multiplying this equation by $\mathrm{d}y$ and taking the first two equations into account, we arrive at the condition:

$$2b\,\mathrm{d}p + c\,\mathrm{d}q = -h\,\mathrm{d}y\,,$$

which in this case replaces condition (33). Finally, for the special Cauchy data (35), we have the following conditions defining a characteristic strip:

$$a = 0; \quad 2b\,\mathrm{d}p + c\,\mathrm{d}q = -h\,\mathrm{d}y; \quad \mathrm{d}u = q\,\mathrm{d}y\,. \tag{36}$$

The condition $a = 0$ shows that it is impossible to find u_{xx} from equation (28). The second condition:

$$2b\frac{\mathrm{d}p}{\mathrm{d}y} + c\frac{\mathrm{d}q}{\mathrm{d}y} + h = 0$$

implies that the p and q given on the straight line $x = x_0$ are such that equation (28) is satisfied, since on this straight line $s = \mathrm{d}p/\mathrm{d}y$ and $t = \mathrm{d}q/\mathrm{d}y$. The third condition gives the obvious expression:

$$q\,|_{x=x_0} = \varphi'(y)\,.$$

133. Higher order derivatives. We have discussed in the previous section the question of finding the second order derivatives on a given strip. We now consider finding the higher order derivatives. Suppose we have the case when determinant (31) is non-zero. We take the total differential of the first two of equations (30) and differentiate the given equation (28) with respect to x and y. This gives us four first degree equations for the four third order derivatives of the required function u on the strip:

$$\begin{aligned}(\mathrm{d}x)^2\,u_{xxx} + 2\mathrm{d}x\,\mathrm{d}y\,u_{xxy} + (\mathrm{d}y)^2\,u_{xyy} &= \dots\\ (\mathrm{d}x)^2\,u_{xxy} + 2\mathrm{d}x\,\mathrm{d}y\,u_{xyy} + (\mathrm{d}y)^2\,u_{yyy} &= \dots\\ au_{xxx} + 2bu_{xxy} + cu_{xyy} &= \dots\\ au_{xxy} + 2bu_{xyy} + cu_{yyy} &= \dots\end{aligned}$$

The determinant of this system is

$$\Delta_1 = \begin{vmatrix} (\mathrm{d}x)^2, & 2\mathrm{d}x\,\mathrm{d}y, & (\mathrm{d}y)^2, & 0 \\ 0, & (\mathrm{d}x)^2 & 2\mathrm{d}x\,\mathrm{d}y, & (\mathrm{d}y)^2 \\ a, & 2b, & c, & 0 \\ 0, & a, & 2b, & c \end{vmatrix},$$

It can be shown that this determinant is equal to the square of determinant (31), i.e. is identically zero. For, writing γ for any root of the equation

$$a + 2b\gamma + c\gamma^2 = 0\,, \tag{37}$$

we add to the elements of the first column of Δ_1 the elements of the second column multiplied by γ, the elements of the third column multiplied by γ^2, and of the fourth column multipled by γ^3. The elements of the first column now become:

$$(\mathrm{d}x + \gamma\,\mathrm{d}y)^2,\ \ \gamma(\mathrm{d}x + \gamma\,\mathrm{d}y)^2,\ \ 0,\ \ 0,$$

whence it is clear that Δ_1, being a homogeneous fourth degree polynomial in $\mathrm{d}x$ and $\mathrm{d}y$, is divisible by $(\mathrm{d}x + \gamma\, \mathrm{d}y)^2$. The coefficient of $(\mathrm{d}x)^4$ in the expression for Δ_1 is c^2, and if we write γ_1, γ_2 for the roots of (37), we can write:

$$\Delta_1 = c^2(\mathrm{d}x + \gamma_1\, \mathrm{d}y)^2\, (\mathrm{d}x + \gamma_2\, \mathrm{d}y)^2,$$

or, taking into account the property of the roots of a quadratic equation:

$$\Delta_1 = (c\, \mathrm{d}x^2 - 2b\, \mathrm{d}x\, \mathrm{d}y + a\, \mathrm{d}y^2)^2 = \Delta^2$$

We have assumed in the proof that (37) has distinct roots. But if $\Delta_1 = \Delta^2$ holds on this assumption, it will also hold when (37) has equal roots. To verify this, we only need to vary somewhat the coefficients a, b, c so that (37) has distinct roots, then pass to the limit in $\Delta_1 = \Delta^2$ as the altered coefficients tend to their initial values, for which the roots of (37) are equal.

Similarly, five first degree equations can be obtained for the five fourth order derivatives, and the determinant of this system also proves to be non-zero, and so on. We suppose the corresponding functions to be analytic and regular. Hence, as in the case of the special Cauchy data and the equation solved for r [127], the derivatives of all orders can likewise be calculated in the general case on a given strip, on the assumption of non-zero Δ. We could form the corresponding Taylor series and prove its convergence as in [126].

We now turn to the case when the given strip is a characteristic strip. We shall confine ourselves here to the special Cauchy initial data (35). These initial data themselves give us s and t for $x = x_0$, and it only remains to find r. But on substituting the initial data obtained in (28), we obtain an identity by (36), and the derivative r is at first sight completely indeterminate at $x = x_0$. We differentiate both sides of (28) with respect to x:

$$ar_x + 2bs_x + ct_x + (a_x + a_u p + a_p r + a_q s)\, r + \\ + (\ldots)\, s + (\ldots)\, t + (\ldots) = 0, \qquad (38)$$

where the brackets with rows of dots contain expressions precisely analogous to the expression in the bracket containing derivatives of a. If we substitute in this equation initial data (35) and the already known second derivatives

$$s\,|_{x=x_0} = \psi'(y); \quad t = |_{x=x_0}\, \varphi''(y),$$

on using the notation $r\,|_{x=x_0} = \omega(y)$, it is easily seen that we get a Riccati equation for the required function $\omega(y)$, i.e. an equation of the form

$$\alpha(y)\,\omega'(y) + \beta(y)\,\omega^2(y) + \gamma(y)\,\omega(y) + \delta(y) = 0,$$

where α, β, γ and δ are known functions of y. If we take any solution of this equation, we then know r at $x = x_0$, and hence know all the third order derivatives at $x = x_0$ except u_{xxx}. To find the initial value of this latter derivative we have to differentiate (38) with respect to x and introduce all the initial data already found into the equation thus obtained. Hence we arrive at a linear differential equation for the required function $u_{xxx}\,|_{x=x_0} = \omega_1(y)$:

$$\alpha_1(y)\,\omega_1'(y) + \beta_1(y)\,\omega_1(y) + \gamma_1(y) = 0.$$

This process can be continued further. New arbitrary constants will be continually introduced on integration of the above Riccati equation and the subsequent linear equations, and the real difficulty of the problem lies in choosing the values of these constants so that the resulting Taylor series is convergent. It can be shown (though we shall not dwell on this) that there is an infinity of ways of doing this for equations of the hyperbolic type, i.e. an infinite number of integral surfaces pass through a characteristic strip. Conditions (36), or in the general case (34), are thus the necessary and sufficient conditions to be satisfied by the initial data in order that integral surfaces exist containing a given characteristic strip.

Let us take as an example the simplest second order equation of the parabolic type:

$$t - u_x = 0, \quad \text{i.e. } u_x = u_{yy}. \tag{39}$$

Here, $a = b = 0$, $c = -1$, and equation (32) gives $dx = 0$, i.e. $x = \text{const}$. There must be some singularity along any line $x = x_0$ when trying to solve Cauchy's problem. Suppose that we have the special Cauchy data (35). On putting $x = x_0$ in (39), we get $\psi(y) = \varphi''(y)$, whence the function $\psi(y)$ is seen to be completely determined by specifying $\varphi(y)$. This corresponds to the necessary fulfilment of the second of conditions (36). Only the first of conditions (35) therefore needs to be stated in the present case.

If we differentiate (39) with respect to x and put $x = x_0$, we completely determine the initial value: $r\,|_{x=x_0} = \varphi^{(IV)}(y)$. Having obtained this initial value, we can obtain the initial value at $x = x_0$ of the

third order derivative with respect to x by differentiating (39) twice with respect to x and setting $x = x_0$, and so on. Here, the initial values of the derivatives with respect to x are uniquely determined, whilst the above-mentioned differential equations degenerate into finite relationships. Having found the initial values of the derivatives of all orders with respect to x at $x = x_0$, we can construct the corresponding Taylor series. It may happen that it is convergent in the neighbourhood of $x = x_0$ only if $\varphi(y)$ is an entire function satisfying some auxiliary condition. We recall that, when considering the problem of heat propagation in an infinite rod [II, 204], we found a solution of equation (39) satisfying the first of conditions (35), in the form of a definite integral. It was obviously not necessary there to assume that $\varphi(y)$ is an entire function. To pass to the earlier notation of [II, 204], x must be replaced by t in (39), and y by x, whilst we took $a^2 = 1$ in the equation of [II, 204].

If we put $\varphi(y) = 0$, we obviously get an identically zero solution of (39). Let us show that a further elementary solution of (39) exists, satisfying the same initial condition $u|_{x=x_0} = 0$ except at the point $y = 0$, $x = x_0$. Suppose that

$$u = \frac{1}{\sqrt{x - x_0}} e^{-\frac{y^2}{4(x-x_0)}} \quad \text{for} \quad x > x_0 \tag{40_1}$$

$$u = 0 \qquad\qquad \text{for} \quad x \leqslant x_0. \tag{40_2}$$

Function (40_1) and all its derivatives tend to zero as x tends to x_0 (from greater values), i.e. the function defined by (40_1) and (40_2) and all its derivatives remain continuous on passing through the straight line $x = x_0$, whilst u and all its derivatives vanish on the line itself. The only exception is the point $x = x_0$, $y = 0$, at which the constructed function has a singularity. It can be seen by direct differentiation of (40_1) that the constructed function satisfies equation (39). At every point of $x = x_0$ the function evidently ceases to be analytic or regular in x, since it is identically zero to the left of the straight line and non-zero to the right. Hence the function cannot be represented by a Taylor series in positive integral powers of $(x - x_0)$. Solution (40_1) differs by a constant factor from the solution providing an elementary heat source [II, 204].

134. Real and imaginary characteristics. Since the coefficients of equation (28) can depend on u, p, q as well as on x and y, we can determine the type of the equation only after fixing some point in five-dimensional

space (x, y, u, p, q). Now if $b^2 - ac > 0$, we have the hyperbolic type, if $b^2 - ac < 0$ the elliptic, and if $b^2 - ac = 0$ the parabolic. Let (29) be a given strip, which we assume to be real. If our equation is elliptic along this strip, the expression on the left-hand side of condition (32) cannot vanish; consequently no real strip can be a characteristic strip. We shall only consider the hyperbolic type below. Equation (32) is a quadratic equation in dy/dx. In the case of the hyperbolic type it has two real distinct roots, which we shall denote by $\mu_1(x, y, u, p, q)$ and $\mu_2(x, y, u, p, q)$, so that the above equation splits into two: $dy = \mu_i \, dx$ $(i = 1, 2)$. We can therefore write two systems of equations in place of (34):

$$\left.\begin{aligned} &dy - \mu_i \, dx = 0 \\ &a\mu_i \, dp + h\mu_i \, dx + c \, dq = 0 \qquad (i = 1,\ 2) \\ &du = p \, dx + q \, dy, \end{aligned}\right\} \tag{41}$$

to which there correspond two systems of characteristics.

The situation is particularly simple when the equation is purely linear in the second order derivatives, i.e. when the coefficients a, b and c depend only on the independent variables (x, y). In this case the fundamental equation (32) becomes an ordinary first order differential equation with variables x and y:

$$a(x, y) \, dy^2 - 2b(x, y) \, dx \, dy + c(x, y) \, dx^2 = 0.$$

In the hyperbolic case it defines two families of curves on the (x, y) plane, which are generally called *characteristic curves* or *characteristics* of equation (28). The special feature of any characteristic is that, if we lay down certain Cauchy data along this curve, i.e. the function u and its first order derivatives, the strip thus obtained either leads to an incompatible system (30) for the second order derivatives, or else is a characteristic strip. In the case of any non-characteristic curve, any Cauchy data lead to definite values of the second and higher order derivatives. In the elliptic case, (32) has imaginary roots for dy/dx, and there are no characteristic curves on the (x, y) plane. If we pass to complex values of the variables (x, y), imaginary characteristics can be obtained from (32). All the functions are naturally assumed to be analytic here. Finally, in the parabolic case, (32) gives us one family of characteristics on the (x, y) plane. It will be seen on turning to the results of (32) that, when reducing an equation to the canonical form, we chose a family of characteristics as the coordinate lines on the (x, y) plane.

135. Fundamental theorems. Just as in the case of a first order equation, the characteristic manifold plays a fundamental role in the solution of an equation. The fundamental theorems are exactly like those that hold for first order equations.

Suppose that two integral surfaces of equation (28) have finite order contact along some curve l of space (x, y, u), i.e. they have a common tangent plane along the curve though certain derivatives of higher order than the first may prove to be different along the curve. It may easily be seen that the curve plus the tangent plane along it must represent a characteristic strip. For, if this were not the case, it would follow from the discussion of (133) that completely determinate values would be obtained for the derivatives of all orders along l. We thus have the theorem:

THEOREM 1. *If two integral surfaces have finite order contact along a curve l, this curve together with the corresponding tangent plane represents a characteristic strip.*

The basic property of a characteristic strip is that, along it, the equation leads to an indeterminate system (30) for the second order derivatives. This property obviously does not depend on the choice of independent variables, and we thus have the theorem:

THEOREM 2. *Characteristic strips transform to characteristic strips for any reversible and smooth change of variables (x, y).*

Let S be an integral surface of equation (28). On the surface u, p, q are definite functions of the independent variables (x, y). On substituting for u, p, q their expressions in terms of (x, y) in the coefficients of (28), we get definite expressions for these coefficients in terms of (x, y), and equation (32) becomes a differential equation of the first order, defining two systems of curves on S. Equations (23) and (32) will be satisfied along each of these curves l, and it may easily be seen that the second of equations (34) must also be satisfied along l. For, if it were not satisfied, we should have an incompatible system for the second order derivatives, and this contradicts the fact that the strip defined by l and the tangent plane to S lies on S. We thus have the third theorem:

THEOREM 3. *Any integral surface can be covered by a family of characteristic strips.*

It may be mentioned that, if we stay in the real domain, this result only holds for the hyperbolic or parabolic cases; in the hyperbolic case we can cover the integral surface by two families of characteristic strips.

We now prove the following converse:

THEOREM 4. *If a family of characteristic strips $u = u(x, y)$, where $u(x, y)$ has continuous derivatives up to the second order, forms a surface S, this surface is an integral surface of equation* (28).

Let S be a surface covered by a family of strips, along which equations (34) are satisfied. We have along each of these strips:

$$\mathrm{d}p = r\,\mathrm{d}x + s\,\mathrm{d}y; \quad \mathrm{d}q = s\,\mathrm{d}x + t\,\mathrm{d}y.$$

On substituting these expressions for $\mathrm{d}p$ and $\mathrm{d}q$ in the second of equations (34), we arrive at the following two equations:

$$as\,\mathrm{d}y^2 + (ar + ct + h)\,\mathrm{d}x\,\mathrm{d}y + cs\,\mathrm{d}x^2 = 0$$
$$a\,\mathrm{d}y^2 - 2b\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}x^2 = 0.$$

On multiplying the second by s and subtracting from the first, we arrive at the fundamental equation (28), where it must be observed that the product $\mathrm{d}x\mathrm{d}y$ differs from zero, since x and y are the independent variables.

In the case of a first order equation, the characteristic strips were given by an ordinary system of ordinary differential equations, and in view of this, the problem of solving a first order partial differential equation reduced the solution of a system of ordinary differential equations. In the present case system (34) is a system of three exact differential equations for five required functions. Levi has shown (*Math. Annal.*, vol. 97) how system (34) can be extended so as to obtain a system of five differential equations of the first order with five unknown functions — a system having a special form. A solution of the Cauchy problem can be constructed in a unique manner for this system, which leads to the solution of the Cauchy problem for equation (28) also.

The next section discusses particular cases when system (34) has a solution.

136. Intermediate integrals. For convenience in future working, we transform (34) for the characteristic strips into a new form. On recalling the basic property of the roots of a quadratic equation, we can write $\mu_1\,\mu_2 = c/a$, and this equation can be used to rewrite (34) with $i = 1$ as:

$$\mathrm{d}y - \mu_1\,\mathrm{d}x = 0; \quad \mathrm{d}p + \mu_2\,\mathrm{d}q + \frac{h}{a}\,\mathrm{d}x = 0; \quad \mathrm{d}u - (p + q\mu_1)\,\mathrm{d}x = 0. \quad (42)$$

The second system (with $i = 2$) is obtained from the first by interchanging the symbols μ_1 and μ_2. We shall seek a function $V(x, y, u, p, q)$ such that its

total differential vanishes by virtue of (42):

$$V_x \, dx + V_y \, dy + V_u \, du + V_p \, dp + V_q \, dq = 0. \tag{43}$$

Having found dy, du and dp from system (42) and after substituting in the left-hand side of the last equation, we must be able to equate to zero the coefficients of the remaining differentials dx and dq. Hence we find that the necessary and sufficient condition for V to be a solution of system (42):

$$V(x, y, u, p, q) = C, \tag{44}$$

is that V satisfies two linear homogeneous first order partial differential equations:

$$\left.\begin{aligned} V_x + \mu_1 V_y + (p + \mu_1 q) V_u - \frac{h}{a} V_p &= 0, \\ V_q - \mu_2 V_p &= 0. \end{aligned}\right\} \tag{45}$$

If μ_1 and μ_2 are interchanged in these equations, an analogous system is obtained, expressing the necessary and sufficient condition for V to be a solution of the second system of characteristic strips. The methods of seeking the solutions of system (45) have been described in [120]. Suppose we have managed to find a solution of this system which is non-trivial (not a constant). We show that *every solution of first order equation* (44), *which is not a singular solution, is now also a solution of equation* (28). In fact, the total differential of V must vanish here by virtue of (42), i.e. must be a linear combination of the left-hand sides of these equations:

$$dV = \alpha(dy - \mu_1 dx) + \beta\left(dp + \mu_2 dq + \frac{h}{a} dx\right) + \gamma(du - p\,dx - q\,dy), \tag{46}$$

Let S be an integral surface of (44). On this surface, u, p, q are definite functions of (x, y), and by solving the first order equation $dy - \mu_1 dx = 0$, we get a family of curves covering S. In addition, we must evidently have $du = p dx + q dy$ along these curves. Since, by what has just been said, the factors of α and γ in (46) must vanish along our curves, the following equation must hold along the curves, i.e. on the surface S:

$$\beta\left(dp + \mu_2 dq + \frac{h}{a} dx\right) = 0.$$

By hypothesis, the integral surface S is not a singular solution, so that the coefficient of dp either or dq will be non-zero on the left-hand side of (43). It follows from this that $\varrho \neq 0$, i.e. all three of equations (42) are satisfied along our curves, so that S is in fact covered by characteristic strips of equation (28). But now, by the fourth theorem of [135], this surface is an integral surface of (28). Hence, knowing solution (44), we obtain a class of solutions of (28) by integrating the first order equation (44). Suppose we have found two independent solutions of system (45): V_1 and V_2. The expression $V_1 - \Phi(V_2)$, where Φ is an arbitrary function, will now also be a solution of system (45), and we have the following solution of system (42):

$$V_1 - \Phi(V_2) = 0, \tag{47}$$

containing the arbitrary function Φ. Suppose we want to find the integral surface of (28) containing a given strip (29). On replacing x, y, u, p, q by their expressions (29) in functions V_1 and V_2, we get two definite functions of the parameter t: $v_1(t)$ and $v_2(t)$. Equation (47) now reduces to the form $v_1(t) - \Phi[v_2(t)] = 0$. We introduce instead of t the new variable $\sigma = v_2(t)$. On solving this equation for t, we get $t = \omega(\sigma)$, and the previous equation, expressed in the variable σ, defines the form of our function $\Phi(\sigma) = v_1[\omega(\sigma)]$. After finding the form of $\Phi(\sigma)$, equation (47) will become a definite first order equation. On solving the Cauchy problem for it, with initial data (29), we get a solution of the Cauchy problem for (28) also. Every solution of system (42) or of the analogous system obtained by interchanging μ_1 and μ_2 is usually called an *intermediate integral* of equation (28). We remark that, if system (45) is complete, it has three independent solutions. It can be shown that this can only be the case when $\mu_1 = \mu_2$.

Note. Suppose that $h = 0$, and that coefficients a, b and c are constant or depend only on p and q. Now, μ_1 and μ_2 also depend only on p and q, and we can find a solution of system (45) if we seek V depending only on p and q. The first equation will now be satisfied for any choice of $V(p, q)$, since $h = 0$, and we get a single equation for V: $V_q - \mu_2(p, q) V_p = 0$. Having found a solution $V_1(p, q)$ of this equation, we obtain the first order equation $V_1(p, q) = \text{const.}$, every solution of which also satisfies the original second order equation. Instead of $\mu_2(p, q)$, we could have used the second root $\mu_1(p, q)$ of equation (32), and another first order equation: $V_2(p, q) = \text{const}$ would have been obtained.

137. The Monge–Ampere equations. The whole of the above theory of characteristic strips and intermediate integrals can be extended at once to equations of a more general type, viz., equations which are linear in r, s, t and $rt - s^2$, i.e. of the form:

$$ar + 2bs + ct + g\,(rt - s^2) + h = 0 \qquad (g \neq 0)$$

This is general known as a *Monge–Ampere equation.* If a strip is given, the second order derivatives will be uniquely determined along this strip if the expression

$$A = a\,\mathrm{d}y^2 - 2b\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}x^2 + g\,(\mathrm{d}x\,\mathrm{d}p + \mathrm{d}y\,\mathrm{d}q)$$

differs from zero. If this expression vanishes, whilst

$$B = a\,\mathrm{d}p\,\mathrm{d}y + h\,\mathrm{d}x\,\mathrm{d}y + c\,\mathrm{d}q\,\mathrm{d}x + g\,\mathrm{d}p\,\mathrm{d}q$$

differs from zero, determination of the second order derivatives will lead to an incompatible system. The characteristic strip is given by the following three equations:

$$A = 0;\ B = 0;\ \mathrm{d}u = p\,\mathrm{d}x + q\,\mathrm{d}y.$$

If we write μ_1 and μ_2 for the roots of

$$\mu^2 + 2b\mu + ac - gh = 0,$$

the two systems of characteristic strips can be defined by the equations:

$$g\,\mathrm{d}p + c\,\mathrm{d}x + \mu_1\,\mathrm{d}y = 0;\quad g\,\mathrm{d}q + a\,\mathrm{d}y + \mu_2\,\mathrm{d}x = 0;\quad \mathrm{d}u = p\,\mathrm{d}x + q\,\mathrm{d}y.$$

The second system can be obtained from the one written by interchanging μ_1 and μ_2. All these results are obtained by working similar to the above. The fundamental theorems of [135] also remain valid for Monge–Ampere equations.

When seeking the intermediate integrals, instead of (45) we arrive at the system:

$$V_x + pV_u - \frac{c}{g} V_p - \frac{\mu_2}{g} V_q = 0,$$

$$V_y + qV_u - \frac{\mu_1}{g} V_p - \frac{a}{g} V_q = 0.$$

The second system can be obtained from the above by interchanging μ_1 and μ_2 as previously. All the properties described above for intermediate integrals still hold.

138. Characteristics with any number of independent variables. We now consider an equation of the second order in any number of independent variables:

$$\sum_{i,\,k=1}^{n} a_{ik} u_{x_i x_k} + \ldots = 0 \qquad (a_{ik} = a_{ki}) \tag{48}$$

where the unwritten terms do not contain second order derivatives. The coefficients a_{ik} will be assumed for the present to be functions of the independent variables x_s only. We shall confine ourselves in the present case to discussing the condition for unique determination of the second order derivatives to be impossible from equation (48) and initial Cauchy data, i.e. an incompatible or indeterminate system is obtained when seeking these derivatives. This condition is analogous to (32) for the case of two independent variables. We shall start our discussion of the problem with the case when the initial Cauchy data have the special form:

$$u\,|_{x_1 = x_1^{(0)}} = \varphi(x_2, \ldots, x_n); \quad u_{x_1}\,|_{x_1 = x_1^{(0)}} = \psi(x_2, \ldots, x_n).$$

These initial data enable us to define all the first order derivatives and all the second order derivatives except $u_{x_1 x_1}$ on the hyperplane $x_1 = x_1^{(0)}$. To find $u_{x_1 x_1}$ we have to make use of (48) itself, after putting $x_1 = x_1^{(0)}$. If it happens here that $a_{11} \neq 0$, we obtain a definite value for the derivative in question. Whereas if $a_{11} = 0$ after the substitution, we either arrive at an impossible equation or obtain an identity. Thus the required condition has the form, in the case of special Cauchy data:

$$a_{11} = 0. \tag{49}$$

We now turn to the general case, when the initial Cauchy data are given on some hypersurface:

$$\omega_1(x_1, \ldots, x_n) = 0. \tag{50}$$

In addition to the function ω_1 of the last expression, we introduce a further $(n-1)$ functions $\omega_s(x_1, \ldots, x_n)$ $(s = 2, \ldots, n)$ such that we can perform the change of independent variables:

$$x'_s = \omega_s(x_1, \ldots, x_n) \qquad (s = 1, \ldots, n), \tag{51}$$

i.e. such that these last equations are soluble for the x_s. We express the derivatives with respect to the old variables in terms of the derivatives with respect to the new, only the terms containing the derivatives of interest to us being written down:

$$u_{x_i} = u_{x'_1}\frac{\partial\omega_1}{\partial x_i} + \ldots; \qquad u_{x_i x_k} = u_{x'_1 x'_1}\frac{\partial\omega_1}{\partial x_i}\frac{\partial\omega_1}{\partial x_k} + \ldots$$

The transformed equation will have the form:

$$a'_{11}u_{x'_1x'_1} + \ldots = 0, \quad \text{where } a'_{11} = \sum_{i,\,k=1}^{n} a_{ik}\frac{\partial\omega_1}{\partial x_i}\frac{\partial\omega_1}{\partial x_k}, \tag{52}$$

and the unwritten terms do not contain the derivative $u_{x'_1x'_1}$. By (51), the initial data for the transformed equation are given on the hyperplane $x' = 0$, i.e. they have the special form. We can therefore make use of condition (49) in our present case, though only in the new independent variables. On taking (52) into account, we can thus say that the necessary and sufficient condition for the initial Cauchy data on the hypersurface (50) to lead to incompatibility or indeterminancy when seeking the second order derivatives is that the function ω_1 satisfies the equation:

$$\sum_{i,\,k=1}^{n} a_{ik}\frac{\partial\omega_1}{\partial x_i}\frac{\partial\omega_1}{\partial x_k} = 0, \tag{53}$$

where this equation must be satisfied for $\omega_1 = 0$, i.e. in other words, by virtue of equation (50). We shall call every hypersurface satisfying this condition *a characteristic surface or characteristic of equation* (48).

If we fix some point $M_0(x_1^{(0)}, \ldots, x_n^{(0)})$, the coefficients a_{ik} will have fixed values at this point, which we shall denote by $a_{ik}^{(0)}$. The direction of the vector, the real components $\alpha_1, \ldots, \alpha_n$ of which satisfy the equation:

$$\sum_{i,\,k=1}^{n} a_{ik}^{(0)}\alpha_i\alpha_k = 0, \tag{53$_1$}$$

will be called the characteristic normal direction at the point M_0. Equation (53) is equivalent to the fact that, at every point of the surface $\omega_1(x_1, \ldots, x_n) = 0$ the direction of the normal to the surface is the characteristic normal direction. If the surface $S(\omega_1 = 0)$ is such that the normal direction is characteristic at no point of the surface, i.e. the left-hand side of (53) is non-zero throughout the surface, it follows from what has been said above that the change of variables (51) will enable us to rewrite (48) as:

$$u_{x_1'x_1'} = \sum_{i,k=2}^{n} a''_{ik} u_{x_i'x_k'} + \sum_{i=2}^{n} a''_{1i} u_{x_1'x_i'} + \ldots, \tag{48_1}$$

where the surface S becomes the plane $x_1' = 0$. This enables us to transform a Cauchy problem with initial data on surface S to a Cauchy problem with initial data on the plane $x_1' = 0$. If (48) has an analytic character, e.g. is linear and has analytic coefficients, the surface S is non-characteristic and ω_1 is an analytic function, then the transformed Cauchy problem can be solved, given suitable conditions, in accordance with Kovalevskaya's theorem. If S is a characteristic surface, the function u and its first order partial derivatives must be connected on it by some relationship. For, u and its partial derivatives on S are expressible in terms of the same magnitudes on the plane $x_1' = 0$ and vice versa. Let

$$u = \varphi_0(x_2', \ldots, x_n'); \quad u_{x_1'} = \varphi_1(x_2', \ldots, x_n') \quad \text{for} \quad x_1' = 0$$
$$u_{x_k'} = \frac{\partial \varphi_0}{\partial x_k'} \qquad (k = 2, \ldots, n).$$

If S is a characteristic surface, $a_{11}' = 0$ for $x_1' = 0$ in the transformed equation, and we have:

$$\sum_{i,k=2}^{n} a'_{ik} u_{x_i'x_k'} + \sum_{i=2}^{x} a'_{1i} u_{x_1'x_i'} + \ldots = 0,$$

where the unwritten terms contain only first order derivatives. The following connection is therefore obtained between functions φ_0 and φ_1:

$$\sum_{i,k=2}^{n} a'_{ik} \frac{\partial^2 \varphi_0}{\partial x_i' \partial x_k'} + \sum_{i=2}^{n} a'_{1i} \frac{\partial \varphi_1}{\partial x_i'} + \ldots = 0.$$

This relationship does not in general lead to an identity in φ_0 and φ_1.

We now suppose that coefficients a_{ik} depend on u and u_{x_s} as well as on x_s. The initial Cauchy data on the $(n - 1)$-manifold (50) depend

on $(n-1)$ parameters. Let us assume that $x_2, \ldots, x_n$ are these parameters. On substituting these expressions for the initial data in the coefficients a_{ik}, we obtain equation (53) as before, where the equation must be satisfied by virtue of (50), and we can find whether or not $\omega_1 = 0$ is a characteristic surface for the given initial data.

We shall confine ourselves in future to the case when the coefficients a_{ik} depend only on the x_s. It may be remarked that, if (48) is an equation of the elliptic type, equation (53) can have no real solutions apart from $\omega_1 = \text{const.}$, as in the case of two independent variables. The constant solution is obviously of no interest for our problem.

139. Bicharacteristics. Equation (53) must be fulfilled by virtue of (50). Let us require that this equation be satisfied as an identity with respect to the x_s. Equation (53) now becomes an ordinary first order partial differential equation, and every solution of it differing from a constant will yield a whole family instead of a single characteristic:

$$\omega_1(x_1, \ldots, x_n) = C, \tag{54}$$

where C is an arbitrary constant. Conversely, the necessary and sufficient condition for the last equation to define a family of characteristics with an arbitrary constant C is that the function ω_1 satisfies equation (53). Precisely as in [100], it can be shown that every characteristic can be included in a family of form (54) and that the solutions of (53) therefore give us all the characteristics.

In the equations of mathematical physics, the independent variable time plays a special role as compared with the remaining variables, which usually give the spatial coordinates. We shall assume in future that this special independent variable is x_n and we shall write $x_n = t$. We introduce the notation $x_1, \ldots, x_m$ for the remaining variables, i.e. we assume $n = m + 1$.

Let us write surface (50) in the explicit form with respect to t: $t - \omega(x_1, \ldots, x_m) = 0$, and let us assume that the coefficients a_{ik} do not depend on t.

On substituting the left-hand side of the equation $t - \omega = 0$ in (53), we get the following equation for the function ω:

$$\sum_{i,k=1}^{m} a_{ik}\frac{\partial\omega}{\partial x_i}\frac{\partial\omega}{\partial x_k} - 2\sum_{i=1}^{m} a_{in}\frac{\partial\omega}{\partial x_i} + a_{nn} = 0. \tag{55}$$

This equation must strictly speaking be satisfied by virtue of $t = \omega$. But it does not contain the letter t at all, so that we can say that it

must be satisfied identically. On returning to the general case, let us consider equation (53) and write the corresponding first order Cauchy system. Equation (53) does not contain the function ω_1 itself, so that we shall not write down the relationship that contains $d\omega_1$ in the corresponding Cauchy system. We thus obtain the following system of ordinary differential equations:

$$\frac{dx_k}{ds} = 2\sum_{i=1}^{n} a_{ki}p_i\,; \tag{56_1}$$

$$(k = 1, 2, \ldots, n)$$

$$\frac{dp_k}{ds} = -\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_k} p_i p_j, \tag{56_2}$$

where s is an auxiliary parameter. We take a family of characteristic hypersurfaces $\omega_1(x_1, \ldots, x_n) = C$ and put $p_k = \partial\omega_1/\partial x_k$. Now, p_i is expressed in terms of $(x_1, \ldots, x_n)$, and, on substituting these expressions in the right-hand sides of equations (56_1), we get a first order system for $(x_1, \ldots, x_n)$. If we take any solution of this system and substitute in the above-mentioned expressions for p_k in terms of $(x_1, \ldots, x_n)$, it may easily be verified that the functions obtained will satisfy equations (56_2). In fact:

$$\frac{dp_k}{ds} = \sum_{i=1}^{n} \frac{\partial^2\omega_1}{\partial x_k\,\partial x_i}\frac{dx_i}{ds} = 2\sum_{i,j=1}^{n} \frac{\partial^2\omega_1}{\partial x_k\,\partial x_i} a_{ij}p_j = 2\sum_{i,j=1}^{n} \frac{\partial p_i}{\partial x_k} a_{ij}p_j. \tag{57}$$

We replace the subscript k by j in (53) and differentiate both sides with respect to x_k:

$$\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_k} p_i p_j + \sum_{i,j=1}^{n} a_{ij}\frac{\partial p_i}{\partial x_k} p_j + \sum_{i,j=1}^{n} a_{ij}p_i\frac{\partial p_j}{\partial x_k} \equiv 0.$$

Since $a_{ij} = a_{ji}$, the last two sums are equal, and we can use the last identity to rewrite (57) as:

$$\frac{dp_k}{ds} = -\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_k} p_i p_j\,,$$

which is in fact the same as (56_2). We remark that equation (54) is now a solution of system (56_1). For:

$$\frac{d\omega_1}{ds} = \sum_{k=1}^{n} \frac{\partial\omega_1}{\partial x_k}\cdot\frac{dx_k}{ds} = 2\sum_{i,j=1}^{n} a_{ik}\frac{\partial\omega_1}{\partial x_k}\frac{\partial\omega_1}{\partial x_i}\,,$$

and the last sum is identically zero, by (53). The curves of space R_n with coordinates $(x_1, \ldots, x_n)$, which are obtained as a result of

integrating system (56_1), in which we put $p_i = \partial\,\omega_1/\partial\,x_i$, are called *bicharacteristics corresponding to the system* $\omega_1 = C$ *of characteristic surfaces.*

If we take a point on a hypersurface $\omega_1 = C_0$ as the initial values of the x_k when integrating system (56_1), the whole of the corresponding bicharacteristic will lie on the hypersurface, i.e. *every characteristic surface of equation* (48) *can be formed from bicharacteristics.* We now indicate the conditions in which the solutions of systems (56_1), (56_2) form a characteristic hypersurface. Surface (54) is an $(n-1)$-manifold in R_n. The parameter s appears in the equation of the bicharacteristic, so that we must form the characteristic hypersurface (54) by taking a family of bicharacteristics depending on $(n-2)$ parameters. We shall assume that the initial values $x_k^{(0)}$ and $p_k^{(0)}$ of the variables appearing in system (56_1), (56_2) depend on $(n-2)$ parameters $t_1, \ldots, t_{n-2}$. On repeating the arguments of [106], it is easily seen that the necessary and sufficient condition for the family of bicharacteristics obtained to give a characteristic hypersurface is that the above-mentioned initial values satisfy the following relationships [110]:

$$\sum_{i,k=1}^{n} a_{ik}^{(0)} p_i^{(0)} p_k^{(0)} = 0\,, \tag{58}$$

$$\sum_{s=1}^{n} p_s^{(0)} \frac{\partial x_s^{(0)}}{\partial t_j} = 0 \quad (j = 1\,, \ldots, n-2)\,, \tag{59}$$

where the $a_{ik}^{(0)}$ are the results of substituting $x_s = x_s^{(0)}$ in the a_{ik}. We are assuming here that at least one of the functional determinants of order $(n-1)$ of variables $(x_1, \ldots, x_n)$ with respect to $(s, t_1, \ldots, t_{n-2})$ differs from zero.

All the above results follow directly from the Cauchy method of solving first order equations [110]. There is a minor complication in the present case, in that the equation of the integral surface is sought in the implicit form $\omega_1(x_1, \ldots, x_n) = C$, whereas the Cauchy system (56) does not contain ω_1 itself.

A fundamental role in mathematical physics is played by a singular integral surface of equation (53), namely the *characteristic conoid* of this equation. This characteristic surface is obtained by the above method if we take the $x_k^{(0)}$ as fixed, i.e. independent of the parameters (the vertex of the conoid), and subject the $p_k^{(0)}$ to condition (58). We remark that the n quantities $p_k^{(0)}$ are defined from this equation as functions of $(n-1)$ parameters. Since (58) is homogeneous, one of

the parameters appears as a factor in $p_k^{(0)}$. But it is easily verified that equations (56_1) and (56_2) are unchanged if we replace s by $(1/a)s$ and p_k by ap_k, where a is independent of s. Hence the parameter appearing as a factor in the $p_k^{(0)}$ can be taken say equal to unity.

If the coefficients a_{ik} are constant, equations (56_2) show that the p_k must be constant, whilst we see from equations (56_1) that the x_k are first degree polynomials in s, i.e. *if the a_{ik} are constant, the bicharacteristics are straight lines in R_n.*

Let us take an important particular case. We introduce the above-mentioned notation $x_n = t$ and consider an equation of the special form:

$$u_{tt} - \sum_{i,\,k=1}^{m} a_{ik}\, u_{x_i x_k} + \ldots = 0. \tag{60}$$

where $m = n - 1$, and the a_{ik} do not contain t, i.e. depend only on $(x_1, \ldots, x_m)$. We shall assume that the quadratic form

$$\sum_{i,\,k=1}^{m} a_{ik}\, \xi_i\, \xi_k$$

is positive definite for all values of ξ_s. Equation (53) now becomes:

$$\left(\frac{\partial \omega_1}{\partial t}\right)^2 - \sum_{i,\,k=1}^{m} a_{ik} \frac{\partial \omega_1}{\partial x_i} \frac{\partial \omega_1}{\partial x_k} = 0.$$

We shall seek the characteristic hypersurface in the explicit form in t:

$$\omega(x_1, \ldots, x_m) - t = 0 \quad \text{or} \quad t = \omega(x_1, \ldots, x_m). \tag{61}$$

Here, $p_0 = \partial\omega_1/\partial t = -1$, and we obtain for the functions ω the first order equation:

$$\sum_{i,\,k=1}^{m} a_{ik}\, \omega_{x_i}\, \omega_{x_k} = 1 \tag{62}$$

or

$$\sum_{i,\,k=1}^{m} a_{ik}\, p_i\, p_k = 1. \tag{63}$$

The Cauchy system corresponding to this equation will be

$$\frac{dx_k}{2\sum_{i=1}^{m} a_{ki}\, p_i} = \frac{dt}{2\sum_{i,k=1}^{m} a_{ik}\, p_i\, p_k} = \frac{dp_k}{-\sum_{i,j=1}^{m} \frac{\partial a_{ij}}{\partial x_k}\, p_i\, p_j}.$$

If we take some concrete characteristic hypersurface (61), it follows from (63) and the last system that the bicharacteristics forming it

must satisfy the following system:

$$\frac{dx_k}{dt} = \sum_{i=1}^{m} a_{ki} p_i \quad (p_i = \omega_{x_i}) \qquad (k = 1, \ldots, m). \tag{64}$$

Instead of regarding surface (61) as a fixed surface in n-dimensional space R_n with coordinates $(x_1, \ldots, x_m, t)$, we can regard it as a surface moving in the course of time in m-dimensional space R_m with coordinates $(x_1, \ldots, x_m)$. In this case, we regard the solutions of system (64) as curves λ in space R_m, defined parametrically with the aid of the parameter t (time). Now, of course, a curve λ in space R_m will no longer lie on the moving surface (61).

If, for instance, we had the cone

$$x_1^2 + x_2^2 - c^2t^2 = 0,$$

in space R_3 with coordinates (x_1, x_2, t), we must regard it as a circle on the (x_1, x_2) plane with centre at the origin and with variable radius ct. If the straight generators of this cone were bicharacteristics, the curvesλ in the (x_1, x_2) plane consist of a pencil of straight lines issuing from the origin. This example corresponds, as we shall see later, to the case when the given equation is the wave equation:

$$u_{tt} - c^2 (u_{x_1x_1} + u_{x_2x_2}) = 0.$$

140. The connection with variational problems. Let A be the matrix of coefficients a_{ik}. On solving equations (64) for the p_i, we get $p_i = A^{-1}\, dx_k/dt$, where, as usual, A^{-1} is the inverse of matrix A. On substituting the expressions obtained for the p_i in the left-hand side of (63), we transform the quadratic form in p_i into a quadratic form in dx_k/dt, i.e. we have

$$\sum_{i,k=1}^{m} b_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt} = \sum_{i,k=1}^{n} a_{ik} p_i p_k = 1, \tag{65}$$

where the matrix B of coefficients b_{ik} is given in terms of the matrix A by the expression [III, 32]:

$$B = (A^{-1})^* A A^{-1} = (A^{-1})^*,$$

or, allowing fo rthe fact that A is symmetric: $B = A^{-1}$.

We introduce a metric into space R_m, defined by

$$d\sigma^2 = \sum_{i,k=1}^{m} b_{ik}\, dx_i\, dx_k.$$

The integral

$$\int \mathrm{d}\sigma = \int \sqrt{\sum_{i,k=1}^{m} b_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k} = \int_{t_0}^{t_1} \sqrt{\sum_{i,k=1}^{m} b_{ik}\,x_i'\,x_k'\,\mathrm{d}t}\,, \tag{66}$$

taken over any bicharacteristic that forms part of the characteristic hypersurface (61) is equal, by (65), to the difference in the values of t corresponding to the ends of the path of integration, i.e. the length of any arc of the bicharacteristic is defined, given metric (66), by the difference in the values of time corresponding to the ends of the arc.

On comparing the above results with those of [80], we see that (63) is the equation of the basic field function for integral (66). The family of hypersurfaces $\omega(x_1, \ldots, x_m) = t$ is therefore a family of transversal surfaces of some field of the variational problem for integral (66). Further, it may readily be seen that the bicharacteristics, corresponding to the family in question and defined by equations (64), will be extremals of the field. This is proved simply by using (64) to show that the bicharacteristics cut the hypersurfaces $\omega(x_1, \ldots, x_m) = t$ transversally.

For, the transversality condition here reduces to the proportionality of $p_i = \omega_{x_i}$ and the derivatives of the integrand of (66) with respect to x_i' [80], i.e. to the proportionality of p_i and $\sum_{k=1}^{m} b_{ik}\,x_k'$. But we obtain on solving (64) for the p_i:

$$p_i = \sum_{k=1}^{k} b_{ik}\,x_k',$$

which in fact proves our assertion regarding the transversal intersection of the family of characteristic hypersurfaces by the corresponding bicharacteristics.

We remark that the case of a characteristic conoid corresponds to a quasisphere in space R_m with centre $(x_1^{(0)}, \ldots, x_m^{(0)})$ corresponding to the vertex of the conoid, and with radius t.

If equation (60) corresponds to a wave process in space R_m, the first order equation (63) defines the geometrical optics of this process with the aid of the characteristic surface, and the bicharacteristics are rays defining the same geometrical optics. The above considerations lead to a geometrical optics having a direct connection with a variational problem. If we are given the wave front S_0 at $t = 0$, to obtain the wave front S_t at any instant t, we must construct a family of quasispheres with centres on S_0 and radius t, and take

the envelope of this family (the Huyghens construction). This construction corresponds to what was said in [109] regarding the solution of Cauchy's problem for first order equations with the aid of the characteristic conoids of the equation. We shall not dwell on the proof of the construction. It can be carried out on the basis of the theory of the complete integral. We remark that the envelope of the quasispheres of radius t can consist of two hypersurfaces. Only one of them will give the wave front at the instant t.

All the above discussion can be carried out in space R_n, where t is included among the coordinates, instead of in space R_m. For greater symmetry, we take the general case of equation (48):

$$\sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} + \ldots = 0 \qquad (a_{ki} = a_{ik}), \tag{67}$$

where a_{ik} are given functions of $(x_1, \ldots, x_n)$. The characteristic surfaces will be given by the equation

$$D(x_1, \ldots, x_n, p_1, \ldots, p_n) = \sum_{i,k=1}^{n} a_{ik} p_i p_k = 0 \qquad \left(p_i = \frac{\partial \omega_1}{\partial x_i}\right), \tag{68}$$

where $D(x_1, \ldots, x_n, p_1, \ldots, p_n)$ denotes the left-hand side of the equation. The Cauchy system corresponding to this equation, i.e. the system of ordinary equations defining the bicharacteristics, is given by equations (56_1) and (56_2). On replacing the auxiliary parameter s by $s/2$, we can write this system as

$$\frac{dx_k}{ds} = \frac{1}{2} D_{p_k}; \qquad \frac{dp_k}{ds} = -\frac{1}{2} D_{x_k} \qquad (k = 1, \ldots, n). \tag{69}$$

The first equations of this system have the form:

$$\frac{dx_k}{ds} = \sum_{i=1}^{n} a_{ki} p_i \qquad (k = 1, 2, \ldots, n).$$

On solving these equations for p_i and substituting in (68), we get:

$$\sum_{i,k=1}^{n} b_{ik} \frac{dx_i}{ds} \frac{dx_k}{ds} = 0, \tag{70}$$

where the matrix B of coefficients b_{ik} is given by $B = A^{-1}$. We introduce the metric into space R_n:

$$d\sigma_1^2 = \sum_{i,k=1}^{n} b_{ik} \, dx_i \, dx_k.$$

The main difference from the previous metric is that the right-hand side can take negative as well as positive values for an equation of the hyperbolic type (an alternating quadratic form [III, 35]), so that $d\sigma_1$ can be imaginary.

It follows from (70) that the special feature of a bicharacteristic is that $d\sigma_1 = 0$, i.e. given the above metric, the length of any segment of a bicharacteristic is zero. The non-real nature of the present metric must be borne in mind here.

141. The propagation of a surface of discontinuity. Suppose that the second order derivatives of a solution u of equation (48) have discontinuities of the first kind on the surface

$$\psi(x_1, \ldots, x_n) = 0, \tag{71}$$

while the solution itself and its first order derivatives remain continuous on passing through this surface. We shall consider the solution u from the two different sides of surface (71) as two different solutions of equation (48). These solutions have the same Cauchy data on this surface, though their second order derivatives have different values; we can therefore say that (71) must be a characteristic surface of equation (67). We should have arrived at the same result if we had assumed that the second order partial derivatives, as well as u itself and its first order derivatives, remain continuous on passing through surface (71), whilst only the derivatives of order higher than the second are discontinuous. In general, a solution of the second order equation (67) is said to have a *weak discontinuity* on surface (71) if, on passing through the surface, u and its first derivatives remain continuous, whilst certain derivatives of order higher than the first have discontinuities of the first kind on the surface. It follows from the above discussion that *a characteristic surface must be a surface of weak discontinuity.*

On distinguishing the independent variable $x_n = t$ as above, instead of (71) we have a moving surface of weak discontinuity in space R_m:

$$\psi(x_1 \ldots, x_m, t) = 0. \tag{72}$$

Let us find the displacement velocity of this surface. We take a point M on surface (72) and draw the normal to the surface at M towards the side where $\psi > 0$. We take the segment MM_1 along this direction of the normal, from M to the point of intersection M_1 with

the surface corresponding to the instant $(t + \Delta t)$. The limit of the ratio $|MM_1| : \Delta t$ as $\Delta t \to 0$ is usually called the displacement velocity of surface (72). On introducing the notation:

$$g = \sqrt{\sum_{i=1}^{m} \psi_{x_i}^2}, \tag{73}$$

we have the following expression for the direction-cosines of the normal:

$$\cos(n, x_i) = \frac{\psi_{x_i}}{g}. \tag{74}$$

We differentiate (72):

$$\sum_{i=1}^{m} \psi_{x_i} \, \mathrm{d}x_i + \psi_t \, \mathrm{d}t = 0.$$

The quantity $\mathrm{d}x_i$ can be taken as the projection on the coordinate axis, of the infinitesimal displacement MM_1 along the normal and we can therefore write:

$$\sum_{i=1}^{m} \psi_{x_i} |MM_1| \cos(n, x_i) + \psi_t \, \mathrm{d}t = 0.$$

On taking (74) into account, we get the following expression for the rate of displacement of surface (72):

$$P = -\frac{\psi_t}{g}. \tag{75}$$

In the case $m = 2$, we have a curve moving on the (x_1, x_2) plane, in the case $m = 3$ we have a surface moving in three-dimensional space (x_1, x_2, x_3).

Let us take as an example the wave equation with $m = 1$:

$$u_{tt} - a^2 u_{xx} = 0.$$

The fundamental equation (53) becomes:

$$\psi_t^2 - a^2 \psi_x^2 = 0 \quad \text{or} \quad \frac{\psi_t}{\psi_x} = \pm a,$$

and it shows that every weak discontinuity must move along the x axis with velocity $\pm a$. The characteristics on the (x, t) plane are two families of straight lines $x \pm at = c$. We consider also the equation

$$u_{tt} - f(u_x, u_t) u_{xx} = 0,$$

which is encountered when considering the motion of a compressible fluid in the one-dimensional case. Equation (53) becomes:

$$\psi_t^2 - f(u_x, u_t) \psi_x^2 = 0.$$

Suppose that the fluid is at rest on the x axis beyond the discontinuity, i.e. we have $u_x = u_t = 0$ on this side of the discontinuity and at the point of discontinuity itself. The above condition becomes $\psi_t^2 - f(0, 0)\,\psi_x^2 = 0$, and the speed of propagation of the discontinuity is given by:

$$P = \pm \sqrt{f(0, 0)}. \tag{76}$$

We now turn to the wave equation in three independent variables:

$$u_{tt} - a^2 (u_{x_1 x_1} + u_{x_2 x_2}) = 0.$$

Equation (53) now becomes:

$$\psi_t^2 - a^2 (\psi_{x_1} + \psi_{x_2}^2) = 0,$$

or, using (73), the last equation can be written as $\psi_t^2 - a^2 g^2 = 0$, and this first order equation expresses the fact that every characteristic curve on the (x_1, x_2) plane must move with velocity a. A similar result is obtained for the characteristic surface in three-dimensional space (x_1, x_2, x_3), if we start from the wave equation:

$$u_{tt} - a^2 (u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}) = 0.$$

We remark that the coefficient a^2 can be assumed dependent on the coordinates (x_1, x_2, x_3).

142. Strong discontinuities. We have assumed in our study of discontinuous solutions of second order equations that the function itself and its first order derivatives remain continuous on passing through the surface of discontinuity and that only derivatives of the second order upwards have discontinuities (weak discontinuities). But we were able to assert with this assumption that the surface of discontinuity must be a characteristic surface. We now turn to a study of *strong discontinuities*. These imply, in the case of a second order equation, that even the first order derivatives have discontinuities. Our aim is to discover the circumstances in which a surface of discontinuity is necessarily a characteristic surface, as before. Let us take the wave equation in three independent variables. We introduce an operator on the left-hand side of this equation:

$$\Box\, u = u_{xx} + u_{yy} - \frac{1}{a^2} u_{tt}. \tag{77}$$

This expression is usually known as a *Lorentz operator*. We bring in a further operator, containing the first order derivatives:

$$P(u) = u_x \cos(n, x) + u_y \cos(n, y) - \frac{1}{a^2} u_t \cos(n, t), \qquad (78)$$

where n is a direction in space (x, y, t). Let D be a domain in space (x, y, t), S the surface bounding it, and n the direction of the outward normal to S. On applying the usual Gauss formula, we can write the following Green's formula for the Lorentz operator precisely as in [II, 193]:

$$\iiint_D [v \Box u - u \Box v] \, d\tau = \iint_D [vP(u) - uP(v)] \, dS, \qquad (79)$$

where u and v are two functions having continuous derivatives up to the second order in D. Suppose that the domain D is split by a surface σ into two parts D_1 and D_2, where σ is a surface of discontinuity for the first order derivatives of the function u. Let us consider the conditions that this discontinuity must satisfy for (79) to remain valid as before throughout D as regards a function u with discontinuous derivatives and as regards any function v with continuous derivatives up to the second order. We shall first of all assume that u itself remains continuous on passing through σ. Let M be a point of the surface σ and l any direction lying in the tangent plane to σ at the point M. We shall assume that the derivative $\partial u/\partial l$ has the same limit as we approach M from both sides of σ, and that this limit is equal to the derivative of the values of u on σ itself taken with respect to the direction l. This condition is sometimes called the *kinematic compatibility condition*. If n is a fixed direction normal to σ at M, we shall assume that $\partial u/\partial n$ has definite limits as we approach M from either side of the surface, but that these limits may be different for the two sides.

We now turn to the formulation of a condition which is known as the *dynamic compatibility condition*. We shall assume that expression (78) has the same limits on both sides of the surface as we approach any point of the surface (n is the direction of the normal at this point), provided we take the same normal direction n in both cases. We assume further that (79) is applicable separately to the parts D_1 and D_2 of D. This will certainly be true if u has continuous derivatives up to the second order as far as the surface in D_1 and D_2. If we apply (79) for D_1 and D_2, we shall have directly opposite directions of the outward normal in both these cases on the surface σ, so that the

expression $P(u)$ will differ in sign for the two integrals. On adding these two formulae, we obtain (79) for the whole of D, since the two integrals over σ cancel each other. Thus, given our assumptions regarding the strong discontinuity of the function u, we find that (79) holds for the whole of D.

We now deduce some important consequences of our propositions. Let $\mathbf{n}$ be the unit vector normal to σ, and let us consider the vector product grad $\mathbf{u} \times \mathbf{n}$. If $\mathbf{l}$ denotes the unit vector in the direction of the projection of grad $\mathbf{u}$ on the tangent plane to σ, so that grad $u = \partial u/\partial l \cdot \mathbf{l} + \partial u/\partial n \cdot \mathbf{n}$, the vector product in question is equal to $\partial u/\partial l\,(\mathbf{l} \times \mathbf{n})$, so that it is continuous on passing through the surface σ. If we form the three components of the vector product, we get the following expressions, which must be continuous on passing through σ by virtue of the kinematic compatibility conditions:

$$\left.\begin{aligned} u_x \cos(n, y) - u_y \cos(n, x) &= M_1 \\ u_y \cos(n, t) - u_t \cos(n, y) &= M_2 \\ u_t \cos(n, x) - u_x \cos(n, t) &= M_3. \end{aligned}\right\} \tag{80}$$

In addition, the condition stated above gives us a fourth expression, which must also remain continuous on passing through σ:

$$u_x \cos(n, x) + u_y \cos(n, y) - \frac{1}{a^2} u_t \cos(n, t) = M_4. \tag{81}$$

We shall regard equations (80) and (81) as four first degree equations in u_x, u_y, u_t. If the matrix of the coefficients of this system proved to have a rank equal to three, i.e. if at least one third order determinant of the matrix differed from zero, we should be able to solve the corresponding three equations with respect to the above-mentioned derivatives, and these derivatives would be expressed in terms of the continuous functions M_k. All the first order derivatives of u would then remain continuous on passing through σ, and we should not have a strong discontinuity. We can therefore assert that the rank of the matrix of coefficients must be less than three, i.e. all the third order determinants of the matrix

$$\left\| \begin{matrix} \cos(n, y), & -\cos(n, x), & 0 \\ 0 & \cos(n, t), & -\cos(n, y) \\ -\cos(n, t), & 0 & \cos(n, x) \\ \cos(n, x), & \cos(n, y), & -\dfrac{1}{a^2}\cos(n, t) \end{matrix} \right\| \tag{82}$$

must vanish. On striking out say the first row, we arrive at the condition:

$$\cos(n, t)\cos^2(n, x) + \cos(n, t)\cos^2(n, y) - \frac{1}{a^2}\cos^3(n, t) = 0.$$

We assume that $\cos(n, t) \neq 0$, and we thus obtain the following equation:

$$\cos^2(n, x) + \cos^2(n, y) - \frac{1}{a^2}\cos^2(n, t) = 0. \qquad (83)$$

If $\psi(x, y, t) = 0$ is the equation of the surface σ, this equation may obviously be rewritten in the form:

$$\psi_x^2 + \psi_y^2 - \frac{1}{a^2}\psi_t^2 = 0,$$

and we now see that the surface σ must in fact be a characteristic surface of the equation $\square\, u = 0$ in the present case of a strong discontinuity. The condition $\cos(n, t) \neq 0$ is obviously equivalent to $\psi_t \neq 0$. If condition (83) is fulfilled, it is easily shown that all the third order determinants of matrix (82) must vanish and that M_4 is a linear combination of M_1, M_2 and M_3; in fact, we now obviously have:

$$\cos(n, t)\, M_4 = \cos(n, y)\, M_2 - \cos(n, x)\, M_3.$$

Hence it will be seen that, if the kinematic compatibility conditions are fulfilled so that M_1, M_2, M_3 are continuous and σ is a characteristic surface of the equation $\square\, u = 0$, the continuity of M_4 must follow from this, i.e. the dynamic compatibility condition is fulfilled. It is worth remarking that we arrived in the previous discussions at the equation of the characteristic surface without being at all concerned with an investigation of the solutions of the equation $\square\, u = 0$; we merely started from equation (79), containing the expression $\square\, u$ on the left-hand side.

143. Riemann's method. We now turn to the solution of the Cauchy problem and start with the case of a linear equation in two independent variables, which will be assumed to be already reduced to the normal form:

$$L(u) = u_{xy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = f(x, y). \qquad (84)$$

The arguments of the coefficients and the function f will often be omitted in what follows. We have written $L(u)$ for the left-hand side of the equation. We recall that the basic condition (32), defining a

characteristic, becomes $dxdy = 0$ for the above equation, so that the characteristics of (84) are the straight lines $x = \text{const.}$ and $y = \text{const.}$, parallel to the axes. Along with the operator $L(u)$, we consider the so-called *conjugate operator*, which is defined as follows:

$$M(v) = v_{xy} - (av)_x - (bv)_y + cv.$$

The coefficients a and b are obviously assumed to be continuously differentiable in this case. Using the expressions for $L(u)$ and $M(v)$, the following elementary identity is readily proved:

$$2\left[vL(u) - uM(v)\right] = (u_x v - v_x u + 2buv)_y + (u_y v - v_y u + 2auv)_x. \tag{85}$$

We take a domain D with contour λ on the (x, y) plane and let u and v have continuous first order derivatives in D and a continuous second order mixed derivative. On integrating both sides of identity (85) over the domain D and using the familiar formula [II, 69]:

$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int\limits_\lambda P\, dx + Q\, dy,$$

we now obtain the following *Green's formula:*

$$2\iint\limits_D \left[vL(u) - uM(v)\right] dS$$
$$= \int\limits_\lambda -(u_x v - v_x u + 2buv)\, dx + (u_y v - v_y u + 2auv)\, dy. \tag{86}$$

After these preliminary remarks, we now proceed to the solution of Cauchy's problem for equation (84).

Suppose that l is a given curve on the (x, y) plane that is cut in not more than one point by any straight line parallel to an axis. The equation of this curve can be written as $x = x(y)$ or $y = y(x)$. We assume that non-zero derivatives $x'(y)$ and $y'(x)$ exist on the part of l in question. The solution of equation (84) is sought satisfying a Cauchy specification on l, i.e. the values of the function u and of its partial derivatives u_x, u_y are given on l, where the usual condition $du = u_x\, dx + u_y\, dy$ must be observed. We can suppose that u, u_x, u_y are given as functions of x only or y only along l.

It is assumed here that the function giving the value of u on l has a continuous derivative, whilst u_x and u_y are continuous functions. The coefficients a and b, as we mentioned above, have continuous

partial derivatives by hypothesis, whilst c and f are continuous in the domain containing l to which our future discussions relate. We shall prove below that the problem has a solution, given our assumptions. Our immediate problem is to obtain a formula for the solution on the assumption that such a solution exists.

We take as the domain B the part of the (x, y) plane bounded by an arc of curve l and two straight lines parallel to the axes and issuing from a fixed point $P(x, y)$ (Fig. 5). Suppose that we know the solution in this domain of the homogeneous conjugate equation:

$$M(v) = 0. \tag{87}$$

On applying formula (86) to the required solution u of the Cauchy problem and to the solution just mentioned of equation (87), we obtain with the aid of (84):

$$-2\iint_D vf\,d\sigma = \int_{AB} + \int_{BP} + \int_{PA}. \tag{88}$$

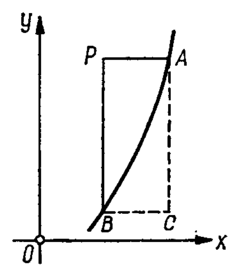

FIG. 5

Integration over the contour λ is split into integration over the arc AB of l and over the straight lines BP and PA, parallel to the axes. The integral over the arc of l must be assumed known, since we are given the value of the required function u and of both its first order partial derivatives on this arc. We consider the integrals over the straight lines. Only x varies along PA, so that the integration over PA gives:

$$-\int_{PA} (u_x v - v_x u + 2buv)\,dx.$$

We can rewrite the integrand as:

$$u_x v - v_x u + 2buv = (uv)_x + 2u\,(bv - v_x),$$

so that we have:

$$-\int_{PA} (u_x v - v_x u + 2buv)\,dx = (uv)_P - (uv)_A - \int_{PA} 2u\,(bv - v_x)\,dx,$$

where for instance $(uv)_P$ is the value of the product uv at the point P.

Similarly, the integration over BP gives us the result:

$$\int_{BP} (u_y v - v_y u + 2auv)\,dy = (uv)_P - (uv)_B + \int_{BP} 2u\,(av - v_y)\,dy.$$

Formula (88) can be rewritten as follows:

$$2v(P)\,u(P) = \int_{AB} \left[(u_x v - xv_x u + 2buv)\,\mathrm{d}x - (u_y v - v_y u + 2auv)\,\mathrm{d}y\right] + u(A)\,v(A) + u(B)\,v(B) + \int_{PA} 2u(bv - v_x)\,\mathrm{d}x + \int_{PB} 2u(av - v_y)\,\mathrm{d}y - 2\iint_D fv\,\mathrm{d}\sigma. \tag{89}$$

Suppose that we know, not just any solution of (87), but a solution satisfying the following conditions on the straight lines PA and PB:

$$bv - v_x = 0 \quad \text{on} \quad PA \quad \text{and} \quad av - v_y = 0 \quad \text{on} \quad PB,$$

and such that, in addition, $v(P) = 1$. In this case the integrals over PA and PB fall out in (89), and we get the following expression for the value of the required function $u(P)$ at the point P, the coordinates of which are denoted by (x_0, y_0):

$$2u(x_0, y_0) = u(A)\,v(A) + u(B)\,v(B) + \int_{AB} (u_x v - v_x u + 2buv)\,\mathrm{d}x - (u_y v - v_y u + 2auv)\,\mathrm{d}y - 2\iint_D fv\,\mathrm{d}\sigma. \tag{90}$$

We shall now discuss in more detail the conditions which must be satisfied by the solution v of equation (87). Along PA we must have:

$$v_x = b(x, y_0)\,v.$$

This equation can be regarded as an ordinary differential equation in the independent variable x, and integration of it gives us the following value for v on PA:

$$v(x, y_0) = e^{\int_{x_0}^{x} b(x,y_0)\,\mathrm{d}x} \quad (\text{on } PA). \tag{91}$$

Similarly, we obtain on PB:

$$v(x_0, y) = e^{\int_{y_0}^{y} a(x_0,y)\,\mathrm{d}y} \quad (\text{on } PB). \tag{92}$$

We now have $v(x_0, y_0) = 1$ at the point $P(x_0, y_0)$ itself. Thus the solution v of equation (87) must have given values defined by (91) and (92) on PA and PB. It will obviously depend on the choice of the point (x_0, y_0), i.e. in essence it is a function of a pair of points. We denote it by

$$v(x, y;\ x_0, y_0). \tag{93}$$

This solution of equation (87), satisfying conditions (91) and (92), is known as *Riemann's function*. This function depends neither on the

Cauchy data on l, nor on the form of this contour. The point (x, y) plays the role of argument for it, whilst the point (x_0, y_0) has the role of parameter. We remark that the existence of a solution of the problem could have been proved by showing directly that (90) in fact gives a function $u(x_0, y_0)$ which satisfies (84) and the conditions on l. This proof presents certain difficulties, and a different proof of the existence of a solution of the Cauchy problem will be found in later sections.

The Riemann method described above reduces the solution of the Cauchy problem to finding the Riemann function (93). This function is itself a solution of homogeneous equation (87) of the same type as equation (84), but with supplementary conditions entirely different from the Cauchy conditions; in fact, as we saw above, we are only given the value of the function v itself on the two characteristics PA and PB issuing from a given point P. We shall prove below the existence of the Riemann function. We remark further that the basic formula (90) has been obtained on the assumption that a solution of the problem exists. Therefore, if a solution exists, it must necessarily be given by formula (90), and the uniqueness of the solution of Cauchy's problem is thereby proved. But it still remains to show that (90) in fact yields a solution of the problem. We shall prove later, not merely the existence of the Riemann function, but also the existence of the solution of the Cauchy problem, and hence at the same time prove that (90) in fact yields the solution of the problem.

If we assume for the moment that all the above-mentioned existence theorems have been proved, we can pass to a discussion of some consequences of formula (90). As we have just mentioned, this formula proves the uniqueness of the solution of the problem. In addition, it follows at once from this formula that, if the Cauchy data are given a sufficiently small variation on the contour l, the solution may be varied by as little as desired, i.e. the solution of the Cauchy problem depends continuously on the initial data. Furthermore, a direct consequence of (90) is that the value of the required function u at the point P depends only on the initial data, distributed over the arc AB of l. If we continue the initial data specified on arc AB by two different methods beyond the arc, whilst preserving the continuity of the initial data at points A and B, we obtain outside the curvilinear triangle PAB two different solutions of the Cauchy problem, i.e. to be more precise, we get two different systems of Cauchy data, which will correspond to the two different solutions of the Cauchy problem, though

these solutions will coincide in the curvilinear triangle PAB because the initial data in both problems are the same along the arc AB. The characteristics PA and PB will be those curves along which the solutions, which are the same in the triangle, split up into two distinct solutions.

None of the arguments of the present section assume that the functions are analytic. Mention must be made of the role of the condition that a straight line parallel to an axis, i.e. a characteristic, intersect the curve l in not more than one point. We take a curve l_1 (Fig. 6), which is cut in two points by a straight line parallel to the x axis, and suppose that the initial Cauchy data are specified on it. On using Riemann's method we can find the value of the required function u at the point P, either by using the curvilinear triangle PAB or the curvilinear triangle PBC. The two formulae obtained in general give different results for u at the point P, so that the problem proves incapable of solution.

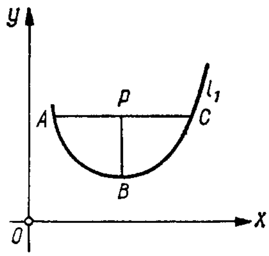

FIG. 6

144. Characteristic initial data. We now take the problem to which we were led when constructing the Riemann function. Only the case of a homogeneous equation will be considered here. Suppose we want to find the solution of the equation:

$$u_{xy} + au_x + bu_y + cu = 0\,, \tag{94}$$

if only the value of the required function u is given on the straight lines CA and CB, parallel to the axes (Fig. 5). Let $(\xi,\ \eta)$ be the coordinates of the point C. We observe that, if u is given on CB, u_x must also be known along CB. But now, on substituting $y = \eta$ and the known functions u and u_x, equation (94) becomes an ordinary first order linear equation for the function u_y along CB. Integration of this equation will give us the partial derivative u_y along CB. Similarly, knowing the value of u along CA, we can find both the first order partial derivatives of u along CA. The arbitrary constants obtained by solving the ordinary first order equations can be determined, since u_y and u_x can be assumed known at the point C. These arguments show us why it is sufficient to specify u only along the characteristics CA and CB. The Riemann method described above is immediately

applicable to the present problem, and we get the expression for the required function:

$$2u(P) = u(A)\,v(A) + u(B)\,v(B) + \int\limits_{CA} (u_y v - v_y u + 2auv)\,\mathrm{d}y + \\ + \int\limits_{CB} (u_x v - v_x u + 2buv)\,\mathrm{d}x\,,$$

where, as above, v is the Riemann function (93).

We carry out the integration in the first of the above integrals after rewriting the integrand as:

$$u_y v - v_y u + 2auv = -(uv)_y + 2v(au + u_y).$$

A similar transformation can be used for the second integral. As a result, we obtain the formula:

$$u(P) = u(C)\,v(C) + \int\limits_{CA} v(au + u_y)\,\mathrm{d}y + \int\limits_{CB} v(bu + u_x)\,\mathrm{d}x. \qquad (95)$$

Let us use this formula to prove a property of the Riemann function. We remark first of all that the conjugate to the operator $M(v)$ will be the initial operator $L(u)$. For:

$$M(v) = v_{xy} - av_x - bv_y + (c - a_x - b_y)\,v\,,$$

and the conjugate operator becomes:

$$L_1(u) = u_{xy} + (au)_x + (bu)_y + (c - a_x - b_y)\,u \\ = u_{xy} + au_x + bu_y + cu = L(u).$$

We apply (95) to the Riemann function u of the operator $M(v)$. The coefficients of v_x and v_y in the operator $M(v)$ are equal to $(-a)$ and $(-b)$, so that the Riemann function of this operator is the solution of (94) satisfying the equations: $au + u_y = 0$ and $bu + u_x = 0$, on the straight lines CA and CB and in addition, we must have $u(C) = 1$. The point $C(\xi, \eta)$ will play the role here of the point $P(x_0, y_0)$ of function (93).

On using formula (95) for this particular case, we arrive at the expression:

$$u(x_0, y_0;\ \xi, \eta) = v(\xi, \eta;\ x_0, y_0)\,,$$

i.e. the Riemann function (93) of operator $L(u)$ transforms to the Riemann function of the conjugate operator $M(v)$ if we interchange the points (x, y) and (x_0, y_0) in it. If the expression $M(v)$ is the same as $L(v)$, the expression or operator $L(u)$ is said to be *self-conjugate*; the Riemann function for a self-conjugate operator is a symmetrical function of the two points on which it depends. By using the expres-

sions for $L(u)$ and $M(v)$, the conditions may easily be written down for $L(u)$ to be self-conjugate: $a = b \equiv 0$. The problem of finding the solution of (94) when the function itself is specified on two characteristics is usually known as *the problem with characteristic initial data.* Just as in the case of the Cauchy problem, formula (95) shows that the problem with characteristic initial data can only have a *single* solution.

145. Existence theorems. It remains for us to prove the theorems establishing the *existence* of the solutions of Cauchy's problem and of the problem with characteristic initial data. We start with the latter problem and confine ourselves to the case of a homogeneous equation. Suppose we want to find the solution of equation (94) taking specified values on the characteristics $x = x_0$ and $y = y_0$:

$$u\big|_{x=x_0} = \psi(y); \quad u\big|_{y=y_0} = \varphi(x) \quad [\psi(y_0) = \varphi(x_0)]. \tag{96}$$

Assuming that the coefficient b has a continuous derivative with respect to y, we can rewrite (94) as a system of two first order equations:

$$u_x + bu = w; \tag{97}$$

$$w_y + aw = du, \tag{98}$$

where

$$d = ab + b_y - c,$$

and the following initial condition is obtained for the new function w:

$$w\big|_{y=y_0} = \varphi'(x) + b(x, y_0)\varphi(x) = \omega(x). \tag{99}$$

If we regard (97) as an ordinary linear differential equation and take the first of conditions (96) into account, we get an expression for the function $u(x, y)$ in terms of $w(x, y)$:

$$u(x, y) = e^{-\int\limits_{x_0}^{x} b(\xi, y)\, d\xi}\left[\int\limits_{x_0}^{x} e^{\int\limits_{x_0}^{\xi} b(\xi', y)\, d\xi'} w(\xi, y)\, d\xi + \psi(y)\right].$$

Similarly, equation (98) gives us:

$$w(x, y) = e^{-\int\limits_{y_0}^{y} a(x, \eta)\, d\eta}\left[\int\limits_{y_0}^{y} e^{\int\limits_{y_0}^{\eta} a(x, \eta')\, d\eta'} d(x, \eta)\, u(x, \eta)\, d\eta + \right.$$
$$\left. + \varphi'(x) + b(x, y_0)\varphi(x)\right].$$

These equations are equivalent to equations (97) and (98) with initial conditions (96). On introducing the notation:

$$K_1(x, y; \xi) = e^{\int\limits_{x}^{\xi} b(\xi', y)\, d\xi'}, \quad K_2(x, y; \eta) = e^{\int\limits_{y}^{\eta} a(x, \eta')\, d\eta'} d(x, \eta), \tag{100}$$

we can rewrite the above equations in the form:

$$\left.\begin{aligned} u(x, y) &= e^{-\int\limits_{x_0}^{x} b(\xi, y)\,d\xi}\,\psi(y) + \int\limits_{x_0}^{x} K_1(x, y;\ \xi)\, w(\xi, y)\, d\xi \\ w(x, y) &= e^{-\int\limits_{y_0}^{y} a(x, \eta)\,d\eta}\,\omega(x) + \int\limits_{y_0}^{y} K_2(x, y;\ \eta)\, u(x, \eta)\, d\eta . \end{aligned}\right\} \tag{101}$$

The method of successive approximations, with the usual argument for establishing the convergence, enables us to prove the existence and uniqueness of the solution of the last system. In order to be able to return from equations (97) and (98) to (94), a continuous mixed derivative u_{xy} must exist. It is clear from equations (101), which are satisfied by the continuous functions $u(x, y)$ and $w(x, y)$, that u_{xy} does in fact exist provided $b(x, y)$ has continuous first order partial derivatives and $\psi(y)$ has a continuous derivative. If we substitute for $w(x, y)$ from the second of equations (101) in the first, we obtain an ordinary Volterra equation with iterated integral for $u(x, y)$.

We turn to the proof of the existence of a solution of the Cauchy problem. As we saw above, the equation of the curve l that supports the Cauchy data can be written in the form $x = x(y)$ or $y = y(x)$, where $x(y)$ and $y(x)$ have continuous non-zero derivatives. The Cauchy data on l can be regarded as functions either of the independent variable x or of the independent variable y. We write these data in the form:

$$u\,|_{x=x(y)} = \psi(y) = \psi_1(x); \quad u_x\,|_{y=y(x)} = \varphi_1(x).$$

When solving equations (97) and (98), we must take into account the initial data:

$$u\,|_{x=x(y)} = \psi(y); \quad w\,|_{y=y(x)} = \varphi_1(x) + b\,[x, y(x)]\,\psi_1(x) = \omega_1(x).$$

We thus obtain, as above, the following system of integral equations:

$$\begin{aligned} u(x, y) &= e^{-\int\limits_{x(y)}^{x} b(\xi, y)\,d\xi}\,\psi(y) + \int\limits_{x(y)}^{x} K_1(x, y;\ \xi)\, \omega(\xi, y)\, d\xi, \\ w(x, y) &= e^{-\int\limits_{y(x)}^{y} a(x, \eta)\,d\eta}\,\omega_1(x) + \int\limits_{y(x)}^{y} K_2(x, y;\ \eta)\, u(x, \eta)\, d\eta, \end{aligned} \tag{102}$$

where $K_1(x, y;\ \xi)$, $K_2(x, y;\ \eta)$ are given by (100). The proof of the convergence of the method of successive approximations is performed for

the system in the usual way, and an existence theorem may thence be proved for the solution. If the point P is located as shown in Fig. 5, the length of the path of integration can be replaced by $(a - x)$ when integrating with respect to ξ in inequalities, and by $(\beta - y)$ when integrating with respect to η, where a and β are the greatest values of x and y in the rectangle with sides parallel to the axes in which the solution of the problem is considered, and where the coefficients satisfy the conditions established above. These conditions were e.g. continuous first order partial derivatives for coefficients a and b, and continuous c and f, these latter being required when carrying out Riemann's method. We could equally have discussed the non-homogeneous equation (84). Here, it is sufficient to add the function $f(x,y)$ to the right-hand side of equation (98). The uniqueness of the solution is easily proved, by using system (102) instead of having recourse to Riemann's method.

146. Method of successive approximations. We could have proved the existence theorems by applying the method of successive approximations direct to equation (94), just as we did in the case of ordinary differential equations. Let us start with the case of characteristic initial data (96). Equation (94) with initial data (96) is equivalent to the integro-differential equation:

$$u(x, y) = \varphi(x) + \psi(y) - \varphi(x_0) - \int_{x_0}^{x}\int_{y_0}^{y} [a(\xi, \eta)\, u_\xi(\xi, \eta) + \\ + b(\xi, \eta)\, u_\eta(\xi, \eta) + c(\xi, \eta)\, u(\xi, \eta)]\, d\xi,\, d\eta, \tag{103}$$

where the term outside the integral satisfies initial data (96) by virtue of the obvious condition $\varphi(x_0) = \psi(y_0)$.

We can take as the first approximation the function

$$u_0(x, y) = \varphi(x) + \psi(y) - \varphi(x_0).$$

The remaining approximations are worked out successively from the formulae:

$$u_n(x, y) = u_0(x, y) - \int_{x_0}^{x}\int_{y_0}^{y}\left[a(\xi, \eta)\frac{\partial u_{n-1}(\xi, \eta)}{\partial \xi} + (\xi, \eta)\frac{\partial u_{n-1}(\xi, \eta)}{\partial \eta} + \right. \\ \left. + c(\xi, \eta)\, u_{n-1}(\xi, \eta)\right] d\xi\, d\eta \qquad (n = 1, 2, \ldots). \tag{104}$$

By using elementary inequalities, it can be shown that the successive functions:

$$u_n(x, y), \quad \frac{\partial u_n(x, y)}{\partial x}, \quad \frac{\partial u_n(x, y)}{\partial y}$$

are uniformly convergent in the rectangle R illustrated in Fig. 5, on the assumption that the coefficients of the equation are continuous functions in this rectangle. On passing to the limit in (104), we easily see that the limit function of the

sequence $u_n(x, y)$ satisfies equation (103), and therefore satisfies equation (94) and initial data (96).

We now turn to the Cauchy problem. Let R be a rectangle containing in its interior the part of the curve l on which the Cauchy data are specified and such that the coefficients of the equation are continuous functions inside R. Let (x, y) be a point inside R. Let us write D_{xy} for the curvilinear triangle PAB, bounded by the arc AB of l and by the two straight lines PA and PB, parallel to the axes and issuing from the point $P(x, y)$. The initial Cauchy data can be written as:

$$u\,|_l = \varphi(x) + \psi(y); \quad u_x\,|_l = \varphi'(x), \quad u_y\,|_l = \psi'(y). \tag{105}$$

For, as we have already remarked, it can always be assumed that the initial data for u_x and u_y are expressed in terms of x or y. On integrating these functions, we also obtain initial data for u in the above-mentioned form. Equation (94) with initial data (105) is equivalent to the equation:

$$u(x, y) = \varphi(x) + \psi(y) + \iint_{D_{xy}} [a(\xi, \eta)\, u_\xi(\xi, \eta) + b(\xi, \eta)\, u_\eta(\xi, \eta) + c(\xi, \eta)\, u(\xi, \eta)]\, d\xi\, d\eta.$$

In (103) we wrote the iterated integral with an indication of the limits, and with this form of writing it is a matter of indifference where the point $P(x, y)$ is situated with respect to the characteristics $x = x_0$ and $y = y_0$. In the last formula we write a double integral and take the mutual disposition of the point, curve and axes shown in Fig. 5. We take as the first approximation:

$$u_0(x, y) = \varphi(x) + \psi(y),$$

and the following approximations are worked out from the formulae:

$$u_n(x, y) = u_0(x, y) + \iint_{D_{xy}} \left[a(\xi, \eta)\, \frac{\partial u_{n-1}(\xi, \eta)}{\partial \xi} + b(\xi, \eta)\, \frac{\partial u_{n-1}(\xi, \eta)}{\partial \eta} + c(\xi, \eta)\, u_{n-1}(\xi, \eta)\right] d\xi\, d\eta.$$

It can be proved as above, with the aid of elementary inequalities for the integrals, that the sequence $u_n(x, y)$ is uniformly convergent in the rectangle R to a limit function which is in fact the solution of the Cauchy problem.

We saw earlier [II, 51] that the method of successive approximations can be used to prove an existence theorem in the case of a non-linear differential equation. Similarly, the method of successive approximations just described can be used for a non-linear partial differential equation of the form:

$$u_{xy} = f(x, y, u, p, q). \tag{106}$$

Suppose that the initial Cauchy data on the curve $l\,(y = y(x))$ are expressed in terms of the independent variable x: $u(x), p(x), q(x)$, where we must have $u'(x) = p(x) + y'(x)\, q(x)$. We shall assume that the functions have continuous derivatives. We form the auxiliary function

$$\omega(x, y) = u(x) + [y - y(x)]\, q(x),$$

which obviously has continuous derivatives ω_x and ω_y. This function satisfies the required initial data on l. On replacing u by the new function $u_1 = u - \omega$

we obtain zero initial Cauchy data for it on l. Obviously, we now have to transform to the new required function in equation (106). We can therefore assume that we have zero initial Cauchy data for equation (106). The function f on the right-hand side of the equation is assumed to have continuous first order derivatives with respect to all its arguments for values of (x, y) sufficiently close to l, and for values of (u, p, q) sufficiently close to zero. Equation (106) with zero initial data becomes:

$$u(x, y) = -\iint_{D_{xy}} f(\xi, \eta, u, p, q)\, d\xi\, d\eta ,$$

and the usual method of successive approximations is applicable to this equation if we confine ourselves to (x, y) lying in some neighbourhood of the curve l. We must take as the first approximation $u_0 = p_0 = q_0 = 0$, and the following approximations are worked out from the formulae:

$$u_n(x, y) = -\iint_{D_{xy}} f(\xi, \eta, u_{n-1}, p_{n-1}, q_{n-1})\, d\xi\, d\eta ,$$

$$p_n(x, y) = \int_{BP} f(x, \eta, u_{n-1}, p_{n-1}, q_{n-1})\, d\eta ,$$

$$q_n(z, y) = \int_{AP} f(\xi, y, u_{n-1}, p_{n-1}, q_{n-1})\, d\xi .$$

We remark that a non-homogeneous equation could obviously be also considered when applying the method of successive approximations for a linear equation, and we could reduce the initial data in the Cauchy problem or in the problem with characteristic initial data to zero, precisely as above for equation (94). The initial homogeneous equation would here become non-homogeneous for the transformed function.

147. Green's formula. The solution of the Cauchy problems becomes a good deal more difficult for second order equations when the number of independent variables is greater than two, and our treatment will be confined to the general outlines of the subject.

We recall that the method of solution of the Cauchy problem has been indicated [II, 171] for the wave equation when the initial conditions are specified at $t = 0$. This special method cannot be generalized, however, to equations with variable coefficients. We shall mention another method of solution of the Cauchy problem for equations with constant coefficients in the present section. This method is a generalization of the Riemann method, and like the latter, is based on a peculiar use of Green's formula. It gives a solution of the Cauchy problem when the initial conditions are specified not only on the plane $t = 0$, but also on several non-characteristic surfaces. It resembles in its basic idea the method used for equations with variable coefficients.

We take the example of the wave equation with variable coefficents in the next section and describe the method of solution of the Cauchy problem for linear equations with variable coefficients due to I. S. Sobolev.

We have succeeded in constructing Green's formula in the case when the left-hand side of the equation has a special form [143]. We now deduce Green's formula for the general case of a linear operator with partial derivatives up to the second order. We shall assume in future that all the derivatives mentioned exist and are continuous.

Let

$$L(u) = \sum_{i,k=1}^{m} a_{ik} u_{x_i x_k} + \sum_{k=1}^{m} b_k u_{x_k} + cu \qquad (a_{ki} = a_{ik}) \tag{107}$$

and

$$M(v) = \sum_{i,k=1}^{m} \frac{\partial^2 (a_{ik} v)}{\partial x_i \, \partial x_k} - \sum_{k=1}^{m} \frac{\partial (b_k v)}{\partial x_k} + cv, \tag{108}$$

where a_{ik}, b_k and c are given functions of the independent variables. Let D be a bounded domain in m-dimensional space $(x_1, \ldots, x_m)$ and S its boundary surface.

Green's formula expresses the m-tuple integral

$$\int \underset{D}{\cdots} \int \left[vL(u) - uM(v)\right] d\tau \tag{109}$$

in terms of an $(m-1)$-tuple integral over the surface S. The following identity is easily verified by direct differentiation:

$$vL(u) - uM(v) =$$

$$= \sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ik}\left(v\frac{\partial u}{\partial x_k} - u\frac{\partial v}{\partial x_k}\right) - \frac{\partial a_{ik}}{\partial x_k} uv\right] + \sum_{i=1}^{m} \frac{\partial}{\partial x_i}(b_i uv),$$

and we obtain by applying Ostrogradskii's formula:

$$\int \underset{D}{\cdots} \int \left[vL(u) - uM(v)\right] d\tau = \int \underset{S}{\cdots} \int \left[vP(u) - uP(v) + uvQ\right] dS, \tag{110}$$

where

$$\left.\begin{aligned} P(u) &= \sum_{i,k=1}^{m} a_{ik} \frac{\partial u}{\partial x_k} \cos(n, x_i), \\ Q &= \sum_{i=1}^{m} \left(b_i - \sum_{k=1}^{m} \frac{\partial a_{ik}}{\partial x_k}\right) \cos(n, x_i), \end{aligned}\right\} \tag{111}$$

and n is the direction of the outward normal to S. We define a direction ν at points of S: this is done by putting

$$N = \sqrt{\sum_{k=1}^{m}\left[\sum_{i=1}^{m} a_{ik}\cos(n, x_i)\right]^2} \tag{112}$$

and defining the direction ν by the formulae:

$$\cos(\nu, x_k) = \frac{1}{N}\sum_{i=1}^{m} a_{ik}\cos(n, x_i) \qquad (k = 1, 2, \ldots, m). \tag{113}$$

The first of formulae (111) can now be rewritten as:

$$P(u) = N\sum_{k=1}^{m}\frac{\partial u}{\partial x_k}\cos(\nu, x_k) = N\frac{\partial u}{\partial \nu},$$

and Green's formula (110) can finally be rewritten as:

$$\int \underset{D}{\cdots} \int [vL(u) - uM(v)]\,d\tau =$$
$$= \int \underset{S}{\cdots} \int \left[N\left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) + uvQ\right] dS. \tag{114}$$

We remark that, if

$$\sum_{k=1}^{m}\frac{\partial a_{ik}}{\partial x_k} = b_i \qquad (i = 1, 2, \ldots m),$$

Q vanishes, the operator $M(v)$ coincides with $L(v)$, and we can write $L(u)$ as:

$$L(u) = \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\sum_{k=1}^{m} a_{ik}u_{x_k} + cu.$$

The operator $L(u)$ is said to be *self-conjugate* in this case.

Suppose that L is a characteristic hypersurface of the equation $L(u) = 0$ or $L(u) = f$, where f is a given function of the independent variables. Let $\omega(x_1, \ldots, x_m) = 0$ be the equation of this hypersurface. The quantities $\cos(n, x_i)$ are proportional to the partial derivatives $p_i = \omega_{x_i}$ and by (113), the direction-cosines of ν are proportional to:

$$\sum_{i=1}^{m} a_{ik}p_i.$$

These sums are the right-hand sides of the equations of the bicharacteristics [139]:

$$\frac{dx_k}{ds} = \sum_{i=1}^{m} a_{ik}p_i,$$

forming the characteristic hypersurface S, and we can therefore say that: *if S is a characteristic hypersurface, the direction ν on it coincides at every point with the direction of the bicharacteristic lying on S and passing through this point.* Consequently, the direction ν lies in the present case in the tangent plane to S; ν is sometimes called the *conormal* to S. We now show the significance of Green's formula (114) for solving the Cauchy problem. Suppose we want to find the solution of the equation

$$L(u) = -f, \tag{115}$$

if u and the conormal derivative $\partial u/\partial \nu$ are given on some surface S_1. We shall assume that S_1 is such that the direction ν on it is *not* in the tangent plane. Then the fact that u and $\partial u/\partial \nu$ are given on S_1 implies that the derivative of u with respect to any direction can be found on S_1. We use the following procedure to find u at a point $M_0(x_1^0, \ldots, x_n^0)$ outside S_1. We draw the characteristic conoid of equation (115) with vertex M_0 and suppose that half of this conoid forms with the part of S_1 a bounded domain D of space $(x_1, \ldots, x_m)$ (Fig. 7). Then Green's formula (114) is applicable to the domain D, u being taken as the required solution of equation (115) and v as some singular solution of the conjugate equation $M(v) = 0$. The surface of D consists of the piece of S_1 on which u and $\partial u/\partial \nu$ are given, and the lateral surface Γ of the characteristic conoid. The direction ν coincides on Γ with the direction of the tangent to the bicharacteristic lying on Γ, and this enables us to integrate by parts when integrating over Γ.

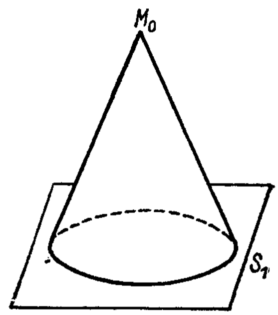

FIG. 7

Let us carry out this method for the wave equation

$$L(u) = u_{xx} + u_{yy} - u_{tt} = -f(x, y, t). \tag{116}$$

The characteristic conoid is a circular cone in this case, for which the angle between the generator and the height is equal to $\pi/4$. The operator $L(u)$ is self-conjugate, formula (112) gives $N = 1$, and we get from (113):

$$\cos(\nu, x) = \cos(n, x); \quad \cos(\nu, y) = \cos(n, y); \quad \cos(\nu, t) = -\cos(n, t).$$

whence it is clear that the direction ν is the mirror image of the direction n in the plane $t = 0$. The equation of the characteristic cone with vertex (x_0, y_0, t_0) will be:

$$(x - x_0)^2 + (y - y_0)^2 - (t - t_0)^2 = 0. \tag{117}$$

We use the following solution of the equation $L(v) = 0$:

$$v = \log\left[\sqrt{\frac{(t-t_0)^2}{r^2} - 1} - \frac{t - t_0}{r}\right], \tag{118}$$

where

$$r^2 = (x - x_0)^2 + (y - y_0)^2.$$

We take the half of cone (117) turned towards decreasing values of t. We have $(t - t_0)/r = -1$ on the lateral surface Γ of this cone, and the solution (118) vanishes on this surface. Differentiation with respect to ν on Γ is differentiation with respect to the conormal direction on Γ, i.e. with respect to the direction of the generator of the cone, so that we have $\partial v/\partial \nu = 0$ as well as $v = 0$ on Γ. But solution (118) has a singularity at $r = 0$, i.e. the straight line through the vertex of the cone parallel to the t axis is a singular curve of solution (118). We separate this curve by a circular cylinder T_ε of radius ε. Let D' denote the remaining part of domain D. The boundary of this domain will consist of the lateral surface T_ε of the cylinder (Fig. 8) in addition to S_1 and Γ. Let S_1' be the part of S_1 lying inside our cone, after subtracting the part that lies in T_ε. We now apply formula (114). On observing that $L(v) = M(v)$, $L(u) = -f(x, y, t)$, $L(v) = 0$ and that $v = \partial v/\partial \nu = 0$ on Γ, we get:

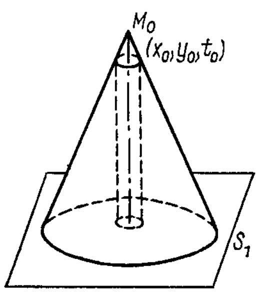

FIG. 8

$$\iint\limits_{T_\varepsilon + S_1'} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) \mathrm{d}S = -\iiint\limits_{D'} fv\,\mathrm{d}\tau. \tag{119}$$

The direction ν coincides on T_ε with the direction of the outward normal, i.e. is opposite to the direction r measured from the t axis. If φ denote the polar angle in the coordinate system: $x - x_0 = r\cos\varphi$ and $y - y_0 = r\sin\varphi$, we get:

$$\iint\limits_{T_\varepsilon} v\frac{\partial u}{\partial \nu}\,\mathrm{d}S = \iint\limits_{T_\varepsilon} v\frac{\partial u}{\partial \nu}\,\varepsilon\,\mathrm{d}\varphi\,\mathrm{d}t. \tag{120}$$

We have $r = \varepsilon$ on T_ε and, by (118), v will be of order $\log \varepsilon$. Since $\varepsilon \log \varepsilon \to 0$ as $\varepsilon \to 0$, we can say that integral (120) tends to zero along with ε. Further, we have:

$$\frac{\partial v}{\partial \nu} = -\frac{\partial v}{\partial r} = -\frac{t - t_0}{r\sqrt{(t - t_0)^2 - r^2}},$$

where the radical must be taken as positive. On T_ε:

$$\sqrt{(t - t_0)^2 - r^2} = \sqrt{(t - t_0)^2 - \varepsilon^2},$$

and as $\varepsilon \to 0$ this radical tends to $(t_0 - t)$, since $t < t_0$. We thus have:

$$\lim_{\varepsilon \to 0} \iint_{T_\varepsilon} u \frac{\partial v}{\partial \nu} \, dS = -\lim_{\varepsilon \to 0} \iint_{T_\varepsilon} \frac{(t - t_0)\, u}{\sqrt{(t - t_0)^2 - \varepsilon^2}} \, d\varphi \, dt$$

$$= 2\pi \int_{t'}^{t_0} u(x_0, y_0, t)\, dt,$$

where t' is the value of t obtained at the point of intersection of the straight line $r = 0$ with the surface S_1. Formula (119) therefore gives:

$$2\pi \int_{t'}^{t_0} u(x_0, y_0, t)\, dt = \iint_{S_2} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\right) dS + \iiint_D fv \, d\tau,$$

where S_2 is the part of S_1 lying inside the above-mentioned cone. The right-hand side consists of given values, and differentiation with respect to t_0 gives us the final result:

$$u(x_0, y_0, t_0) = \frac{1}{2\pi} \frac{\partial}{\partial t_0} \left[\iint_{S_2} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\right) dS + \iiint_D fv \, d\tau \right]. \tag{121}$$

We have obtained this formula on the assumption that a solution of the problem exists. Strictly speaking, we must also show that the right-hand side satisfies all the conditions of the problem. This is extremely difficult, since the position of cone (117) changes as t_0 varies. If S_2 is the plane $t = 0$, the solution is one which we obtained previously. This method of solution of the Cauchy problem is due to Volterra. A detailed account of it can be found in Webster and Szegő: *Partial Differential Equations of Mathematical Physics* (Vol. II, ch. 6).

Another method of solution of the Cauchy problem is connected with Green's formula, namely Hadamard's method. When this method is used, we take the solution of the equation $M(v) = 0$ that becomes infinite throughout the lateral surface of the characteristic conoid, or of cone (117) in the case of equation (116). This fact calls for special

precautions when applying Green's formula and leads naturally to a special new concept of integral.

In the case of the equation:

$$L(u) = \sum_{s=1}^{m} u_{x_s x_s} - u_{tt} = -f$$

Hadamard's singular solution has the form:

$$v = \left[(t - t_0)^2 - \sum_{s=1}^{m} (x_s - x_s^{(0)})^2 \right]^{\frac{1}{2} - \frac{m}{2}}.$$

A detailed treatment of Hadamard's method as applied to linear equations with variable coefficients can be found in his book *Le Probléme de Cauchy et Les Equations aux Dérivées Partielles Linéaires Hyperboliques* (Paris, 1932). The application of Hadamard's method to equations with constant coefficients is dealt with in Courant and Hilbert: *Methods of Mathematical Physics*, Vol. II.

148. Sobolev's formula. We have had Kirchhoff's formula [II,202] in the case of the wave equation in four independent variables. Let u be the solution of the wave equation having continuous derivatives up to the second order in a domain D of space (x_1, x_2, x_3), bounded by a surface S. Kirchoff's formula gives u at any point inside D in terms of an integral over S, containing retarded values of u and its first order derivatives. We have also seen that, with a special choice of surface S, Kirchhoff's formula leads to a solution of Cauchy's problem when the initial data are given at $t = 0$ [II, 202]. Kirchhoff's formula can be generalized to the case of the wave equation with any even number of independent variables:

$$u_{tt} = u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_{2k+1}\, x_{2k+1}} ,$$

and, as above, it yields a solution of Cauchy's problem for this equation [cf. 153].

We shall now generalize Kirchhoff's formula to the case of the wave equation with variable coefficients:

$$u_{tt} = c^2 (x, y, z) (u_{xx} + u_{yy} + u_{zz}) , \tag{122}$$

where $c(x, y, z)$ is a positive function having a sufficient number of derivatives. In future we shall often write $c(M)$ instead of $c(x, y, z)$, where M is the point with coordinates (x, y, z).

If we use the theory of characteristics for equation (122), we naturally arrive at an extremum problem for the functional:

$$J = \int_{M_0}^{M_1} \frac{ds}{c\,(x, y, z)} = \int_{M_0}^{M_1} \frac{\sqrt{dx^2 + dy^2 + dz^2}}{c\,(x, y, z)}\,. \tag{123}$$

In this case the transversality condition is the same as the orthogonality condition, and we can construct a field for the variational problem as indicated in [79]. Let $\tau(M; M_0)$ be the basic function of the central field with centre M_0. This function gives the magnitude of integral (123) over the extremal from M_0 to M. The equation $\tau(M; M_0) = \text{const}$ gives quasispheres with centre M_0 with the metric defined by (123). We have for the function $\tau(M; M_0)$:

$$\operatorname{grad}^2 \tau\,(M;\ M_0) = \frac{1}{c^2\,(M)}\,, \tag{124}$$

i.e.

$$\tau_x^2 + \tau_y^2 + \tau_z^2 = \frac{1}{c^2\,(M)}\,. \tag{124_1}$$

The function $\tau\,(M; M_0)$ is obviously symmetric in M_0 and M. If c is a constant, $\tau(M; M_0) = r/c$, where r is the distance from M_0 to M. In the general case we use τ instead of r/c for finding the retarded values of any function $u(M, t)$ and, as in [II, 202], we introduce the notation:

$$u\,(M, t - \tau) = [u\,(M, t)]\,.$$

Let $u(M, t)$ be a solution of (122), and let us simplify the writing by putting $u(M, t - \tau) = u_1(M, t)$.

We pass to retarded values in equation (122):

$$[u_{tt}] = c^2\,(M)\,[\Delta u], \tag{125}$$

where Δ is the Laplace operator. We express $[\Delta u]$ in terms of u_1. We have:

$$\left.\begin{aligned}
&\operatorname{grad} u_1 = [\operatorname{grad} u] - [u_t] \operatorname{grad} \tau \\
&\quad \Delta u_1 = \operatorname{div} \operatorname{grad} u_1 = \\
&\qquad = [\Delta u] - 2\,[\operatorname{grad} u_t] \cdot \operatorname{grad} \tau - [u_t]\,\Delta\tau + [u_{tt}] \operatorname{grad}^2 \tau,
\end{aligned}\right\} \tag{126}$$

and, on substituting for $[\Delta u]$ from the last equation in (125), we obtain with the aid of (124):

$$\frac{1}{c^2\,(M)}\,[u_{tt}] = \Delta u_1 + 2\,[\operatorname{grad} u_t] \cdot \operatorname{grad} \tau + [u_t]\,\Delta\tau - [u_{tt}]\,\frac{1}{c^2\,(M)}\,.$$

We have, in analogy with the first of expressions (126):

$$\operatorname{grad}\frac{\partial u_1}{\partial t} = [\operatorname{grad} u_t] - [u_{tt}]\operatorname{grad}\tau,$$

and, on substituting for $[\operatorname{grad} u_t]$ from the last equation in the previous formula, we obtain the following important expression:

$$\Delta u_1 = -2\operatorname{grad}\tau\cdot\operatorname{grad}\frac{\partial u_1}{\partial t} - \Delta\tau\,\frac{\partial u_1}{\partial t}$$

We multiply both sides of this equation by an as yet undetermined function $\sigma(M)$:

$$\sigma\Delta u_1 = -2\sigma\operatorname{grad}\tau\cdot\operatorname{grad}\frac{\partial u_1}{\partial t} - \sigma\Delta\tau\,\frac{\partial u_1}{\partial t} \tag{127}$$

and choose $\sigma(M)$ so that the right-hand side is the divergence of a vector of the form $(-\partial u_1/\partial t)\,\mathbf{w}$, where $\mathbf{w}$ is a vector independent of u_1:

$$\sigma\Delta u_1 = \operatorname{div}\left(-\frac{\partial u_1}{\partial t}\,\mathbf{w}\right). \tag{128}$$

We expand the right-hand side:

$$\sigma\Delta u_1 = -\frac{\partial u_1}{\partial t}\operatorname{div}\mathbf{w} - \operatorname{grad}\frac{\partial u_1}{\partial t}\cdot\mathbf{w}.$$

On comparing with (127), it will be seen that (128) will hold and $\mathbf{w}$ will be independent of u_1 if the following two equations are satisfied:

$$\mathbf{w} = 2\sigma\operatorname{grad}\tau\,;\quad \operatorname{div}\mathbf{w} = \sigma\Delta\tau. \tag{129}$$

On substituting the first of these equations in the second, we obtain an equation for σ:

$$\operatorname{div}(2\sigma\operatorname{grad}\tau) = \sigma\Delta\tau,$$

i.e.

$$2\operatorname{grad}\sigma\cdot\operatorname{grad}\tau + \sigma\Delta\tau = 0\,, \tag{130}$$

or in coordinate form:

$$2\,(\sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z) + \sigma\Delta\tau = 0, \tag{131}$$

i.e. we get a linear first order equation for σ. Having found σ, we can determine the vector $\mathbf{w}$ from the first of expressions (129). Let D be a domain of three-dimensional space (x, y, z) and S its boundary surface. Suppose that the functions σ and u_1 have continuous derivatives up to the second order in D. We apply Green's formula:

$$\iiint_D (\sigma\Delta u_1 - u_1\Delta\sigma)\,dv = \iint_S \left(\sigma\frac{\partial u_1}{\partial n} - u_1\frac{\partial\sigma}{\partial n}\right)dS,$$

where n is the direction of the outward normal to S. On using (128) and (129), we can rewrite Green's formula as:

$$-\iiint\limits_D u_1 \Delta\sigma \, dv - \iiint\limits_D \operatorname{div}\left(2\sigma \frac{\partial u_1}{\partial t} \operatorname{grad} \tau\right) dv = \iint\limits_S \left(\sigma \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \sigma}{\partial n}\right) dS,$$

and we obtain on applying Ostrogradskii's formula to the integral containing the divergence:

$$\iint\limits_S \left(\sigma \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \sigma}{\partial n} + 2\sigma \frac{\partial \tau}{\partial n} \frac{\partial u_1}{\partial t}\right) dS + \iiint\limits_D u_1 \Delta\sigma \, dv = 0 .$$

On returning to the function u and observing that

$$\frac{\partial u_1}{\partial n} = \left[\frac{\partial u}{\partial n}\right] - \left[\frac{\partial u}{\partial t}\right] \frac{\partial \tau}{\partial n},$$

we obtain the following fundamental formula:

$$\iint\limits_S \left\{\sigma \left[\frac{\partial u}{\partial n}\right] - [u] \frac{\partial \sigma}{\partial n} + \sigma \frac{\partial \tau}{\partial n} \left[\frac{\partial u}{\partial t}\right]\right\} dS + \iiint\limits_D [u] \Delta\sigma \, dv = 0 . \qquad (132)$$

We could have taken any field, rather than a central field, in all the above working. The function σ which has to satisfy equation (130) obviously depends on τ, i.e. on the choice of field. We shall in future be concerned only with a central field and σ will be denoted by $\sigma(M; M_0)$. All our discussion relates only to a neighbourhood of M_0 in which the extremals of integral (123) do not cut themselves and form a field. If c is a constant, $\tau = r/c$ as already mentioned, and the function $\sigma = 1/r$ is easily shown to satisfy (130).

149. Sobolev's formula (continued). Suppose we have succeeded in constructing a function $\sigma(M; M_0)$ with continuous derivatives up to the second order in the neighbourhood of the point M_0, having a singularity at M_0 and satisfying the following conditions:

(1) the product $\sigma(M, M_0)\, \tau(M; M_0)$ has continuous derivatives up to the second order, including the point M_0, and

$$\lim_{M \to M_0} \sigma(M;\ M_0)\, \tau(M;\ M_0) = \frac{1}{c(M_0)}; \qquad (133)$$

(2) $$\sigma(M_0;\ M) = \sigma(M;\ M_0); \qquad (134)$$

(3) the Laplace operator of $\sigma(M; M_0)$ satisfies the inequality:

$$|\Delta\sigma(M;\ M_0)| \leqslant \frac{K}{\tau(M;\ M_0)}, \tag{135}$$

where K is a constant (which does not depend on M);

(4) if S_1 is a closed surface, having M_0 as an interior point, and n is the direction of the outward normal to S_1, the following equation holds in the limit as S_1 contracts indefinitely to M_0:

$$\lim_{S_1 \to M_0} \iint_{S_1} \frac{\partial\sigma(M;\ M_0)}{\partial n} \mathrm{d}S = -4\pi. \tag{136}$$

If c is a constant, the function $\sigma = 1/r$ satisfies all these conditions.

We now use a function $\sigma(M; M_0)$ with the above properties to construct a formula for solutions of equation (122). Let $u(M, t)$ be such a solution in the domain D, bounded by the surface S, and let M_0 be an interior point of D. Suppose that a central field exists with centre M_0 containing the domain D, and that we have a function $\sigma(M; M_0)$ with the above properties.

We exclude from D a small sphere S_ε with centre M_0 and radius ε. We can apply (132) to the remaining domain D':

$$\iint_S \left\{\sigma\left[\frac{\partial u}{\partial n}\right] - [u]\frac{\partial\sigma}{\partial n} + \sigma\frac{\partial\tau}{\partial n}\left[\frac{\partial u}{\partial t}\right]\right\}\mathrm{d}S + \iint_{S_\varepsilon}\{\quad\}\,\mathrm{d}S + \\ + \iiint_{D'} [u]\,\Delta\sigma\,\mathrm{d}v = 0. \tag{137}$$

We show that the integral over S_ε gives us $-4\pi u(M_0, t)$ as $\varepsilon \to 0$. For, the quantities:

$$\left[\frac{\partial u}{\partial t}\right] \text{and} \frac{\partial\tau}{\partial n}\left[\frac{\partial u}{\partial t}\right]$$

are bounded on approaching M_0; $\tau(M; M_0)$ is of order ε on S_ε, and by (133), $\sigma(M; M_0)$ is of order $1/\varepsilon$ on S_ε, whilst the area of S_ε is of order ε^2. It follows from this that the integrals:

$$\iint_{S_\varepsilon} \sigma\left[\frac{\partial u}{\partial n}\right]\mathrm{d}S \quad \text{and} \quad \iint_{S_\varepsilon} \sigma\frac{\partial\tau}{\partial n}\left[\frac{\partial u}{\partial t}\right]\mathrm{d}S$$

tend to zero along with ε. There remains the integral:

$$-\iint_{S_\varepsilon} [u]\frac{\partial\sigma}{\partial n}\mathrm{d}S = -\iint_{S_\varepsilon} u(M, t-\tau)\frac{\partial\sigma}{\partial n}\mathrm{d}S.$$

Here the normal is taken as outward with respect to D', i.e. inward with respect to the sphere S_ε. On the sphere, $u(M, t - \tau)$ tends to $u(M_0 t)$ as $\varepsilon \to 0$, and on taking (136) into account and recalling what has been said about the normal direction, we see that the last integral in fact gives $-4\pi u(M_0, t)$ in the limit. Formula (137) gives the required formula in the limit:

$$u(M_0;\ t) = \frac{1}{4\pi}\iint\limits_S \left\{\sigma\left[\frac{\partial u}{\partial n}\right] - [u]\frac{\partial \sigma}{\partial n} + \sigma\frac{\partial \tau}{\partial n}\left[\frac{\partial u}{\partial t}\right]\right\} dS + \frac{1}{4\pi}\iiint\limits_D [u]\,\Delta\sigma\, dv, \quad (138)$$

which was obtained by Sobolev.

If c is constant, then $\sigma = 1/r$ and $\Delta\sigma = 0$, the triple integral falls out, and we get the usual Kirchhoff formula. In the case of variable c (non-homogeneous sphere) the value of u at M_0 is obtained as a result of the retarded illumination not only of points of the surface S, but of the whole of the domain D.

Formula (138) can be used in solving the Cauchy problem for equation (122). Let the solution of (122) be required that satisfies the given initial conditions:

$$u(M, t)\big|_{t=0} = f_0(M);\quad u_t(M, t)\big|_{t=0} = f_1(M). \quad (139)$$

We apply formula (138) to the required solution, the quasisphere S_t with centre M_0 and radius t being taken as the surface S, i.e. we suppose that S has the form: $\tau(M; M_0) = t$. Now, the values of the functions

$$[u],\ \left[\frac{\partial u}{\partial n}\right],\ \left[\frac{\partial u}{\partial t}\right]$$

on the right-hand side must be taken at the instant $t - \tau(M; M_0)$ or, since $\tau(M; M_0) = t$, at the instant $t = 0$. On taking into account the initial data (139), we can rewrite (138) as:

$$u(M_0, t) = \frac{1}{4\pi}\iint\limits_{S_t} \left\{\sigma\frac{\partial f_0}{\partial n} - f_0\frac{\partial \sigma}{\partial n} + \sigma\frac{\partial \tau}{\partial n} f_1\right\} dS + \frac{1}{4\pi}\iiint\limits_{D_t} [u]\,\Delta\sigma\, dv,$$

where D_t is the domain bounded by the quasisphere S_t. The double integral on the right is a known function, which we denote by $F(M_0, t)$. We have thus obtained the integral equation for $u(M, t)$:

$$u(M_0, t) = F(M_0, t) + \frac{1}{4\pi}\iiint\limits_D [u]\,\Delta\sigma(M;\ M_0)\, dv. \quad (140)$$

We have had to assume, when deducing this equation, that t is such that a central field with centre M_0 and a function $\sigma(M; M_0)$ with the above-mentioned properties exist in D_t.

We remark that, when M_0 and t vary, the domain D_t also varies, and equation (140) is analogous to a Volterra equation. It can be shown that this equation has a unique solution for t sufficiently close to zero; this solution can be obtained by the ordinary method of successive approximations, and is at the same time the solution of our Cauchy problem for equation (122). If the space is infinite, the closeness of t to zero is demanded by the possible appearance of singularities in the field of the variational problem on the extension of D_t. When boundaries are present, we naturally have to reckon with the arrival of excitations reflected from the boundaries, which also substantially limits the possible interval of variation of t.

150. Construction of the function σ. We turn to the construction of a function σ with the above-mentioned properties. We shall prove that this function has an explicit expression in the closed form if the extremals forming the central field are assumed known. We must first prove two lemmas.

LEMMA 1. *Given the system of differential equations:*

$$\frac{dx_k}{dt} = X_k(t, x_1, x_2, x_3) \qquad (k = 1, 2, 3) \tag{141}$$

for which the general solution

$$x_k = \varphi_k(t, a_1, a_2, a_3) \qquad (k = 1, 2, 3) \tag{142}$$

is known, we have:

$$\frac{d}{dt} \log \frac{D(\varphi_1, \varphi_2, \varphi_3)}{D(a_1, a_2, a_3)} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_1}{\partial x_2} + \frac{\partial X_3}{\partial x_3}. \tag{143}$$

The expression behind the logarithm in this formula is the functional determinant of functions (142) with respect to a_1, a_2, a_3, whilst the x_k on the right-hand side have to be replaced by functions (142). We write down the determinant just mentioned and differentiate it with respect to t. On recalling the fundamental definition of a determinant as a sum of the products of its elements, we can say that the determinant can be differentiated simply by differentiating each column

separately and adding all the determinants obtained [III$_2$, 120]. We thus obtain:

$$\frac{d}{dt}\frac{D(\varphi_1,\varphi_2,\varphi_3)}{D(a_1,a_2,a_3)} = \begin{vmatrix} \frac{\partial^2\varphi_1}{\partial a_1\,\partial t}, & \frac{\partial\varphi_2}{\partial a_1}, & \frac{\partial\varphi_3}{\partial a_1} \\ \frac{\partial^2\varphi_1}{\partial a_2\,\partial t}, & \frac{\partial\varphi_2}{\partial a_2}, & \frac{\partial\varphi_3}{\partial a_2} \\ \frac{\partial^2\varphi_1}{\partial a_3\,\partial t}, & \frac{\partial\varphi_2}{\partial a_3}, & \frac{\partial\varphi_3}{\partial a_3} \end{vmatrix} + \begin{vmatrix} \frac{\partial\varphi_1}{\partial a_1}, & \frac{\partial^2\varphi_2}{\partial a_1\,\partial t}, & \frac{\partial\varphi_3}{\partial a_1} \\ \frac{\partial\varphi_1}{\partial a_2}, & \frac{\partial^2\varphi_2}{\partial a_2\,\partial t}, & \frac{\partial\varphi_3}{\partial a_2} \\ \frac{\partial\varphi_1}{\partial a_3}, & \frac{\partial^2\varphi_2}{\partial a_3\,\partial t}, & \frac{\partial\varphi_3}{\partial a_3} \end{vmatrix} +$$

$$+ \begin{vmatrix} \frac{\partial\varphi_1}{\partial a_1}, & \frac{\partial\varphi_2}{\partial a_1}, & \frac{\partial^2\varphi_3}{\partial a_1\,\partial t} \\ \frac{\partial\varphi_1}{\partial a_2}, & \frac{\partial\varphi_2}{\partial a_2}, & \frac{\partial^2\varphi_3}{\partial a_2\,\partial t} \\ \frac{\partial\varphi_1}{\partial a_3}, & \frac{\partial\varphi_2}{\partial a_3}, & \frac{\partial^2\varphi_3}{\partial a_3\,\partial t} \end{vmatrix}. \tag{144}$$

On observing that functions (142) must satisfy system (141), we get the following identities in t and a_k:

$$\frac{\partial\varphi_k}{\partial t} = X_k(t, \varphi_1, \varphi_2, \varphi_3) \qquad (k = 1, 2, 3).$$

Differentiation of these identities with respect to a_s gives us:

$$\frac{\partial^2\varphi_k}{\partial a_s\,\partial t} = \sum_{i=1}^{3}\frac{\partial X_k}{\partial x_i}\frac{\partial\varphi_i}{\partial a_s}.$$

On substituting these expressions for the second derivatives in the right-hand side of (144) and expanding each determinant into a sum of three determinants, we obtain:

$$\frac{d}{dt}\frac{D(\varphi_1,\varphi_2,\varphi_3)}{D(a_1\,a_2,a_3)} = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3}\right)\cdot\frac{D(\varphi_1,\varphi_2,\varphi_3)}{D(a_1,a_2,a_3)},$$

which in fact gives (143).

LEMMA 2. *Let* **t** *be the unit tangent vector to a family of curves depending on two parameters and filling three-dimensional space or some part of it, and let Δ be the functional determinant of a transformation from Cartesian to curvilinear coordinates. These curvilinear coordinates are the two parameters a_1 and a_2 defining a curve of the family, and the length of arc s along this curve, measured either from a surface which is cut by all the curves of the family or from a point where all the curve intersect. We now have*

$$\operatorname{div}\mathbf{t} = \frac{\partial\log\Delta}{\partial s}. \tag{145}$$

Let $X(x, y, z)$, $Y(x, y, z)$, $Z(x, y, z)$ be the components of the vector $\mathbf{t}$ at the point (x, y, z). The curves of the family now satisfy the system of differential equations:

$$\frac{dx}{ds} = X\,; \quad \frac{dy}{ds} = Y\,; \quad \frac{dz}{ds} = Z\,.$$

Since the right-hand sides do not contain s, one of the arbitrary constants s_0 will appear as a term added to s, and the general solution of the system will have the form:

$$x = \varphi_1 (s + s_0, a_1, a_2)\,; \quad y = \varphi_2 (s + s_0, a_1, a_2)\,; \quad z = \varphi_3 (s + s_0, a_1, a_2)\,.$$

We obtain by applying the previous lemma:

$$\operatorname{div} \mathbf{t} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = \frac{\partial}{\partial s} \log \frac{D (\varphi_1, \varphi_2, \varphi_3)}{D (s_0, a_1, a_2)}\,,$$

and if we recall that

$$\frac{\partial \varphi_k}{\partial s_0} = \frac{\partial \varphi_k}{\partial s}\,,$$

we in fact obtain (145).

We now return to the central field of extremals with centre M_0 and to equation (130), which must be satisfied by the function σ. The vector grad τ touches an extremal, and it follows from (124) that $\partial\tau/\partial s = n(M)$, where $n(M) = 1 : c(M)$. On taking this into account, we can rewrite (130) as

$$2\, \frac{\partial \sigma (M)}{\partial s}\, n (M) + \sigma (M)\, \Delta \tau (M) = 0\,,$$

where the arc length s is measured from M_0.

We use Lemma 2 to find $\Delta\tau(M)$. We have:

$$\operatorname{grad} \tau (M) = n (M)\, \mathbf{t},$$

where $\mathbf{t}$ is the unit tangent to an extremal of the field. Hence:

$$\operatorname{div} \operatorname{grad} \tau (M) = \Delta \tau (M) = \mathbf{t} \cdot \operatorname{grad} n (M) + n (M) \operatorname{div} \mathbf{t}\,.$$

The first term on the right is the derivative of $n(M)$ with respect to s, and the second, by Lemma 2, is equal to

$$n (M)\, \frac{\partial \log \Delta}{\partial s}\,,$$

and the equation for $\sigma(M)$ can be rewritten as

$$2\, \frac{\partial \sigma}{\partial s}\, n (M) + \sigma \left[\frac{\partial n (M)}{\partial s} + n (M)\, \frac{\partial \log \Delta}{\partial s} \right] = 0\,,$$

or

$$2\,\frac{\partial \log \sigma}{\partial s} = -\,\frac{\partial \log n}{\partial s} - \frac{\partial \log \Delta}{\partial s}\,,$$

whence we obtain by integration:

$$\sigma(M) = \frac{\Psi(a_1, a_2)}{\sqrt{n(M)\,\Delta}}\,,$$

where $\psi(a_1, a_2)$ is an arbitrary function of its arguments. We take as parameters a_1 and a_2 the angular coordinates θ_0, φ_0 of a spherical system of the direction of the tangent to the extremal at the point M_0. The last expression now becomes

$$\sigma(M) = \frac{\Psi(\vartheta_0, \varphi_0)}{\sqrt{n(M)\,\dfrac{D(x, y, z)}{D(s, \vartheta_0, \varphi_0)}}}\,. \tag{146}$$

We find the form of the function $\Psi(\theta_0, \varphi_0)$ from the first of the conditions of [149] for the function $\sigma(M)$. This condition has the form:

$$\lim_{M \to M_0} \sigma(M)\,\tau(M) = n(M_0)\,.$$

On taking the form of integral (123) into account, we can write

$$\tau(M) = \int_0^s n(M)\,\mathrm{d}s\,,$$

where the integration is carried out along an extremal. On using the mean value theorem, we get:

$$\lim_{s \to 0} \frac{\tau(M)}{s} = n(M_0),$$

and the previous condition for $\sigma(M)$ can be written as:

$$\lim_{s \to 0} \sigma(M)\,s = 1\,. \tag{147}$$

We remark that the point M tends to M_0 as $s \to 0$.

To investigate the functional determinant appearing in (146), we return to the formulae established in [81] for the canonical variables in the problem on geodesics. In the present case:

$$\varphi = n^2(M)\,(x'^2 + y'^2 + z'^2),$$

and the canonical variables have the form:

$$p_1 = 2n^2(M)\,x'; \quad p_2 = 2n^2(M)\,y'; \quad p_3 = 2n^2(M)\,z'.$$

We have the following initial conditions:

$$x_0' = \sin\vartheta_0 \cos\varphi_0; \quad y_0' = \sin\vartheta_0 \sin\varphi_0; \quad z_0' = \cos\vartheta_0$$

and

$$\begin{gathered} p_{10} = 2n^2(M_0)\sin\vartheta_0\cos\varphi_0; \quad p_{20} = 2n^2(M_0)\sin\vartheta_0\sin\varphi_0; \\ p_{30} = 2n^2(M_0)\cos\vartheta_0. \end{gathered} \tag{148}$$

The equations for an extremal of the field become

$$x = \varphi_1(r_1, r_2, r_3, x_0, y_0, z_0); \quad y = \varphi_2(\quad); \quad z = \varphi_3(\quad), \tag{149}$$

where $r_k = sp_{k0}$ and φ_k are functions with continuous derivatives up to a certain order. On differentiating the first of the formulae with respect to s and then putting $s = 0$, we obtain:

$$\sin\vartheta_0\cos\varphi_0 = \left(\frac{\partial\varphi_1}{\partial r_1}\right)_{s=0} p_{10} + \left(\frac{\partial\varphi_1}{\partial r_2}\right)_{s=0} p_{20} + \left(\frac{\partial\varphi_1}{\partial r_3}\right)_{s=0} p_{30}.$$

By using (148) and the fact that θ_0 and φ_0 are arbitrary, we have:

$$\left(\frac{\partial\varphi_1}{\partial r_2}\right)_{s=0} = \left(\frac{\partial\varphi_1}{\partial r_3}\right)_{s=0} = 0; \quad \left(\frac{\partial\varphi_1}{\partial r_1}\right)_{s=0} = 1{:}2n^2(M_0).$$

On making use of the remaining expressions of (149), we finp the following general expressions:

$$\left(\frac{\partial\varphi_k}{\partial r_k}\right)_{s=0} = 1{:}2n^2(M_0); \qquad \left(\frac{\partial\varphi_k}{\partial r_l}\right)_{s=0} = 0 \qquad (l \neq k).$$

By using (148) and (149), we can form the functional determinant of functions φ_k with respect to the variables s, θ_0 and φ_0. When differentiating with respect to θ_0 and φ_0 via r_k, we obtain a factor s, and two columns of the determinant will contain this factor. After dividing the determinant by s^2, we pass to the limit as s tends to zero. We thus obtain, with the aid of the previous expressions:

$$\lim_{s\to 0}\frac{1}{s^2}\,\frac{D(x,y,z)}{D(s,\vartheta_0,\varphi_0)} = \begin{vmatrix} \sin\vartheta_0\cos\varphi_0, & \cos\vartheta_0\cos\varphi_0, & -\sin\vartheta_0\sin\varphi_0 \\ \sin\vartheta_0\sin\varphi_0, & \cos\vartheta_0\sin\varphi_0, & \sin\vartheta_0\cos\varphi_0 \\ \cos\vartheta_0, & -\sin\vartheta_0, & 0 \end{vmatrix} = \sin\vartheta_0.$$

To define the arbitrary function in (146), we multiply both sides of it by s and let s tend to zero. We obtain by using the last expression and (147):

$$1 = \frac{\Psi(\vartheta_0,\varphi_0)}{\sqrt{n(M_0)\sin\vartheta_0}}, \quad \text{i.e.} \quad \Psi(\vartheta_0,\varphi_0) = \sqrt{n(M_0)\sin\vartheta_0},$$

and we finally have the following expression for the function σ:

$$\sigma(M, M_0) = \sqrt{\frac{n(M_0)\sin\vartheta_0}{n(M)\,\dfrac{D(x, y, z)}{D(s, \vartheta_0, \varphi_0)}}}\,. \tag{150}$$

It may be verified that this function has all the properties indicated in [149]. If $n(M) = \text{const.}$, (s, θ_0, φ_0) are the usual spherical coordinates of the point M, and the last formula gives $\sigma = 1/r$.

151. The general case of initial data. Now let the initial data be given on a surface with the equation $t = \varphi(M)$, instead of on the plane $t = 0$:

$$u|_{t=\varphi(M)} = f_0(M); \quad n_t|_{t=\varphi(M)} = f_1(M). \tag{151}$$

We shall solve the problem for $t > \varphi(M)$. Instead of the hypersphere $\tau(M; M_0) = t$, we consider the surface

$$\tau(M;\ M_0) + \varphi(M) = t, \tag{152}$$

and suppose that, for all positive values of the difference $[t - \varphi(M)]$ sufficiently close to zero, (152) is a closed surface of three-dimensional space containing M_0 as an interior point; the part of the space enclosed by the surface is given by the inequality

$$\tau(M;\ M_0) + \varphi(M) < t. \tag{153}$$

We now apply (138), taking the surface (152) as S. We shall now have in the integrand of the integral over S:

$$[u] = u(M; t - \tau) = u[M;\ \varphi(M)] = f_0(M); \quad [u_t] = f_1(M).$$

We show that $[\partial u/\partial n]$ is also expressible in terms of the initial data. We have:

$$\frac{\partial f_0(M)}{\partial n} = \frac{\partial u[M; \varphi(M)]}{\partial n}$$

$$= \frac{\partial u(M; t)}{\partial n}\bigg|_{t=\varphi(M)} + \frac{\partial u(M; t)}{\partial t}\bigg|_{t=\varphi(M)} \cdot \frac{\partial \varphi(M)}{\partial n},$$

whence

$$\frac{\partial u(M; t)}{\partial n}\bigg|_{t=\varphi(M)} = \frac{\partial f_0(M)}{\partial n} - \frac{\partial u(M, t)}{\partial t}\bigg|_{t=\varphi(M)} \cdot \frac{\partial \varphi(M)}{\partial n},$$

i.e.

$$\left[\frac{\partial u(M; t)}{\partial n}\right] = \frac{\partial f_0(M)}{\partial n} - \frac{\partial \varphi(M)}{\partial n} f_1(M).$$

On introducing the notation

$$F(M_0; t) = \frac{1}{4\pi} \iint\limits_{\tau(M;\, M_0)+\varphi(M)=t} \left\{ \sigma \frac{\partial f_0(M)}{\partial n} + \right.$$

$$\left. + \sigma \left(\frac{\partial \tau}{\partial n} - \frac{\partial \varphi}{\partial n} \right) f_1(M) - \frac{\partial \sigma}{\partial n} f_0 \right\} \mathrm{d}S,$$

we get an equation for $u(M_0; t)$ analogous to (140):

$$u(M_0;\ t) = F(M_0;\ t) + \frac{1}{4\pi} \iiint\limits_{\tau(M;\, M_0)+\varphi(M)<t} [u]\, \Delta\sigma(M;\ M_0)\, \mathrm{d}v. \tag{154}$$

As above, it can be solved by the method of successive approximations and gives the solution of the Cauchy problem under conditions (151). All these proofs strictly speaking require the existence of a certain number of continuous partial derivatives of the functions $c(M)$, $f_0(M)$, $f_1(M)$ and $\varphi(M)$.

Let us consider the connection of surface (152) with the theory of characteristics. The characteristic conoid of equation (122) with vertex (M_0, t) has the equation in four-dimensional space $(M; t_1)$:

$$t_1 = t - \tau(M;\ M_0), \tag{155}$$

where t_1 and $M(x, y, z)$ are the current coordinates, and t and m_0 are parameters. Surface (152) is the locus of the points of three-dimensional space which have the same coordinates (x, y, z) as the points of intersection of the characteristic conoid (155) with the surface $t_1 = \varphi(M)$ of four-dimensional space, i.e. surface (152) is the projection of the intersection in question on three-dimensional space (x, y, z). To obtain a clearer picture, let us imagine that everything occurs in three-dimensional space (x, y, t_1). Equation (155) corresponds to an ordinary surface of the conical type. This surface intersects the surface $t_1 = \varphi(x, y)$ along a curve. The projection of this curve on the (x, y) plane must be a closed curve l, which is the analogue of surface (152). The projection of the vertex of the conoid on the (x, y) plane must fall inside l, and the analogue of the triple integral of (154) is a double integral over the part of the (x, y) plane inside l. This domain naturally depends on the position of the vertex (x_0, y_0, t) of the conoid. If this vertex approaches some point (x_0', y_0', t') on the surface $t_1 = \varphi(x, y)$, the curve must contract to the point (x_0', y_0'). Similarly, the closed surface S must contract to the point M_0 if the vertex of conoid (155) tends to a point (M_0', t') on the surface $t_1 = \varphi(M)$.

All these geometrical properties of the surface S, needed for a strict proof of the existence of the Cauchy problem, are bound up with the fact that the tangent plane to the surface $t = \varphi(M)$ must not deviate too much from the plane $t = 0$. It can be shown that this condition can be written in the form:

$$\operatorname{grad}^2 \varphi(M) < \frac{1}{c^2(M)} . \tag{156}$$

It is essential here that the function $c^2(M)$ be connected with $\tau(M; M_0)$ by equation (124). When condition (156) is fulfilled, the surface $t = \varphi(M)$ is said to be spatially orientated. In the case of the more general equation of the hyperbolic type:

$$u_{tt} - \sum_{i,j=1}^{m} a_{ij} u_{x_i x_j} + \ldots = 0,$$

where u is a function of the independent variables $x_1, x_2, \ldots, x_m$, the surface $t = \varphi(x_1, x_2, \ldots, x_m)$ is said to be spatially orientated at a point if we have at this point:

$$\sum_{i,j=1}^{m} a_{ij} \varphi_{x_i} \varphi_{x_j} < 1.$$

The development of this method and a full description of its application to the solution of the Cauchy problem for equation (122) may be found in articles by Sobolev *(Trudy Seismologicheskego Instituta Akademii Nauk SSSR,* no. 6 and no. 42, 1930 and 1934). This method has been used by Gogoladze for more general linear equations of the hyperbolic type in four independent variables *(Dokl. Akad. Nauk SSSR,* 1934). The method was further extended by Sobolev *(Matemat. Sbornik,* 1 (43), vyp. 1, 1936) to general linear equations of the hyperbolic type in an even number of independent variables. We shall indicate in the next section the modifications introduced into the original method by Gogoladze, then show how the method is extended to any even number of independent variables in the case of the wave equation with constant coefficient c^2.

152. Generalized wave equation. Instead of (122) we shall consider the more general equation:

$$\frac{1}{c^2} u_{tt} = \sum_{i=1}^{3} a_i u_{x_i x_i} + \sum_{i=1}^{3} b_i u_{x_i} + hu, \tag{157}$$

where the coefficients a_i, b_i, c and h are functions of the independent variables x_1, x_2, x_3, and the a_i are greater than a certain positive number. We construct instead of function (123):

$$J = \int_{M_0}^{M_1} \frac{ds}{c}, \tag{158}$$

where

$$ds^2 = \sum_{i=1}^{3} \frac{dx_i^2}{a_i}. \tag{159}$$

The basic function $\tau(M; M_0)$ of the central field satisfies the following equation:

$$\sum_{i=1}^{3} a_i \tau_{x_i}^2 = \frac{1}{c^2}. \tag{160}$$

The retarded value of any function $u(M; t)$ may be determined as in [148]. We obtain the following equation for the function σ, instead of (131):

$$2\sum_{i=1}^{3} a_i \tau_{x_i} \sigma_{x_i} + \sigma \sum_{i=1}^{3} \left[a_i \tau_{x_i x_i} + \left(2\frac{\partial a_i}{\partial x_i} - b_i \right) \tau_{x_i} \right] = 0. \tag{161}$$

Condition (133) takes the form:

$$\lim_{M \to M_0} \sigma(M; M_0)\, \tau(M; M_0) = \frac{n(M_0)}{\sqrt{a_1^0 a_2^0 a_3^0}} \qquad \left(n(M) = \frac{1}{c(M)} \right), \tag{162}$$

where a_i^0 are the values of functions a_i at the point M_0. Instead of (135), we have:

$$|L(\sigma)| \leqslant \frac{K}{\tau(M; M_0)}, \tag{163}$$

where $L(u)$ is the right-hand side of equation (157) and K is a constant, whilst (136) becomes:

$$\lim_{S_1 \to M_0} \iint_{S_1} \sum_{i=1}^{3} a_i \sigma_{x_i} \cos(n, x_i)\, dS = -4\pi. \tag{164}$$

Instead of (138) we have:

$$u(M_0; t) = \frac{1}{4\pi} \iint_S \left\{ \sigma P([u]) - [u] P(\sigma) + \sigma \left[\frac{\partial u}{\partial t} \right] P(\tau) + \sigma R[u] \right\} dS$$

$$+ \frac{1}{4\pi} \iiint_D [u] M(\sigma)\, dv. \tag{165}$$

where

$$P(v) = \sum_{i=1}^{3}{}' a_i v_{x_i} \cos(n, x_i), \qquad R = \sum_{i=1}^{3}{}' \left(\frac{\partial a_i}{\partial x_i} - b_i \right) \cos(n, x_i),$$

$$M(\sigma) = \sum_{i=1}^{3}{}' \left(\frac{\partial^2 a_i \sigma}{\partial x_i^2} - \frac{\partial b_i \sigma}{\partial x_i} + h\sigma \right),$$

$M(\sigma)$ being the operator conjugate to $L(\sigma)$. By using (165), we can carry out a similar reduction to that of [149] of the Cauchy problem with initial data (139) to the integral equation:

$$u(M_0;\ t) = F(M_0;\ t) + \frac{1}{4\pi} \iiint_D [u]\, M(\sigma)\, \mathrm{d}v.$$

We remark that there is a certain indeterminacy in the writing of equation (157), bound up with the fact that we can exclude the factor c^2 in different ways. In particular, by multiplying both sides of the equation by c^2 and including this function in the coefficients of the equation, we can take $c = 1$.

We obtain a formula for $\sigma(M; M_0)$ analogous to (150):

$$\sigma^2(M,\ M_0) = \frac{n(M_0) \sin\vartheta_0\, e^{\int_0^s \sum_{i=1}^{3}{}' \left(b_i - \frac{\partial a_i}{\partial x_i} \right) \frac{1}{a_i} \frac{\mathrm{d}x_i}{\mathrm{d}s} \mathrm{d}s}}{n(M) \dfrac{D(x_1, x_2, x_3)}{D(s, \vartheta_0, \varphi_0)} \sqrt{a_1^0 a_2^0 a_3^0}}, \tag{166}$$

where s is the arc length of the extremal joining M_0 and M, $\mathrm{d}s^2$ being calculated from (159).

153. The case of any number of independent variables. To apply Sobolev's method in the case of a large number of independent variables, several functions σ have to be introduced. The application of the method will be described for the wave equation with constant coefficients:

$$\frac{1}{c^2} u_{tt} = \sum_{i=1}^{2k+1}{}' u_{x_i x_i} \tag{167}$$

[see Sobolev, *On a Generalization of Kirchhoff's Formula* (Ob odnom obobschenii formuly Kirkhgofa), *Dokl. Akad. Nauk. SSSR,* 1933].

Let M denote the point in space R_{2k+1} with coordinates $(x_1, \ldots, x_{2k+1})$. We shall consider in addition the space R_{2k+2} with coordinates

$(x_1, \ldots, x_{2k+1}, t_1)$ or $(M; t_1)$. The characteristic cone for (167) with vertex $(M_0; t)$ has the equation in R_{2k+2}:

$$t_1 = t - \frac{r}{c}, \tag{168}$$

where

$$r^2 = \sum_{i=1}^{2k+1} (x_i - x_i^0)^2. \tag{169}$$

We write $[\varphi]$ as usual for the retarded value of the function φ:

$$[\varphi(M; t)] = \varphi\left(M, t = \frac{r}{c}\right),$$

i.e. the value of φ on the half of cone (168) which is lower as regards t. As we have already mentioned [138], certain relationships between the function u satisfying (167) and its derivatives hold on the characteristic surface. We establish these relationships for the derivatives of u with respect to t:

$$u_s = \left[\frac{\partial^s u}{\partial t^s}\right] \quad (s = 0, 1, 2, \ldots), \tag{170}$$

where the u_s will be regarded as functions in R_{2k+1}. We remark first of all that the fundamental equation (124) has the form in the present case:

$$\left(\operatorname{grad} \frac{r}{c}\right)^2 = \frac{1}{c^2}. \tag{171}$$

On carrying out the differentiation of u_s with respect to the coordinates both directly and via the argument $(t - r/c)$, we can use the fairly obvious expression:

$$\operatorname{grad} u_{s+1} \cdot \operatorname{grad} \frac{r}{c} = \left[\operatorname{grad} \frac{\partial^{s+1} u}{\partial t^{s+1}}\right] \cdot \operatorname{grad} \frac{r}{c} - \left[\frac{\partial^{s+2} u}{\partial t^{s+2}}\right] \operatorname{grad}^2 \frac{r}{c},$$

where the point denotes the scalar product in R_{2k+1}, to obtain the equation:

$$\Delta u_s = -2 \operatorname{grad} u_{s+1} \cdot \operatorname{grad} \frac{r}{c} - u_{s+1} \Delta \frac{r}{c}, \tag{172}$$

On introducing the operator:

$$\begin{aligned} L(v) &= -2 \operatorname{grad} v \cdot \operatorname{grad} \frac{r}{c} - v \Delta \frac{r}{c} \\ &= -\frac{2}{c} \sum_{i=1}^{2k+1} \frac{x_i - x_i^{(0)}}{r} v_{x_i} - \frac{v}{c} \Delta r, \end{aligned} \tag{173}$$

we can rewrite (172) as

$$\Delta u_s = L(u_{s+1}). \tag{174}$$

These are in fact the relationships which are satisfied on cone (168). The operator L satisfies

$$vL(w) + wL(v) = -\operatorname{div}\left(2vw \operatorname{grad}\frac{r}{c}\right). \tag{175}$$

We have for powers of r:

$$\Delta r^s = (2k + s - 1)\, s r^{s-2}; \quad L(r^s) = -\frac{2}{c}(s + k)\, r^{s-1}. \tag{176}$$

We introduce the functions σ_i:

$$\sigma_i = \frac{(2k-2)(2k-4)\ldots(2i+2)\,2i}{(k-i)!\, c^{k-1}(2k-2)(2k-3)\ldots(k+i-1)}\, r^{-k-i+1}; \ \sigma_k = r^{-2k+1}. \tag{177}$$

$$(i = 1,\ 2, \ldots,\ k-1).$$

We have:

$$L(\sigma_1) = 0; \quad L(\sigma_{i+1}) = \Delta\sigma_i; \quad \Delta\sigma_k = 0 \quad (i = 1,\ 2, \ldots,\ k-1). \tag{178}$$

Let D be a domain in space R_{2k+1} not containing the point M_0. We form the integral of multiplicity $(2k + 1)$:

$$\int_D \sum_{s=1}^{k} (-1)^{s-1} [(u_{s-1}\Delta\sigma_{k-s+1} - \sigma_{k-s+1}\Delta u_{s-1}) + \\ + (\sigma_{k-s+1} L(u_s) + u_s L(\sigma_{k-s+1}))]\, dx_1 \ldots dx_{2k+1}. \tag{179}$$

It follows from (174) and (178) that this integral vanishes. On taking into account (175) and the formula

$$v\,\Delta w - w\,\Delta v = \operatorname{div}(v \operatorname{grad} w - w \operatorname{grad} v),$$

we can transform integral (179) to an integral over the surface S bounding the domain D. If we also take into account the equation:

$$\frac{\partial u_s}{\partial n} = \left[\frac{\partial^{s+1} u}{\partial t^s\, \partial n}\right] - \left[\frac{\partial^{s+1} u}{\partial t^{s+1}}\right]\frac{\partial \frac{r}{c}}{\partial n},$$

we can write:

$$\int_S \sum_{s=1}^{k} (-1)^s \left\{\frac{\partial\sigma_{k-s+1}}{\partial n}\left[\frac{\partial^{s-1} u}{\partial t^{s-1}}\right] - \sigma_{k-s+1}\left[\frac{\partial^s u}{\partial n\, \partial t^{s-1}}\right] - \right. \\ \left. - \sigma_{k-s+1}\frac{\partial \frac{r}{c}}{\partial n}\left[\frac{\partial^s u}{\partial t^s}\right]\right\} dS = 0,$$

where n is the direction of the outward normal on S. If D contains M_0 as an interior point, the above formula can be applied after excluding M_0 by a small sphere. On then passing to the limit in the usual way, we obtain the formula:

$$u(M_0, t) = A \int_S \sum_{s=1}^{k} (-1)^s \left\{ \frac{\partial \sigma_{k-s+1}}{\partial n} \left[\frac{\partial^{s-1} u}{\partial t^{s-1}} \right] - \right.$$
$$\left. - \sigma_{k-s+1} \left[\frac{\partial^s u}{\partial n \, \partial t^{s-1}} \right] - \sigma_{k-s+1} \frac{\partial \frac{r}{c}}{\partial n} \left[\frac{\partial^s u}{\partial t^s} \right] \right\} dS, \tag{180}$$

where the constant A is given by

$$A = \frac{\prod_{i=1}^{2k-1} \Gamma\left(\frac{i+2}{2}\right)}{2(2k-1)\,\pi^{\frac{2k-1}{2}} \prod_{i=1}^{2k-1} \Gamma\left(\frac{i+1}{2}\right)}.$$

When $2k + 1 = 3$, (180) is the same as Kirchhoff's formula. If the surface S is taken as the sphere with centre M_0 and radius ct, the retarded values of the derivatives of u are expressible in terms of the initial data for u and u_t at $t = 0$, and we obtain an explicit solution of the Cauchy problem; this solution was obtained earlier in another form [II, 171]. Similarly, we can use (180) to solve the Cauchy problem for the case when the initial conditions are given on a surface $t_1 = \varphi(M)$. It may be mentioned that derivatives of the initial data also appear under the integral sign, so that, to obtain a solution of the problem, we have to require continuity of the derivatives of the initial data up to a certain order depending on k, as was mentioned earlier.

This method has been extended to non-linear hyperbolic equations by S. A. Khristianovich *(Matematich. Sbornik,* t. II, no. 5, 1937).

154. Basic inequalities. Let us consider a hyperbolic equation of the form

$$\sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} + \sum_{i=1}^{n} b_i u_{x_i} + cu - u_{tt} = f, \tag{181}$$

in which a_{ik}, b_i, c and f depend on $(x_1, \ldots, x_n, t)$, where b_i, c and f are continuous, whilst a_{ik} has continuous first order derivatives in the domains of space $(x_1, \ldots, x_n, t)$ which we shall describe below. Since (181) is of the hyperbolic type, the inequality must hold:

$$\sum_{i,k=1}^{n} a_{ik} \xi_i \xi_k \geqslant \lambda \sum_{i=1}^{n} \xi_i^2 \quad (\lambda > 0), \tag{182}$$

where λ will be assumed to be a positive constant for the domains in question. We shall assume in future, for greater clarity, that $n = 2$, i.e. we consider three-dimensional space R with coordinates (x_1, x_2, t). The discussion can readily be carried over to the general case.

Our problem is to find inequalities satisfied by the solutions of (181), in terms of the initial data and coefficients. It will follow at once from these inequalities that the solution of the Cauchy problem is *unique* and that it *depends continuously* on the initial data. The discussion that follows is similar to our proof in [II 179] of the uniqueness of the solution of the Cauchy problem and boundary value problem for the wave equation.

Let D be a finite domain in space R, bounded by a smooth surface S, and let n be the direction of the outward normal to S. Suppose we have a solution of (181) in D, continuous together with its derivatives up to the second order as far as S. We form the integral:

$$J = \iint_S \Big[\Big(\sum_{i,k=1}^{2} a_{ik} u_{x_i} u_{x_k} + u_t^2\Big)\cos(n,t) - \\ - 2\sum_{i,k=1}^{2} a_{ik} u_{x_k} u_t \cos(n, x_i)\Big]\, dS. \qquad (183)$$

On applying Ostrogradskii's formula and using (181), we obtain

$$J = \iiint_D \Big\{2u_t\Big[\sum_{i=1}^{2}\Big(b_i - \sum_{k=1}^{2}\frac{\partial a_{ik}}{\partial x_k}\Big)u_{x_i} + cu - f\Big] + \\ + \sum_{i,k=2}^{2}\frac{\partial a_{ik}}{\partial t} u_{x_i} u_{x_k}\Big\}\, d\tau. \qquad (184)$$

The integrand in (183) can be written as

$$\frac{1}{\cos(n,t)}\Big[\sum_{i,k=1}^{2} a_{ik}\big(u_{x_i}\cos(n,t) - u_t\cos(n,x_i)\big)\big(u_{x_k}\cos(n,t) - \\ - u_t\cos(n,x_k)\big) + u_t^2\Big(\cos^2(n,t) - \sum_{i,k=1}^{2} a_{ik}\cos(n,x_i)\cos(n,x_k)\Big)\Big]. \qquad (185)$$

Suppose that the domain D is bounded by the planes $t = 0$, $t = C$ $(C > 0)$ and by a characteristic surface, on which $\cos(n,t) > 0$. We now have

$$\cos^2(n,t) - \sum_{i,k=1}^{2} a_{ik}\cos(n,x_i)\cos(n,x_k) = 0 \qquad (186)$$

on this surface, and by (182), (185) is non-negative on the lateral surface. We remark that the same conclusion follows if, instead of the lateral surface being characteristic, the condition is satisfied on it:

$$\cos^2(n, t) - \sum_{i,k=1}^{2} a_{ik}\cos(n, x_i)\cos(n, x_k) \geqslant 0. \tag{187}$$

In this case the surface is said to be *spatially orientated* [cf. 151]. Further, $\cos(n, t) = 1$ for $t = C$ and $\cos(n, t) = -1$ for $t = 0$, and $\cos(n, x_1)$ and $\cos(n, x_2)$ now vanish. On writing

$$K(t) = \int\limits_{B(t)}\int \Big(\sum_{i,k=1}^{2} a_{ik} u_{x_i} u_{x_k} + u_t^2\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{188}$$

where $B(t_0)$ denotes the section of D by the plane $t = t_0$, and recalling that (185) is non-negative throughout the lateral surface, we get:

$$\int\limits_0^t\int\limits_{B(t_1)}\int 2u_t\Big[\sum_{i=1}^{2}\Big(b_i - \sum_{k=1}^{2}\frac{\partial a_{ik}}{\partial x_k}\Big)u_{x_i} + cu - f\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_1 +$$

$$+ \int\limits_0^t\int\limits_{B(t_1)}\int\sum_{i,k=1}^{2}\frac{\partial a_{ik}}{\partial t}u_{x_i}u_{x_k}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_1 \geqslant K(t) - K(0)$$

$$(0 < t \leqslant C). \tag{189}$$

Suppose that

$$\left|\frac{\partial a_{ik}}{\partial t}\right| \leqslant P_0, \tag{190}$$

where P_0 is a positive number, so that

$$\left|\int\limits_{B(t_1)}\int\sum_{i,k=1}^{2}\frac{\partial a_{ik}}{\partial t}u_{x_i}u_{x_k}\,\mathrm{d}x_1\,\mathrm{d}x_2\right| \leqslant P_0\int\limits_{B(t_1)}\int\sum_{i,k=1}^{2}|u_{x_i}|\cdot|u_{x_k}|\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

We have further:

$$\sum_{i,k=1}^{2}|u_{x_i}|\cdot|u_{x_k}| \leqslant 2\sum_{i=1}^{2}u_{x_i}^2.$$

But, by (182),

$$\sum_{i=1}^{2}u_{x_i} \leqslant \frac{1}{\lambda}\sum_{i,k=1}^{2}a_{ik}u_{x_i}u_{x_k},$$

so that

$$\left|\int\limits_0^t\int\limits_{B(t_1)}\int\sum_{i,k=1}^{2}\frac{\partial a_{ik}}{\partial t}u_{x_i}u_{x_k}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_1\right| \leqslant P_1\int\limits_0^t K(t_1)\,\mathrm{d}t_1,$$

where P_1 is a positive constant depending on the coefficients. Similarly:

$$\left|\int_0^t \int\int_{B(t_1)} 2u_t \sum_{i=1}^{2}\left(b_i - \sum_{k=1}^{2}\frac{\partial a_{ik}}{\partial x_k}\right) u_{x_i}\,\mathrm{d}x_1 \mathrm{d}x_2\,\mathrm{d}t_1\right| \leqslant P_2 \int_0^t K(t_1)\,\mathrm{d}t_1,$$

where P_2 is a constant analogous to P_1. We use the notation:

$$L(t) = \int\int_{B(t)} u^2\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{191}$$

On applying the inequality $|\,2ab\,| \leqslant a^2 + b^2$, and observing that

$$\sum_{i,k=1}^{2} a_{ik}\,u_{x_i} u_{x_k} \geqslant 0, \tag{192}$$

we obtain

$$\left|\int_0^t \int\int_{B(t_1)} 2cu_t u\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_1\right| \leqslant P_3 \int_0^t [K(t_1) + L(t_1)]\,\mathrm{d}t_1.$$

If we use the inequality $|\,2fu_t| \leqslant u_t^2 + f^2$ and (192), we obtain

$$\left|\int\int_{B(t_1)} 2fu_t\,\mathrm{d}x_1\,\mathrm{d}x_2\right| \leqslant K(t_1) + M(t_1),$$

where

$$M(t_1) = \int\int_{B(t_1)} f^2\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{193}$$

On substituting the inequalities obtained in (189), we have:

$$\begin{aligned} K(t) \leqslant K(0) + (P_1 + P_2 + P_3 + 1) \int_0^t K(t_1)\,\mathrm{d}t_1 \\ + P_3 \int_0^t L(t_1)\,\mathrm{d}t_1 + \int_0^t M(t_1)\,\mathrm{d}t_1. \end{aligned} \tag{194}$$

We now consider an inequality for $L(t)$. We consider the integral

$$J_1 = \int_0^t \int\int_{B(t_1)} [u^2(x_1, x_2, t_1)]_{t_1}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_1.$$

It can be regarded as a triple integral over the domain D_t bounded from below by the plane $t = 0$, from above by the plane of constant t, and at the sides by the above-mentioned surface S, on which $\cos(n, t) > 0$. On applying Ostrogradskii's formula, we easily obtain the inequality

$$J_1 \geqslant \int\int_{B(t)} u^2\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int\int_{B(0)} u^2\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

i.e.

$$\int\limits_{B(t)}\int u^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \leqslant \int\limits_{B(0)}\int u^2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int\limits_0^t \int\limits_{B(t_1)}\int 2uu_{t_1}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_1, \tag{195}$$

whence, by using the inequality $|2uu_{t_1}| \leqslant u_{t_1}^2 + u^2$ and (192), we obtain

$$L(t) \leqslant L(0) + \int\limits_0^t K(t_1)\,\mathrm{d}t_1 + \int\limits_0^t L(t_1)\,\mathrm{d}t_1. \tag{196}$$

Addition of (194) and (196) gives:

$$K(t) + L(t) \leqslant K(0) + L(0) + P\int\limits_0^t [K(t_1) + L(t_1)]\,\mathrm{d}t_1 + \int\limits_0^t M(t_1)\,\mathrm{d}t_1, \tag{197}$$

where the constant $P = P_1 + P_2 + P_3 + 2$ depends on the sizes of coefficients a_{ik}, b_i, c and of the derivatives of a_{ik}. On introducing the notation

$$w(t) = \int\limits_0^t [K(t_1) + L(t_1)]\,\mathrm{d}t_1,$$

we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-Pt}w(t)] = -P\mathrm{e}^{-Pt}w(t) + \mathrm{e}^{-Pt}[K(t) + L(t)],$$

whence we obtain by using (197):

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-Pt}w(t)] \leqslant \mathrm{e}^{-Pt}\delta + \mathrm{e}^{-Pt}\int\limits_0^t M(t_1)\,\mathrm{d}t_1,$$

where

$$\delta = K(0) + L(0). \tag{198}$$

We integrate the last inequality from 0 to t then multiply both sides by e^{Pt}:

$$w(t) \leqslant \frac{\delta}{P}(\mathrm{e}^{Pt} - 1) + \mathrm{e}^{Pt}\int\limits_0^t \mathrm{e}^{-Pt_2}\Big[\int\limits_0^{t_2} M(t_1)\mathrm{d}t_1\Big]\,\mathrm{d}t_2,$$

i.e.

$$\int\limits_0^t [K(t_1) + L(t_1)]\,\mathrm{d}t_1 \leqslant \frac{\delta}{P}(\mathrm{e}^{Pt} - 1) + \mathrm{e}^{Pt}\int\limits_0^t \mathrm{e}^{-Pt_2}\Big[\int\limits_0^{t_2} M(t_1)\,\mathrm{d}t_1\Big]\,\mathrm{d}t_2.$$

On substituting this in the right-hand side of (197), we obtain

$$K(t) + L(t) \leqslant \delta\mathrm{e}^{Pt} + P\mathrm{e}^{Pt}\int\limits_0^t \mathrm{e}^{-Pt_2}\Big[\int\limits_0^{t_2} M(t_1)\,\mathrm{d}t_1\Big]\,\mathrm{d}t_2 + \int\limits_0^t M(t_1)\,\mathrm{d}t_1. \tag{199}$$

This inequality will hold all the more if the left-hand side contains only $K(t)$ or $L(t)$. For a homogeneous equation ($f \equiv 0$) we have to put $M(t_1) \equiv 0$, and we obtain

$$K(t) + L(t) \leqslant \delta e^{Pt}. \tag{200}$$

On taking (182) into account, we can write the inequality

$$K(t) \geqslant \int\limits_{B(t)}\int (\lambda u_{x_1}^2 + \lambda u_{x_2}^2 + u_t^2)\, dx_1\, dx_2. \tag{201}$$

We can assume without loss of generality that the λ in (182) satisfies $0 < \lambda \leqslant 1$, so that (200) (with $f \equiv 0$) leads to an inequality of the form:

$$\int\limits_{B(t)}\int (u_{x_1}^2 + u_{x_2}^2 + u_t^2)\, dx_1\, dx_2 \leqslant \frac{\delta}{\lambda} e^{Pt}. \tag{202}$$

We remark that the inequalities obtained above hold for all t for which we can construct a domain D_t of the above type.

The above inequalities and their applications to the theory of partial differential equations may be found in well known works by Friedrich, Levy, Schauder and Sobolev.

155. Theorems on the uniqueness and continuous dependence of the solutions. The above inequalities lead readily to theorems on the uniqueness of the solution of the Cauchy problem and on the continuous dependence of the solution on the initial data and the non-homogeneous term of the equation. We consider the difference between two solutions of the Cauchy problem for the same initial data and reduce the uniqueness theorem to the following: if the non-homogeneous term f in equation (181) vanishes and the initial data are

$$u|_{t=0} = u_t|_{t=0} = 0, \tag{203}$$

the solution of the problem must be $u \equiv 0$. We draw the characteristic conoid through any point $(x_1^{(0)}, x_2^{(0)}, t^{(0)})$ and suppose that it forms, in conjunction with the plane $t = 0$, a domain D of the type described above. Let $u(x_1, x_2, t)$ be a solution of the problem for $f \equiv 0$ and with initial conditions (203), which is continuous together with its derivatives up to the second order in the domain D. We can use e.g. inequality (200), where it follows from what has been said that $\delta = 0$. Hence

$$L(t) = \int\limits_{B(t)}\int u^2\, dx_1\, dx_2 = 0,$$

so that $u \equiv 0$ in D. This assertion retains its force if the homogeneous initial conditions (203) hold only on the base $B(0)$ of D instead of throughout the (x, y) plane, since $\delta = 0$ even in this case. We can conclude from this that *the value of the solution of homogeneous equation* (181) *at the point* $(x_1^{(0)}, x_2^{(0)}, t^{(0)})$ *depends on the values of the initial data only on the base* $B(0)$ *of the characteristic conoid with vertex* $(x_1^{(0)}, x_2^{(0)}, t^{(0)})$. We are assuming here that the conoid forms, in conjunction with the plane $t = 0$, a domain D of the type mentioned.

The continuous dependence of the solution on the initial data reduces, precisely as above, to the fact that if $f \equiv 0$, and the functions $\varphi_0(x_1, x_2)$ and $\varphi_1(x_1, x_2)$ appearing in the initial conditions

$$u|_{t=0} = \varphi_0(x_1, x_2); \quad u_t|_{t=0} = \varphi_1(x_1, x_2) \tag{204}$$

are small (in some sense), then the solution $u(x_1, x_2, t)$ is also small (in the usual sense). Suppose that the initial data are small in the sense that integrals $L(0)$ and $K(0)$ are small, i.e. suppose that $L(0) \leqslant \varepsilon$ and $K(0) \leqslant \varepsilon$, where ε is a small positive number. It now follows directly from (200) and (201) that $L(t)$ and $K(t)$ satisfy throughout the domain D:

$$K(t) \leqslant 2\varepsilon e^{Pt}; \quad L(t) \leqslant 2\varepsilon e^{Pt}.$$

The continuity of the dependence on the initial data can be proved, not only in the sense of inequalities for integrals $K(t)$ and $L(t)$, but also in the sense of an inequality for the absolute value of the function itself if $n = 1$, i.e. if we have two independent variables x_1 and t. This follows at once from Riemann's method [143], if the equation is reduced to the canonical form, as employed with Riemann's method. If the number of independent variables is greater than two, we cannot use these inequalities to deduce from the smallness of $|\varphi_0|$ and $|\varphi_1|$ that $|u|$ is small (for $f \equiv 0$). Let us consider the case $n = 1$ with the aid of the inequalities deduced above.

We have in this case the independent variables (x, t), and the domain D is a trapezoid ABB_1A_1 with in general curvilinear lateral sides. The straight line A_1B_1 has the equation $t = C$. Let $x = \xi_1(t)$ be the equation of the side AA_1 and $x = \xi_2(t)$ the equation of BB_1. We assume that the absolute value of the derivative of $\varphi_0(x)$, as well as of $\varphi_0(x)$ and $\varphi_1(x)$, appearing in the initial conditions (204), is small. The integrals of the squares of these magnitudes over the base AB of domain D are now also small, and $L(0)$ and $K(0)$ are therefore small;

we have here:

$$K(t) = \int_{\xi_1(t)}^{\xi_2(t)} [a(x, t)\, u_x^2 + u_t^2]\, dx,$$

where $a(x, t) \geqslant m = 0$. It follows from the above inequalities, since $L(0)$ and $K(0)$ are small, that $L(t)$ and $K(t)$ are small for $0 \leqslant t < C$, whence we can conclude that the integrals

$$\int_{\xi_1(t)}^{\xi_2(t)} u^2 (x, t)\, dx; \quad \int_{\xi_1(t)}^{\xi_2(t)} u_x^2 (x, t)\, dx; \quad \int_{\xi_1(t)}^{\xi_2(t)} u_t^2 (x, t)\, dx, \tag{205}$$

are small. Suppose that these integrals do not exceed some positive number η. Application of Buniakowski's inequality gives:

$$\{u(x, t) - u[\xi_1 (t), t]\}^2 = \Big[\int_{\xi_1(t)}^{x} u_x (x', t)\, dx' \Big]^2$$

$$\leqslant \int_{\xi_1(t)}^{x} u_x^2 (x', t)\, dx \cdot \int_{\xi_1(t)}^{x} 1^2\, dx,$$

whence

$$\{u(x, t) - u(\xi_1 (t), t)\}^2 \leqslant a\eta \qquad [\xi_1 (t) \leqslant x \leqslant \xi_2 (t)], \tag{206}$$

where a is the maximum of the difference $\xi_2(t) - \xi_1(t)$ in D.

We can obtain in the same way the inequality:

$$\left.\begin{array}{l} \Big[\int_{\xi_1(t)}^{x} u(x', t)\, dx' \Big]^2 \leqslant a\eta \\ {[\xi_1 (t) \leqslant x \leqslant \xi_2 (t)].} \end{array}\right\} \tag{207}$$

It follows from (206) that

$$u[\xi_1 (t), t] = u(x, t) + v(x, t) \qquad \big[|v(x, t)| \leqslant \sqrt{a\eta}\,\big]. \tag{208}$$

On integrating both sides with respect to x between the limits $\xi_1(t) \leqslant x \leqslant \xi_2(t)$ and using (207), we get:

$$|u[\xi_1 (t), t]| \leqslant \frac{\sqrt{a\eta}}{b} + \sqrt{a\eta}, \tag{209}$$

where b is the minimum of the difference $| \xi_2(t) - \xi_1(t) |$ in D, i.e. $| u[\xi_1(t), t] | \leqslant c^{\eta^{1/2}}$, where c is a constant. Using (208), we have on the basis of this last inequality:

$$|u(x, t)| \leqslant d\eta^{\frac{1}{2}}, \tag{210}$$

where η is the upper bound of integrals (205) and d is a constant, the same for all points of D. This is in fact the required inequality for

$|u(x, t)|$ throughout D. We now turn to finding an inequality for the solution $u(x, t)$ in terms of the function f.

Let the function f in the homogeneous initial conditions (203) be non-zero. Let $|f| \leqslant M_0$, where M_0 is a positive number; let E denote the maximum area of $B(t)$ in the domain D. Using (193) and (199), we have:

$$K(t) + L(t) \leqslant \frac{M_0^2 E}{P} (e^{Pt} - 1). \tag{211}$$

We can use the inequality for the $M(t_1)$ appearing in (199) instead of the inequality for $|f|$.

The integrals $K(t)$ and $L(t)$ can be made as small as desired for the difference $u_2 - u_1$ between two solutions u_1 and u_2 of equation (181) with different functions f but with the same initial conditions, provided that $|f_2 - f_1|$ is sufficiently small. When $n = 1$, an inequality can also be obtained from this for $|u_2 - u_1|$, as above.

156. The case of the wave equation. We take the homogeneous wave equation:

$$u_{x_1x_1} + u_{x_2x_2} - u_{tt} = 0.$$

We know how to solve the Cauchy problem for this when the initial data are

$$u|_{t=0} = \varphi(x_1, x_2); \quad u_t|_{t=0} = \psi(x_1, x_2),$$

where $\varphi(x_1, x_2)$ has continuous derivatives up to the third order and $\psi(x_1, x_2)$ up to the second order [II, 171, 172]. If $\varphi(x_1, x_2)$ has continuous derivatives up to the sixth order and $\psi(x_1, x_2)$ up to the fifth order, we can use Poisson's formula to assert that any derivatives of u with respect to the coordinates (x_1, x_2) up to the third order is also a solution of the wave equation with initial conditions in which φ and ψ are replaced by the corresponding derivatives. For instance, u_{x_1} is the solution of the Cauchy problem with initial data φ_{x_1} and ψ_{x_1} and so on. Let D be a characteristic cone of the wave equation with vertex $M_1(x_1, y_1, t_1)$ $(t_1 > 0)$, bounded from below by the plane $t = 0$. Any section of it $B(t_0)$ by the plane $t = t_0$, where $0 \leqslant t_0 < t_1$, is a circle. Let the following inequalities hold for the functions φ and ψ and their derivatives:

$$\iint\limits_{B(0)} \left(\frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}\right)^2 dS \leqslant \varepsilon^2; \quad \iint\limits_{B(0)} \left(\frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}\right)^2 dS \leqslant \varepsilon^2.$$

$$(\alpha = 0, 1, 2, 3, 4) \qquad (\alpha = 0, 1, 2, 3)$$

For the wave equation, $K(t)$ reduces to the integral over $B(t)$ of the sum of the squares of the derivatives with respect to x_1, x_2 and t, and by what has been said, a consideration of the Cauchy problem for u and its derivatives with respect to (x_1, x_2) up to the third order gives us $K(0) \leqslant 3\varepsilon^2$ and $L(0) \leqslant \varepsilon^2$, whence $\delta \leqslant 4\varepsilon^2$, and inequality (200) gives:

$$K(t) \leqslant 4\varepsilon^2 e^{Pt}; \qquad L(t) \leqslant 4\varepsilon^2 e^{Pt}.$$

These inequalities hold both for u and for its derivatives with respect to (x_1, x_2) up to the third order. We have, all the more:

$$\iint\limits_{B(t)} \left(\frac{\partial^\alpha u}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\right)^2 dS \leqslant 4\varepsilon^2 e^{Pt}; \qquad \iint\limits_{K(t)} \left(\frac{\partial^\alpha u}{\partial t\,\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\right)^2 dS \leqslant 4\varepsilon^2 e^{Pt}$$

$$(\alpha = 0, 1, 2, 3, 4) \qquad\qquad (\alpha = 1, 2, 3, 4),$$

where $K(t)$ is a circle. Hence, as we shall now prove, inequalities can be obtained for the function itself and its derivatives:

$$|u|,\ u_{x_1}|,\ |u_{x_2}|,\ |u_{x_1x_1}|,\ |u_{x_1x_2}|\ |u_{x_2x_2}|,\ |u_t|,\ |u_{tx_1}|,\ |u_{tx_2}| \leqslant 2\varepsilon c e^{\frac{P}{2}t}$$

in any circle $K_1(t)$, concentric with $K(t)$ and with a smaller radius, where c is a constant. The inequality for u_{tt} follows directly from the wave equation itself. We thus obtain inequalities for the function itself and its derivatives up to the second order, instead of for their mean squares.

Everything that we have said is a direct consequence of the following general theorem, which we shall state and prove for a space with any number of dimensions. We shall require this theorem in the next chapter.

THEOREM. *If the function $f(x_1, \ldots, x_n)$ has continuous derivatives up to some order l inside an n-dimensional sphere D and the inequalities hold*

$$\int\limits_D \left(\frac{\partial^\alpha f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\right)^2 dx_1\ldots dx_n \leqslant A^2 \qquad (\alpha = 0, 1, \ldots, l), \tag{212}$$

we have in any interior concentric sphere D_1, for the function itself and for its derivatives up to order $l-[n/2]-1$, where $[2/n]$ is the integral part of the positive number $n/2$:

$$\left|\frac{\partial^\beta f}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}}\right| \leqslant cA \qquad \left(\beta = 0, 1, \ldots, l-\left[\frac{n}{2}\right]-1\right), \tag{213}$$

where the constant c depends only on the choice of D_1.

We construct the auxiliary function

$$\sigma(x) = \begin{cases} 1 \text{ for } x \leqslant \dfrac{1}{3}, \\ 0 \text{ for } x \geqslant \dfrac{2}{3}, \\ \dfrac{1}{2} + \dfrac{1}{2}\, \dfrac{e^u - e^{-u}}{e^u + e^{-u}} \text{ for } \dfrac{1}{3} < x < \dfrac{2}{3}, \end{cases} \tag{214}$$

where

$$u = \frac{\dfrac{1}{2} - x}{\left(\dfrac{2}{3} - x\right)\left(x - \dfrac{1}{3}\right)}.$$

Obviously, $u \to +\infty$ as x tends to 1/3 from greater values, and $u \to -\infty$ as x tends to 2/3 from smaller values. Now, $\sigma(x)$ tends to unity and zero respectively, and it is easily seen that all the derivatives of $\sigma(x)$ are continuous for $x = 1/3$ and $x = 2/3$. Let M_0 be a point of D_1 and let h be the difference between the radii of D and D_1. We introduce the spherical system of coordinates with centre M_0:

$$\begin{aligned} x_1 &= r \cos\theta_1; \\ x_2 &= r \sin\theta_1 \cos\theta_2; \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-2} &= r \sin\theta_1 \ldots \sin\theta_{n-3} \cos\theta_{n-2}; \\ x_{n-1} &= r \sin\theta_1 \ldots \sin\theta_{n-2} \cos\psi; \\ x_n &= r \sin\theta_1 \ldots \sin\theta_{n-2} \sin\psi, \end{aligned}$$

where $0 \leqslant \theta_k \leqslant \pi$ and $0 \leqslant \psi < 2\pi$. We have for an elementary volume:

$$d\omega_n = r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}\, dr\, d\theta_1 \ldots d\theta_{n-2}\, d\psi.$$

On striking out dr and putting $r = 1$, we get the elementary surface area $d\sigma_m$ of the unit sphere. We introduce the function:

$$\begin{aligned} F(M) = f(M) \cdot \frac{\partial^{l-1}}{\partial r^{l-1}} \left[\frac{r^{l-1}}{(l-1)!}\sigma\left(\frac{r}{h}\right)\right] - \\ - \frac{\partial f(M)}{\partial r} \cdot \frac{\partial^{l-2}}{\partial r^{l-2}} \left[\frac{r^{l-1}}{(l-1)!}\sigma\left(\frac{r}{h}\right)\right] + \\ + \ldots + (-1)^{l-1} \frac{\partial^{l-1} f(M)}{\partial r^{l-1}} \cdot \left[\frac{r^{l-1}}{(l-1)!}\sigma\left(\frac{r}{h}\right)\right], \end{aligned}$$

where r is the distance $\overline{M_0M}$. The following expressions may be verified directly:

$$F(M_0) = f(M_0); \qquad F(M) = 0 \text{ for } r = h,$$

$$\frac{\partial F(M)}{\partial r} = f(M) \cdot \frac{\partial^l}{\partial r^l}\left[\frac{r^{l-1}}{(l-1)!}\,\sigma\left(\frac{r}{h}\right)\right] + \\ + (-1)^{l-1}\frac{\partial^l f}{\partial r^l} \cdot \left[\frac{r^{l-1}}{(l-1)!}\,\sigma\left(\frac{r}{h}\right)\right], \tag{215}$$

and we can write:

$$f(M_0) = -\int\limits_0^h \frac{\partial F(M)}{\partial r}\,\mathrm{d}r,$$

where the integration is performed along a ray issuing from M_0. On multiplying both sides of this equation by $\mathrm{d}\sigma_n = \mathrm{d}\omega_n : r^{n-1}\,\mathrm{d}r$, and integrating between the limits $0 \leqslant \theta_s \leqslant \pi$; $0 \leqslant \psi \leqslant 2\pi$, we obtain:

$$f(M_0) = -\frac{1}{\sigma_n}\int\limits_{D_0} \frac{\partial F(M)}{\partial r}\,r^{-n+1}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where D_0 is a sphere with centre M_0 and radius h and σ_n is the surface area of the unit sphere in R_n. Putting $k = [n/2]$, we can rewrite the last equation as

$$f(M_0) = -\frac{1}{\sigma_n}\int\limits_{D_0} \frac{1}{r^k}\,\frac{\partial F(M)}{\partial r}\,r^{k-n+1}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

and application of Buniakowski's inequality gives:

$$f^2(M_0) \leqslant \frac{1}{\sigma_n^2}\int\limits_{D_0}\left(\frac{1}{r^k}\,\frac{\partial F(M)}{\partial r}\right)^2 \mathrm{d}x_1 \ldots \mathrm{d}x_n \times \\ \times \int\limits_{D_0} r^{2k-2n+2}\,r^{n-1}\,\mathrm{d}r\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{n-2}\,\mathrm{d}\psi.$$

The power in the second integral is unity when n is even, and zero when n is odd. We therefore obtain:

$$f^2(M_0) \leqslant c_1\int\limits_{D_0}\left(\frac{1}{r^k}\,\frac{\partial F(M)}{\partial r}\right)^2 \mathrm{d}x_1 \ldots \mathrm{d}x_n, \tag{216}$$

where the constant c_1 depends only on h. Let us return to (215). The coefficient of f on the right-hand side vanishes when $r \leqslant h/3$, by (214). On the other hand, we can say by using the rule for differentiation

of a composite function that $\partial^l f/\partial r^l$ is a linear combination of the derivatives of order l with respect to $x_3, \ldots, x_n$ with bounded coefficients. On taking this into account, we can write:

$$\frac{1}{r^k} \frac{\partial F(M)}{\partial r} = af + \sum a_{\alpha_1 \ldots \alpha_n} \frac{\partial^l f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

where a is a bounded continuous function and $| a_{\alpha} \ldots_{\alpha_n} | \leqslant c_2 r^{l-k-1}$. With $l \geqslant k+1$, i.e. with $l \geqslant [n/2]+1$, all the coefficients in the last expression are bounded, and we obtain by (216), on taking into account the inequality $(x_1 + \ldots + x_n)^2 \leqslant n(x_1^2 + \ldots + x_n^2)$ and inequality (212):

$$f^2(M_0) \leqslant c^2 A^2,$$

where the constant c depends only on h. If, for a positive integer β, we have $l - \beta \geqslant [n/2]+1$, i.e. $\beta \leqslant l - [n/2] - 1$, we can apply the whole of the above discussion with f replaced by any partial derivative of f of order β and l replaced by $(l - \beta)$. We thus obtain (213). The theorem is proved. This theorem and the present proof are due to Sobolev.

The theorem can be used to obtain inequalities for the solutions of the non-homogeneous wave equation, on the assumption of sufficient differentiability not only of the functions appearing in the initial conditions, but also of the function f. In addition, the arguments still hold for the generalized wave equation [II, 188]:

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2} + c^2 u$$

and for the case of any number of independent variables in the wave equation.

157. Supplementary propositions. We shall now mention some theorems from the theory of functions that will be required later. These theorems hold in Euclidean space of any number of dimensions. For simplicity, we shall state them for the case of a plane.

Let $f(P) = f(x, y)$ be a continuous function given on a bounded closed set F on the plane. The definition of continuity on F is the same as in a closed domain [I, 67 and 151], and it can be proved in the same way as for a closed domain that $f(P)$ has a maximum and minimum on F. Let $A = \max | f(P) |$ on F.

THEOREM 1. *The function $f(x, y)$, continuous on the bounded closed set F, can be extended to the whole of the plane whilst preserving its continuity and upper bound A.*

We shall first prove the following lemma:

LEMMA. *If D and E are two bounded closed sets on the plane with no common points, and a and b are given numbers $(a < b)$, we can construct on the whole of the plane a continuous function $\varphi_{a,b}(x, y)$, equal to a at points of D and to b at points of E and satisfying the inequality:*

$$a \leqslant \varphi_{a,b}(x, y) \leqslant b.$$

We remark that the distance $\varrho(x, y; F)$ of a point (x, y) from the bounded closed set F is positive if (x, y) does not belong to F, is zero if (x, y) belongs to F, and is a continuous function of (x, y) [II, 89]. If $a = 0$ and $b = 1$, the function

$$\varphi_{0,1}(x, y) = \frac{\varrho(x, y; D)}{\varrho(x, y; D) + \varrho(x, y; E)}$$

obviously satisfies all the requirements of the lemma. In the general case, we only have to put:

$$\varphi_{a,b}(x, y) = (b - a)\,\varphi_{0,1}(x, y) + a.$$

Note. If one of the sets, say E, is absent, it is sufficient to put $\varphi(x, y) = a$ throughout the plane.

We turn to the proof of the theorem. We put $f_0(x, y) = f(x, y)$. Let D_0 and E_0 denote the closed sets consisting of points of the set F at which $f(x, y) \leqslant -A/3$ and $f(x, y) \geqslant A/3$ respectively. By the lemma, we can construct a function $\varphi_0(x, y)$ throughout the plane, equal to $(-A/3)$ in D_0, equal to $A/3$ on E_0 and satisfying the condition $|\varphi_0(x, y)| \leqslant A/3$. Let

$$f_1(x, y) = \varphi_0(x, y) - f_0(x, y) \qquad [(x, y) \text{ on } F].$$

It follows at once from the properties of $\varphi_0(x, y)$ and $f_0(x, y)$ that, if $A_1 = \max |f_1(x, y)|$ on F, then $A_1 \leqslant 2A/3$. We now construct from $f_1(x, y)$ a new function $f_2(x, y)$, just as $f_1(x, y)$ was constructed from $f_0(x, y)$. Let D_1 and E_1 be the sets of the points of F at which $f_1(x, y) \leqslant -A_1/3$ and $f_1(x, y) \geqslant A_1/3$. We construct $\varphi_1(x, y)$, continuous throughout the plane, equal to $(-A_1/3)$ on D_1, equal to $A_1/3$ on E_1 and satisfying $|\varphi_1(x, y)| \leqslant A_1/3$. We next put:

$$f_2(x, y) = \varphi_1(x, y) - f_1(x, y) \qquad [(x, y) \text{ on } F].$$

If $A_2 = \max |f_2(x, y)|$ on F, then $A_2 \leqslant 2A_1/3$. Two sequences of continuous functions may be constructed in this way: $f_n(x, y)$, defined on F, and $\varphi_n(x, y)$, defined throughout the plane, where

$$f_{n+1}(x, y) = f_n(x, y) - \varphi_n(x, y) \qquad [(x, y) \text{ on } F] \tag{217}$$

and

$$|f_n(x, y)| \leqslant \left(\frac{2}{3}\right)^n A \qquad [(x, y) \text{ on } F];$$
$$|\varphi_n(x, y)| \leqslant \left(\frac{2}{3}\right)^n \frac{A}{3} [(x, y) \text{ arbitrary}]. \tag{218}$$

It follows from the last inequality that the series

$$\sum_{n=0}^{\infty} \varphi_n(x, y)$$

is uniformly convergent throughout the plane. Its sum $\varphi(x, y)$ is continuous throughout the plane, and

$$|\varphi(x, y)| \leqslant \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{A}{3} = A.$$

It remains to show that $\varphi(x, y)$ coincides with $f(x, y)$ on F. On summing equation (217) from $n = 0$ to $n = p$ over n, we get:

$$\sum_{n=0}^{p} \varphi_n(x, y) = f_0(x, y) - f_{p+1}(x) \qquad [(x, y) \text{ on } F].$$

We find by virtue of the first of inequalities (218) that $\varphi(x, y) = f_0(x, y)$ on F as $p \to \infty$, i.e. $\varphi(x, y) = f(x, y)$ on F, which is what we had to prove.

The above proof has been taken from P. S. Aleksandrov's *Introduction to the General Theory of Sets and Functions* (Vvedenie v obschuyu teoriyu mnozhestv i funktsii). We now introduce an averaging process for any function $f(x, y)$, given and continuous throughout the plane. It leads us to a sequence of functions $F_n(x, y)$, which have derivatives of all orders and are close to $f(x, y)$ for large n.

Let $\omega(t)$ be a function defined for all real t, having ordinary derivatives of all orders, non-negative in the interval $[-1, 1]$, zero outside this interval, and such that

$$\int_{-\infty}^{+\infty} \omega(t)\, dt = \int_{-1}^{1} \omega(t)\, dt = 1. \tag{219}$$

The following function can be quoted as an example:

$$\omega(t) = ce^{\frac{1}{t^2-1}} \text{ for } |t| < 1 \text{ and } \omega(t) = 0 \text{ for } |t| \geqslant 1, \tag{220}$$

where the constant c is given by the condition

$$c \int_{-1}^{1} e^{\frac{1}{t^2-1}}\,dt = 1.$$

If $t \to 1$ from smaller values, then $1/(t^{2-1}) \to -\infty$, and the derivatives of all orders of the function $\omega(t)$ pass to zero values for $t \geqslant 1$ without losing continuity when t passes through $t = 1$. Similarly, when $t \to -1$ from larger values.

We now construct a sequence of averaging kernels on the plane:

$$\psi_n(x, y;\ \xi, \eta) = n^2\,\omega(nx - n\xi)\,\omega(ny - n\eta).$$

The non-negative function $\psi_n(x, y;\ \xi, \eta)$ has continuous partial derivatives of all orders, depends only on the differences $x - \xi$, $y - \eta$, vanishes outside the two-dimensional interval

$$\Delta_n^{(\xi,\eta)}\left(|x - \xi| \leqslant \frac{1}{n},\ |y - \eta| \leqslant \frac{1}{n}\right)$$

and, by (219):

$$\iint\limits_{\Delta_n^{(\xi,\eta)}} \psi_n(x, y;\ \xi, \eta)\,dx\,dy = 1. \tag{221}$$

Let $f(x, y)$ be defined and continuous throughout the plane. We construct the sequence of *mean functions*.

$$F_n(\xi, \eta) = \iint f(x, y)\,\psi_n(x, y;\ \xi, \eta)\,dx\,dy \quad (n = 1, 2, \ldots). \tag{222}$$

The integrand vanishes outside $\Delta_n^{(\xi,\eta)}$ for any fixed (ξ, η), and the integral can be regarded as over the interval $\Delta_n^{(\xi,\eta)}$ or over the entire plane. The integrand is a continuous function of the pair of points (x, y), (ξ, η) and has continuous derivatives of all orders with respect to (ξ, η). Hence it follows [II, 80] that $F_n(\xi, \eta)$ is continuous and has continuous derivatives of all orders throughout the plane.

Let us show that the $F_n(\xi, \eta)$ *tend uniformly to* $f(\xi, \eta)$ *in any finite closed domain* $\overline{B}$ *of the plane.* Using (221) and (222), we can write:

$$f(\xi, \eta) - F_n(\xi, \eta) = \iint [f(\xi, \eta) - f(x, y)]\,\psi_n(x, y;\ \xi, \eta)\,dx\,dy,$$

whence, since the ψ_n are positive:

$$|f(\xi, \eta) - F_n(\xi, \eta)| \leqslant \iint |f(\xi, \eta) - f(x, y)|\,\psi_n(x, y;\ \xi, \eta)\,dx\,dy. \tag{223}$$

We choose N large enough for $| f(\xi, \eta) - f(x, y) | \leqslant \varepsilon$ if (x, y) belongs to $\Delta_n^{(\xi,\eta)}$ with $n \geqslant N$ and for any choice of (ξ, η) in $\overline{B}$. Such an N exists by virtue of the uniform continuity of $f(x, y)$ in any bounded domain.

On taking (223) into account and the fact that $\psi_n(x, y; \xi, \eta) = 0$ outside $\Delta_n^{(\xi,\eta)}$, we obtain on the basis of (221): $| f(\xi, \eta) - F_n(\xi, \eta) | \leqslant \varepsilon$ for $n \geqslant N$ if (ξ, η) belongs to $\overline{B}$, which in fact proves our assertion.

If $f(x, y)$ satisfies $| f(x, y) | \leqslant A$, then $| F_n(\xi, \eta) | \leqslant A$ for any n. For:

$$|F_n(\xi, \eta)| \leqslant \int\int |f(x, y)| \psi_n(x, y;\ \xi, \eta)\, dx\, dy \leqslant$$
$$\leqslant \int\int A\psi_n(x, y;\ \xi, \eta)\, dx\, dy = A.$$

A further fact must be mentioned. *If $f(x, y)$ vanishes outside a finite domain B_1, the function $F_n(\xi, \eta)$ vanishes at all points whose distance from the boundary of B_1 is greater than $1/n$.* It follows from this that *the function $F_n(\xi, \eta)$ vanishes for all sufficiently large values of n at any point outside B_1. In this case $F_n(\xi, \eta) \to f(\xi, \eta)$ uniformly throughout the plane.*

This method of constructing mean functions can also be used when $f(x, y)$ is merely integrable. This will be discussed in more detail in Vol. V.

158. Generalized solutions of the wave equation. By using Green's formula, it is possible to generalize the concept of the solution of a partial differential equation. We start with the wave equation:

$$\Box u = u_{xx} + u_{yy} - \frac{1}{a^2} u_{tt} = 0. \tag{224}$$

Let D be a bounded domain of three-dimensional space (x, y, t) and S its boundary surface. Green's formula becomes:

$$\int\int\limits_D\int (v \Box u - u \Box v)\, d\tau = \int\limits_S\int [\sigma P(u) - uP(\sigma)]\, dS, \tag{225}$$

where

$$P(u) = u_x \cos(n, x) + u_y \cos(n, y) - \frac{1}{a^2} u_t \cos(n, t). \tag{226}$$

Suppose that the function u has continuous derivatives up to the second order inside D and satisfies equation (224), whilst σ is any function having continuous derivatives up to the second order inside D and vanishing at all points of D whose distance from S does not

exceed some positive number (different for different σ). Equation (225) now gives:

$$\iiint_D u \, \Box \, \sigma \, d\tau = 0. \tag{227}$$

This formula does not contain the derivatives of u, and the foregoing considerations naturally lead us to the following definition: *a function integrable in the domain D is said to be a generalized solution of equation* (224) *if it satisfies equation* (227), *where σ is an arbitrary function with the above-mentioned properties.*

Let $u(M, \lambda)$ be a family of generalized solutions depending on the parameter λ. Condition (227) may be written as

$$\iiint_{D_1} u(M; \lambda) \, \Box \, \sigma(M) \, d\tau = 0, \tag{228}$$

where $M(x, y, t)$ is a variable point. Suppose for definiteness that $u(M, \lambda)$ is a continuous function of the four variables (x, y, t, λ) when λ varies in a fixed finite interval $[a, b]$. On integrating (228) with respect to λ over the interval $[a, b]$, we get:

$$\iiint_{D_1} u_1(M) \, \Box \, \sigma(M) \, d\tau = 0, \text{ where } u_1(M) = \int_a^b u(M, \lambda) \, d\lambda,$$

i.e. the function $u_1(M)$ satisfies condition (227), and is thus also a generalized solution of equation (224). In other words, *if a generalized solution depending on a parameter is integrated with respect to the parameter, a further generalized solution is obtained.*

The theory of generalized solutions of the wave equation was given by Sobolev in *A General Theory of Wave Diffraction at Riemann Surfaces* (Obshchaya teoriya diffraktsii voln na Rimanovykh poverkhnostyakh) (Trudy matematicheskogo instituta im. V. A. Steklova). We shall mention two results obtained in this work. The necessary and sufficient condition for $u(M)$ to be a generalized solution inside D is that there exist a sequence $u_n(M)$ of solutions of equation (224) having continuous derivatives up to the second order in D, and such that

$$\lim_{n \to \infty} \iiint_D |u(M) - u_n(M)| \, d\tau = 0.$$

The second result was the establishment of the Cauchy problem for generalized solutions and the proof of the uniqueness of its solution. The author then employed generalized solutions to solve the diffraction problem for waves on Riemann surfaces. It follows from the

discussion of [142] that a generalized solution of the wave equation can only have a strong discontinuity on a characteristic surface when the kinematic compatibility conditions are fulfilled.

Let us give an example of a generalized solution of the wave equation:

$$\Box u = u_{xx} + u_{yy} + u_{zz} - \frac{1}{a^2} u_{tt} = 0.$$

Let $\omega(\xi)$ be a function which is continuous in the finite interval $J\,(a \leqslant \xi \leqslant b)$, but which has no derivative. Let D be a finite domain of four-dimensional space (x, y, z, t) such that $r = \sqrt{x^2 + y^2 + z^2} > 0$ at all points of D, whilst $(t - r/a)$ belongs to the interval J. There exists a sequence of function $\omega_n(\xi)$ having derivatives of any order for all ξ, which tends uniformly to $\omega(\xi)$ in J [157]. The functions

$$\frac{\omega_n\left(t - \dfrac{r}{a}\right)}{r}$$

have continuous derivatives of all orders in D and satisfy equation (224) [II, 200]. We have by Green's formula:

$$\iiint\limits_D \frac{\omega_n\left(t - \dfrac{r}{a}\right)}{r} \Box\sigma\, d\tau = 0,$$

where σ satisfies the above-mentioned conditions. On passing to the limit as $n \to \infty$, we obtain:

$$\iiint\limits_D \frac{\omega\left(t - \dfrac{r}{a}\right)}{r} \Box\,\sigma\, d\tau = 0, \text{ i.e. the function } u = \frac{\omega\left(t - \dfrac{r}{a}\right)}{r},$$

which does not even possess first order derivatives, is a generalized solution of (224) in D. It can similarly be shown that $\omega(x - at)$ is a generalized solution of the equation $u_{xx} - (1/a^2) \cdot u_{tt} = 0$ in some domain of the (x, t) plane. By using Green's formula and a method of induction, generalized solutions can also be defined for the non-homogeneous wave equation, or even for an equation of the form [147]:

$$L(u) = \sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} + \sum_{k=1}^{n} b_k u_{x_k} + cu = f. \tag{229}$$

We can use Green's formula to arrive, as above, at the following definition of the generalized solution of this equation: *a generalized*

solution of equation (229) *is any continuous function satisfying the condition*

$$\int \underset{D}{\cdots} \int uM(\sigma)\, d\tau = \int \underset{D}{\cdots} \int f\sigma\, d\tau, \tag{230}$$

where σ is any function with continuous derivatives up to the second order inside D and vanishing at all points whose distance from the boundary of D does not exceed some positive number. We have written $M(\sigma)$ for the operator conjugate to $L(u)$ [147]. If u satisfies condition (230) and has continuous derivatives up to the second order inside D, we can use formula (110), together with the fact that v vanishes close to the boundary of D, to obtain:

$$\int \underset{D}{\cdots} \int [\sigma L(u) - uM(\sigma)]\, d\tau = 0.$$

We obtain by taking (230) into account:

$$\int \underset{D}{\int} \int \sigma[L(u) - f]\, d\tau = 0,$$

and hence we can assert [62], in view of the arbitrariness of σ inside D, that $L(u) = f$ inside D. It thus follows from (230) that, when the function u has continuous derivatives up to the second order inside D, it in fact satisfies equation (229) inside D. The actual definition (230) does not even require first order derivatives for u.

The requirement that u be continuous is also superfluous in essence, and can be replaced by the condition that it be integrable. A general discussion of generalized solutions requires the theory of functions of a real variable, and we leave this to Vol. V.

More detailed information regarding generalized solutions may be found in the book by S. L. Sobolev: *Partial Differential Equations of Mathematical Physics*, Pergamon Press, 1964.

159. Equations of the elliptic type. We have so far considered hyperbolic equations when investigating the Cauchy problem. We shall now dwell on the simplest equation of the elliptic type, namely Laplace's equation in two independent variables:

$$u_{xx} + u_{yy} = 0. \tag{231}$$

We know that any solution of this equation is the real part of an analytic function: $f(z) = u(x, y) + v(x, y)i$ [III, 22]. Let us consider the solution of (231) in the neighbourhood of some point, which can be

taken as the origin. Assuming that u has continuous derivatives up to the second order at and in the neighbourhood of this point, we have a power expansion of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

convergent in a circle $|z| < R$, where the $c_n = a_n + b_n i$ are complex numbers. On separating the real part of each term of the series

$$f(z) = \sum_{n=0}^{\infty} (a_n + b_n i)(x + yi)^n,$$

we obtain $u(x, y)$ as a series in homogeneous polynomials in (x, y):

$$u(x, y) = \sum_{n=0}^{\infty} \left\{ a_n \left[x^n - \frac{n(n-1)}{2!} x^{n-2} y^2 + \dots \right] + \right.$$
$$\left. + b_n \left[-nx^{n-1} y + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 - \dots \right] \right\} \quad (232)$$

and this series is absolutely convergent provided that $\sqrt{x^2 + y^2} < R$. Let us write the last series as a double series in positive integral powers of x and y:

$$\sum_{p, q=0}^{\infty} d_{pq} x^p y^q \quad (233)$$

and show that it is also convergent if the real values x and y are sufficiently close to zero. In fact, the absolute values of the terms of series (233) do not exceed the terms of the double series which is obtained from

$$\sum_{n=0}^{\infty} |c_n| (|x| + |y|)^n.$$

But the series

$$\sum_{n=0}^{\infty} |c_n| r^n \quad (r > 0)$$

is convergent for $r < R$ and it follows at once from this that series (233) is absolutely convergent provided $|x| + |y| < R$. We can group the terms in this series, and hence obtain series (232), i.e. the sum of series (233) is equal to $u(x, y)$. Thus every solution of equation (226) can be written as a power series in the neighbourhood of any point (x, y) provided the solution has no singularity at this point, i.e. to put the matter more simply, *every solution of equation* (231) *is an analytic function of* (x, y). Hence it follows immediately that the harmonic

function has derivatives of all orders and that, if two harmonic functions coincide in some two-dimensional part of the (x, y) plane, they coincide everywhere.

We remark that a completely different picture is obtained for the hyperbolic equation

$$u_{yy} - a^2 u_{xx} = 0, \tag{234}$$

where a is a given real number. This equation has the obvious solution [II, 164]:

$$u = \varphi(x + ay), \tag{235}$$

where φ is an arbitrary function having continuous derivatives up to the second order. It is proved in the theory of functions of a real variable that a $\varphi(t)$ can be constructed, having continuous first and second order derivatives and having no third order derivative for any value of t. For such a $\varphi(t)$, equation (234) will have no third order derivative for any (x, y), and hence cannot be an analytic function of (x, y). Let $a = i$ in (234) and (235). Now $a^2 = -1$, and (234) becomes equation (231), whilst formula (235) gives its solution in the form $u = \varphi(x + yi)$. This function must have a continuous derivative with respect to its argument, which is here a complex variable. But a function having a continuous derivative with respect to a complex argument is analytic. On separating the real part in the solution $u = \varphi(x + yi)$, we obtain a further analytic solution of (231). This discussion is of a formal kind and is not strict, but it can be used to provide a very simple explanation of the reason why the solutions of (231) and (235) differ in character, as we saw above.

A Cauchy problem can be formed for equation (231). For instance, we can seek the solution of (231) when u and its derivative u_x are given at $x = 0$:

$$u|_{x=0} = f_0(y); \quad u_x|_{x=0} = f_1(y), \tag{236}$$

where $f_0(y)$ and $f_1(y)$ are given analytic functions of y [127]. This problem has a unique solution in the neighbourhood of $x = 0$. We shall show by an example that the solution of this problem can have a serious defect from the mathematical point of view. Let

$$f_0(y) = 0 \text{ and } f_1(y) = \frac{1}{n}\sin(ny), \tag{237}$$

where n is a given positive number. It is easily shown that the solution of (231) satisfying these initial data is

$$u = \frac{e^{nx} - e^{-nx}}{2n^2}\sin(ny). \tag{238}$$

Let $n \to \infty$. The initially given $f_1(y)$ here tends to zero uniformly with respect to y, since $|\sin(ny)| \leqslant 1$, whilst the solution (238) tends to infinity if $x \neq 0$ and ny differs from a multiple of π. For, if say $x > 0$, then $e^{-nx} \to 0$, and the ratio $e^{nx}/n^2 \to \infty$ as $n \to \infty$, since the exponential function e^{nx} increases more rapidly than n^2. Thus, as the initial data tend to zero, the solution itself will increase indefinitely. In other words, we see from our example that the solution of the Cauchy problem for equation (231) does not possess the property of continuous dependence on the initial data. We always have this continuous dependence, in one form or another, for equations of the hyperbolic type [155].

We have proved that solutions of Laplace's equation in two independent variables are analytic. The same is true in the case of three independent variables:

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Let us prove this. Suppose we have a solution of the equation with continuous derivatives up to the second order at and in the neighbourhood of the origin. The function u is obviously harmonic in some closed sphere with centre at the origin and radius R. We can express the value of u at any interior point (x, y, z) of the sphere in terms of its values at points (ξ, η, ζ) on the surface of the sphere S in accordance with the formula [II, 197]:

$$u(x, y, z) = \frac{1}{4\pi R} \iint\limits_S u(\xi, \eta, \zeta) \frac{R^2 - (x^2 + y^2 + z^2)}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} \, dS. \tag{239}$$

For all x, y, z, sufficiently close to zero, we can expand the function:

$$[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{-3/2} = R^{-3}\left[1 + \frac{(x^2+y^2+z^2) - (2\xi x + 2\eta y + 2\zeta z)}{R^2}\right]^{-\frac{3}{2}}$$

in a power series in positive integral powers of (x, y, z) by using the binomial formula. The entire integrand of (239) can now be written as such a series with coefficients depending on (ξ, η, ζ). Integration of this series term by term over S gives us a power series for $u(x, y, z)$.

It can similarly be shown that the solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \tag{240}$$

are analytic functions of the variables (x, y): this will be discussed in the next chapter.

Proofs that the solutions of a wide class of elliptic equations are analytic may be found in the works of S. H. Bernshtein.

160. Generalized solution of Poisson's equation. The generalized solution of Laplace's equation would have to be defined in three-dimensional space by the relationship

$$\iiint_D u\,\Delta\sigma\,\mathrm{d}\tau = 0, \tag{241}$$

where σ is any function with continuous derivatives up to the second order inside D and vanishing at all points sufficiently close to the boundary of D. We shall prove a theorem showing that every continuous generalized solution of Laplace's equation is an ordinary solution.

THEOREM. *If a function $u(M)$, continuous inside D, satisfies* (241), *it is a harmonic function inside D.*

We first prove a simple lemma.

LEMMA. *If a function $u(M)$, continuous inside D, has the property that its value $u(M_0)$ at any points M_0 inside D is equal to its average value over any sphere with centre M_0 and sufficiently small radius, $u(M)$ is harmonic inside D.*

It is sufficient to show that $u(M)$ is a harmonic function in a sphere with centre at an arbitrary point M_0 inside D. We take such a sphere C with sufficiently small radius and let $U_0(M)$ be harmonic inside C, taking the same values on the boundary of C as $u(M)$. The difference $u(M) - u_0(M)$ has the mean value property indicated in the lemma at every interior point of C, since this property must be possessed by $u(M)$ by hypothesis, and by the harmonic function $u_0(M)$. It follows from this property that the difference attains its maximum and minimum values on the boundary of C. But it vanishes at every point of the boundary, so that $u(M)$ coincides with the harmonic function $u_0(M)$ inside C, which proves the lemma.

Now let us prove the theorem. In view of our lemma, we only need to show that, by virtue of (241), $u(M)$ satisfies the mean value property of the lemma at every point M_0. Let D_ε be a sphere with centre M_0 and radius ε inside D. We construct the function:

$$\sigma(M) = \begin{cases} (r^2 - \varepsilon^2)^3 & \text{if } r \leqslant \varepsilon, \text{ i.e. if } M \text{ belongs to } D_\varepsilon, \\ 0 & \text{if } r > \varepsilon, \text{ i.e. if } M \text{ is outside } D_\varepsilon, \end{cases}$$

where $r = | M_0M |$.

This function satisfies the requirements imposed above on $\sigma(M)$ We have

$$\frac{1}{6}\,\Delta\sigma(M) = \begin{cases} 7r^4 - 10\varepsilon^2 r^2 + 3\varepsilon^4 & \text{for } r \leqslant \varepsilon \\ 0 & \text{for } r > \varepsilon, \end{cases}$$

and (241) gives:

$$\iiint\limits_{r\leqslant\varepsilon} u(M)\,(7r^4 - 10\varepsilon^2 r^2 + 3\varepsilon^4)\,\mathrm{d}\tau = 0\,.$$

We differentiate this equation with respect to ε. Here, we must not only differentiate the integrand with respect to ε, but must also add a double integral over the surface of the sphere $r = \varepsilon$ [II, 171]. But the integrand vanishes for $r = \varepsilon$, and we have:

$$\iiint\limits_{r\leqslant\varepsilon} u(M)\,(12\varepsilon^3 - 20\varepsilon r^2)\,\mathrm{d}\tau = 0.$$

Further differentiation with respect to ε gives:

$$\varepsilon \iint\limits_{r=\varepsilon} u(M)\,\mathrm{d}S - 3\iiint\limits_{r\leqslant\varepsilon} u(M)\,\mathrm{d}\tau = 0$$

This equation can be rewritten as

$$4\pi\varepsilon^3\left[\frac{1}{4\pi\varepsilon^2}\iint\limits_{r=\varepsilon} u(M)\,\mathrm{d}S\right] - 3\iiint\limits_{r\leqslant\varepsilon} u(M)\,\mathrm{d}\tau = 0.$$

Further differentiation with respect to ε gives:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{4\pi\varepsilon^2}\iint\limits_{r=\varepsilon} u(M)\,\mathrm{d}S\right] = 0,$$

whence it is clear that the mean value over the sphere (contained in the squares brackets) does not depend on the radius ε of the sphere, i.e.

$$\frac{1}{4\pi\varepsilon^2}\iint\limits_{r=\varepsilon} u(M)\,\mathrm{d}S = C.$$

We see by letting ε tend to zero that the constant C is equal to $u(M_0)$, i.e. $u(M)$ has in fact the mean value property of the lemma, and the theorem is proved.

Thus there can exist no generalized solutions of Laplace's equation different from the ordinary solutions. However, generalized solutions

can exist for Poisson's equation:

$$\Delta u(M) = f(M), \tag{242}$$

defined by the relationship

$$\iiint_D u \Delta \sigma \, d\tau = \iiint_D f\sigma \, d\tau, \tag{243}$$

with the previous conditions regarding σ, if say the function $f(M)$ does not possess good enough properties. Suppose say that $f(M)$ is continuous in the closed domain $\bar{D}$ but has no derivatives. We can evidently continue $f(M)$ throughout the space whilst preserving its continuity. Let $F_n(x, y, z)$ be mean functions for $f(M)$. They tend uniformly to $f(M)$ in the closed domain $\bar{D}$.

We show that the Newtonian potential

$$u(x, y, z) = -\frac{1}{4\pi} \iiint_D \frac{f(\xi, \eta, \zeta)}{r} \, d\xi \, d\eta \, d\zeta \tag{244}$$

$$\left(r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}\right)$$

satisfies (243), i.e. is a generalized solution of Poisson's equation. At the same time, it may not have second order derivatives [II, 200]. We construct the potentials

$$u_n(x, y, z) = -\frac{1}{4\pi} \iiint_D \frac{F_n(\xi, \eta, \zeta)}{r} \, d\xi \, d\eta \, d\zeta, \tag{245}$$

which satisfy inside D the equation $\Delta u_n(x, y, z) = F_n(x, y, z)$. We write down Green's formula:

$$\iiint_D (u_n \Delta \sigma - \sigma \Delta u_n) \, d\tau = 0,$$

where the fact has been taken into account that σ vanishes at all points sufficiently close to the boundary of D. It follows from this that:

$$\iiint_D u_n \Delta \sigma \, d\tau = \iiint_D F_n \sigma \, d\tau.$$

On passing to the limit, (243) is easily seen to be obtained, where u is given by (244). We shall leave an investigation of the general properties of parabolic equations to the chapter on boundary value problems. We merely remark that it can be shown as for Laplace's equation,

that every generalized solution of the homogeneous heat conduction equation has continuous derivatives and is a solution in the ordinary sense of the word (S. L. Sobolev, *Partial Differential Equations of Mathematical Physics*, Pergamon Press, 1964, lect. 22.).

§ 3. Systems of equations

161. Characteristics of systems of equations. We now turn to an investigation of systems of partial differential equations. We have already discussed in [126] the question of the existence and uniqueness of the solution of the Cauchy problem in the analytic case. This question is much more difficult, as compared with dealing with a single equation, in the non-analytic case. Very general results along these lines have been obtained by I. G. Petrovskii in *On the Cauchy problem for systems of partial differential equations* (O probleme Koshi dlya sistem uravnenii s chastnymi proizvodnymi) (*Matematicheskii sbornik*, t. II, vyp. 5, 1937) and in *On the Cauchy problem for systems of linear partial differential equations in the domain of non-analytic functions* (O probleme Koshi dlya sistemy lineinykh uravnenii s chastynimi proizvodnymi v oblasti neanaliticheskikh funktsii) (*Byulletin Moskovskogo universiteta*, 1938). Some of the results of relevance here are given in Petrovskii's *Lectures on Partial Differential Equations* (Lektsiyakh ob uravneniyakh s chastnymi proizvodnymi). The literature of the subject and a survey of the results may be found in this last work.

We shall confine ourselves to a few systems and start with the theory of characteristics and the related subject of discontinuous solutions.

Let us consider the system:

$$\sum_{j=1}^{m} \sum_{k=1}^{n} a_{ij}^{(k)} \frac{\partial u_j}{\partial x_k} + \overline{\Phi}_i (x_k, u_s) = 0 \qquad (i = 1, 2, \ldots, m). \tag{1}$$

Since this is a first order system, the Cauchy data reduce to specifying the initial values of the function $u_s(x_1, \ldots x_n)$ on a given surface of space $(x_1, \ldots, x_n)$. Suppose that the surface supporting the data is the plane $x_1 = 0$, i.e. that we have the special Cauchy data:

$$u_j|_{x_1=0} = \varphi_j (x_2, \ldots, x_n) \quad (j = 1, \ldots, m). \tag{2}$$

These initial data enable us to evaluate on the plane $x_1 = 0$ all the first order derivatives except for $\partial u_j/\partial x_1$. If system (1) is soluble for

$\partial u_j/\partial x_1$ after substituting $x_1 = 0$ and the other initial data (2), we obtain the values of all the first order derivatives at $x_1 = 0$. Otherwise, $x_1 = 0$ will be a *characteristic plane*. In general, a surface

$$\omega_1(x_1, \ldots, x_n) = 0 \tag{3}$$

together with the initial data defined on it, is described as *characteristic* if these initial data, in conjunction with system (1), do not enable us to find all the first order derivatives uniquely on the surface. When the coefficients $a_{ij}^{(k)}$ contain x_s only, it is of no importance for us to know the initial data for the functions u_j on surface (3). In order to find the conditions which must be satisfied by the characteristic surface (3), we introduce, as in [138], new independent variables x_k' instead of x_k in accordance with the formulae

$$x_k' = \omega_k(x_1, \ldots, x_n) \quad (k = 1, \ldots, n), \tag{4}$$

where the $(n - 1)$ functions $\omega_2, \ldots, \omega_n$ are chosen so that the formulae written are soluble with respect to the x_k. On expressing the derivatives with respect to the old variables in terms of the derivatives with respect to the new, we get:

$$\frac{\partial u_j}{\partial x_k} = \sum_{s=1}^{n} \frac{\partial u_j}{\partial x_s'} \frac{\partial \omega_s}{\partial x_k}.$$

We substitute these expressions in system (1), writing down only the terms that contain the derivatives $\partial u_j/\partial x_1'$:

$$\sum_{j=1}^{m} \sum_{k=1}^{n} a_{ij}^{(k)} \frac{\partial \omega_1}{\partial x_k} \frac{\partial u_j}{\partial x_1'} + \ldots = 0 \quad (i = 1, \ldots, m). \tag{1_1}$$

We have the Cauchy data in the special form in the new variables, i.e. these data refer to the plane $x_1' = 0$. This is a characteristic plane if the last system does not give definite values for the derivatives $\partial u_j/\partial x_1'$, i.e. if the determinant from the coefficients of the $\partial u_j/\partial x_1'$ vanishes. On introducing for brevity the notation:

$$\omega_{ij} = \sum_{k=1}^{n} a_{ij}^{(k)} \frac{\partial \omega_1}{\partial x_k}, \tag{5}$$

we obtain the following first order equation, which must be satisfied by every characteristic surface of system (1):

$$|\omega_{ij}| = \begin{vmatrix} \omega_{11}, & \omega_{12}, & \ldots, & \omega_{1m} \\ \omega_{21}, & \omega_{22}, & \ldots, & \omega_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_{m1}, & \omega_{m2}, & \ldots, & \omega_{mm} \end{vmatrix} = 0. \tag{6}$$

This first order equation is of degree m in the derivatives $\partial\omega_1/\partial x_k$. It is entirely analogous to equation (53) of [138].

Equation (6) must be satisfied by virtue of (3). If we require that it be satisfied as an identity, i.e. if we consider it as an ordinary first order equation for the function $\omega_1(x_1, \ldots, x_n)$, we obtain a family $\omega_1(x_1, \ldots, x_n) = C$ of characteristic surfaces of system (1). It can be shown [cf. 101] that every characteristic surface can be included in this family.

If the function $\omega_1(x_1, \ldots, x_n)$ is such that the left-hand side of (6) differs from zero on the surface $\omega_1 = 0$, by carrying out the change of variables (4), we can solve the transformed system (1_1) for $\partial u_j/\partial x_1'$.

If we replace $\partial\omega_1/\partial x_k$ by a_k on the left-hand side of (6), we obtain an equation of degree m for the components of the vector $(a_1, \ldots, a_n)$, which defines a characteristic normal direction at every point. The normal at every point of a characteristic surface has this characteristic direction.

We can similarly consider a system of second order equations:

$$\sum_{j=1}^{m} \sum_{k,l=1}^{n} a_{ij}^{kl} \frac{\partial^2 u_j}{\partial x_k \partial x_l} + \ldots = 0, \tag{7}$$

where we can assume $a_{ij}^{lk} = a_{ij}^{kl}$ as usual. If we have special Cauchy data on the hyperplane $x_1 = 0$:

$$u_j|_{x_1=0} = \varphi_j(x_2, \ldots, x_n); \quad \left.\frac{\partial u_j}{\partial x_1}\right|_{x_1=0} = \psi_j(x_2, \ldots, x_n) \; (j = 1, \ldots, m),$$

all the first order derivatives, and all the second order derivatives except for $\partial^2 u_j/\partial x_1^2$ will be known on this hyperplane. On substituting the initial data in the coefficients of the system and equating to zero the determinant formed from the coefficients of the $\partial^2 u_j/\partial x_1^2$, we obtain the condition for the hyperplane $x_1 = 0$ to be a *characteristic surface*. In the general case, the functions themselves and their first order derivatives are given on surface (3), and we have to find the condition that system (7), in conjunction with the initial data, does not define the second order derivatives uniquely. We introduce new variables x_k' again instead of x_k, in accordance with (4). The expressions for the derivatives with respect to the old variables in terms of the derivatives with respect to the new are:

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_j}{\partial x_1'} \frac{\partial \omega_1}{\partial x_k} + \ldots$$

$$\frac{\partial u_j}{\partial x_k \partial x_l} = \frac{\partial^2 u_j}{\partial x_1'^2} \frac{\partial \omega_1}{\partial x_k} \frac{\partial \omega_1}{\partial x_l} + \ldots$$

On substituting in (7) and writing down only the terms containing $\partial^2 u_j / \partial x'^2_1$, we obtain the system in the new independent variables:

$$\sum_{j=1}^{m} \sum_{k,l=1}^{n} a_{ij}^{kl} \frac{\partial \omega_1}{\partial x_k} \frac{\partial \omega_1}{\partial x} \frac{\partial^2 u_j}{\partial x'^2_1} + \ldots = 0.$$

In the new variables the initial data refer to the plane $x_1' = 0$, and we have to write down the condition that the latter system does not enable us to determine uniquely the derivatives $\partial^2 u_j / \partial x'^2_1$. On introducing notation similar to the previous:

$$\omega_{ij}' = \sum_{k,l=1}^{n} a_{ij}^{kl} \frac{\partial \omega_1}{\partial x_k} \frac{\partial \omega_1}{\partial x_l}, \tag{8}$$

we can write this condition as:

$$|\omega_{ij}'| = \begin{vmatrix} \omega_{11}', & \omega_{12}', \ldots, \omega_{1m}' \\ \omega_{21}', & \omega_{22}', \ldots, \omega_{2m}' \\ \cdot \; \cdot \; \cdot & \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ \omega_{m1}' & \omega_{m2}', \ldots, \omega_{mm}' \end{vmatrix} = 0. \tag{9}$$

The left-hand side of this first order equation is a homogeneous polynomial of degree $2m$ in the derivatives $\partial \omega_1 / \partial x_k$.

We return to first order systems. If we replace $\partial \omega_1 / \partial x_k$ by a_k on the left-hand side of (6), we get the equation:

$$\bar{\Phi}(a_1, \ldots, a_n) = 0, \tag{10}$$

where $\bar{\Phi}$ is a homogeneous polynomial of degree m in the arguments $a_1, \ldots, a_n$ with coefficients depending on $(x_1, \ldots, x_m)$. If the left-hand side of (10) vanishes only for $a_1 = \ldots = a_n = 0$ in some domain D of space $(x_1, \ldots, x_n)$, system (1) is said to be of the *elliptic type* in D. The elliptic type is defined as regards system (7) in the same way. The term "hyperbolic type" is applied to systems in a rather different sense. We shall return to this question for the case of two independent variables. If, by a suitable linear transformation of the variables d_s, the homogeneous polynomial $\bar{\Phi}(a_1, \ldots, a_n)$ can be reduced at a point $(x_1, \ldots, x_n)$ or in a domain D to a smaller number of variables, system (1) is said to be *parabolically degenerate* at that point or in that domain.

If the coefficients $a_{ij}^{(k)}$ of system (1) contain the functions u_j (the system is *quasilinear*), we can form equation (6) by substituting in these coefficients any functions u_j given on the surface $\omega_1 = 0$, and decide the question as to whether $\omega_1 = 0$ is a characteristic surface.

A similar remark applies for system (7) if its coefficients a_{ij}^{kl} contain the functions u_j and their first order partial derivatives [cf. 128]. We remark that system (7) can be reduced to a system of first order equations if we introduce the mn new functions:

$$\frac{\partial u_j}{\partial x_k} = \omega_{jk} \quad \begin{pmatrix} j = 1, \dots, m \\ k = 1, \dots, n \end{pmatrix}. \tag{10_1}$$

On carrying out substitutions (10_1) in equation (7), we obtain m first order equations in the $(m + mn)$ functions u_j and w_{jk}. The further mn equations (10_1) are added to these equations.

162. Kinematic compatibility conditions. For what follows, we need to prove a proposition on the differentiation of functions along a surface. We shall prove this lemma for the case of three independent variables in order to get a clearer geometrical picture.

Let the function $f(x_1, x_2, x_3)$ be continuous up to a surface S:

$$\psi(x_1, x_2, x_3) = 0$$

from one side of it, and further, let its first order partial derivatives also be continuous from this side of S and have definite limits f_{x_i} on S. If l is a curve given from the same side of S: $x_i = x_i(t)$ $(i = 1, 2, 3)$, where the $x_i(t)$ have continuous derivatives with respect to t, then f is a function of t along l, and we have:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{3} f_{x_k} x_k'(t). \tag{11}$$

LEMMA. *Formula* (11) *holds if l lies on S.*

We can assume that the curve l is sufficiently small. Let N_1 and N_2 be its ends, and N a variable point on l. We draw through N a straight line parallel to the normal n_1 to the surface at the point N_1, the normal being directed towards the side where f is defined; then we mark off segments NN' of the same length δ on each of these straight lines. Let the ends N' of these segments form a curve l', which does not intersect itself and lies in the domain in which f is defined. Points of this curve have the coordinates: $\xi_i = x_i(t) + \delta \cos(n_i, x_i)$. We can apply (11) along l':

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{l'} = \sum_{k=1}^{3} f_{x_k}(\xi_1, \xi_2, \xi_3)\, x_k'(t).$$

We integrate both sides with respect to t between the limits from $t = t_1$ corresponding to the point N_1 to a variable t:

$$f(t)\,|_{l'} - f(t_1)\,|_{l'} = \int_{t_1}^{t} \sum_{k=1}^{3} f_{x_k}(\xi_1, \xi_2, \xi_3)\, x'_k(t)\, \mathrm{d}t,$$

where $f(t_1)$ and $f(t)$ on the left-hand side are the values of f on l' at the points corresponding to the values of t mentioned. By hypothesis, f and f_{x_k} are continuous as far as S, so that the integrand on the right-hand side is a uniformly continuous function of the parameter δ. On passing to the limit in the last formula as $\delta \to 0$, we obtain

$$f(t) - f(t_1) = \int_{t_1}^{t} \sum_{k=1}^{3} f_{x_k}[x_1(t), x_2(t), x_3(t)]\, x'_k(t)\, \mathrm{d}t,$$

where the left-hand side contains the values of f on l. On differentiating both sides with respect to t, we obtain (11). This lemma will be useful in the next chapter as well as in the present section.

We turn to the case of any number of variables and now suppose that the function $f(x_1, \ldots, x_n)$ is continuous on passing through surface S:

$$\psi(x_1, \ldots, x_n) = 0, \tag{12}$$

whilst its first order partial derivatives have definite limits on each side of the surface, these limits being different for the different sides, i.e. more briefly, the first order derivatives of f have discontinuities of the first kind on surface (12). We shall speak of the two sides as the positive and the negative side. We use a $+$ sign to denote a limit obtained on the positive side, and a $-$ sign for a limit on the negative side. For instance, the condition that f is continuous on passing through S can be written as $f^+ = f^-$. We introduce a notation for the jump in the first order derivatives:

$$[f_{x_k}] = f^+_{x_k} - f^-_{x_k}.$$

By hypothesis, f^+ and f^- coincide along any curve l lying on surface (12). Hence we have, by using the lemma:

$$\sum_{k=1}^{n} f^+_{x_k} \mathrm{d}x_k = \sum_{k=1}^{n} f^-_{x_k} \mathrm{d}x_k \quad (\text{on } S). \tag{13}$$

The variables x_k cannot be regarded as independent on surface S. If say the equation of the surface is given explicitly, one of the coordinates will be a function of the rest, whilst these latter can now be regarded as independent variables.

We can rewrite the previous expression as:

$$\sum_{k=1}^{n} [f_{x_k}]\, \mathrm{d}x_k = 0 .$$

We have, in addition:

$$\sum_{k=1}^{n} \psi_{x_k} \mathrm{d}x_k = 0.$$

We multiply the last equation by an as yet undetermined factor h and subtract from the previous equation:

$$\sum_{k=1}^{n} \{[f_{x_k}] - h\psi_{x_k}\}\, \mathrm{d}x_k = 0.$$

We now define the factor h so that the coefficient of the differential of the dependent variable vanishes. The coefficients of the differentials of the independent variables must obviously be zero [I, 167], and we thus arrive at the following n equations:

$$[f_{x_k}] = h\psi_{x_k}, \tag{14}$$

i.e. *the jumps of the first order derivatives must be proportional to the partial derivatives of the left-hand side of* (12) *with respect to the corresponding variables. These conditions are generally known as the kinematic compatibility conditions.*

We now take the case when f itself and its first order derivatives remain continuous on passing through surface (12), whilst the second order derivatives have discontinuities. Our previous discussion is now applicable for each of the functions f_{x_k}. Each of these functions will have its coefficient of proportionality h_k in the kinematic compatibility conditions, and the jump of the derivative of f_{x_k} with respect to each variable x_l must be proportional to ψ_{x_l}, i.e. we have the following equations for the jumps of the second order derivatives:

$$[f_{x_k x_l}] = f^{+}_{x_k x_l} - f^{-}_{x_k x_l} = h_k \psi_{x_l}.$$

On taking into account the independence of the result of differentiation on the order of differentiation both on the positive and on the negative side of the surface, we can write $h_k \psi_{x_l} = h_l \psi_{x_k}$, i.e. $h_k/\psi_{x_k} = h_l/\psi_{x_l}$. In other words the ratio h_k: ψ_{x_k} must not depend on the subscript k. On putting $h_k : \psi_{x_k} = h$, we finally transform the last expression to:

$$[f_{x_k x_l}] = h\psi_{x_k} \psi_{x_l}. \tag{15}$$

These formulae give the kinematic compatibility conditions for the case of a second order discontinuity, i.e. a discontinuity of the second order derivatives.

163. Dynamic compatibility conditions. We return to the system (1) of first order equations and suppose that (3) is a characteristic surface for the system; the solution u is assumed to have a weak discontinuity on this surface, i.e. u itself is continuous, and only the first order derivatives can be discontinuous. Let u^+ be a continuous solution from the positive side of the surface and u^- a continuous solution from the negative side with which u coincides. We can write system (1) for u^+ and u^-. We consider the difference of these equations on surface (3) itself. The terms Φ_i will be continuous on passing through the surface and cancel on subtraction. We therefore arrive at the following m equations, which must be satisfied by the jumps of the first order derivatives:

$$\sum_{j=1}^{m} \sum_{k=1}^{n} a_{ij}^{(k)} \left[\frac{\partial u_j}{\partial x_k}\right] = 0. \tag{16}$$

When deducing these conditions we have essentially made use of system (1) itself, which usually describes some physical process; the conditions obtained for the jumps are called the *dynamic compatibility conditions*. Each of the functions u_j has its coefficient of proportionality h_j in the kinematic compatibility conditions (14):

$$\left[\frac{\partial u_j}{\partial x_k}\right] = h_j \frac{\partial \omega_1}{\partial x_k} \quad (j = 1, 2, \ldots, m). \tag{17}$$

On substituting these expressions in conditions (16) and using the notation (5), we obtain a system of m homogeneous first degree equations for the coefficients h_j:

$$\sum_{j=1}^{m} \omega_{ij} h_j = 0 \quad (i = 1, 2, \ldots, m). \tag{18}$$

It follows directly from the equation of the characteristic surface (6) that the determinant of this system vanishes, so that we can obtain a non-zero solution of the system. In the general case, when the rank of the matrix of the coefficients of system (18) is $(m - 1)$, the general solution of the system is determined up to an arbitrary constant factor, which is of no importance for obtaining a qualitative picture of the discontinuity.

We now turn to a consideration of the system (7) of second order equations. Here, a solution having a weak discontinuity will be one in which the function itself and its first order derivatives are continuous. Precisely as above, we obtain dynamic compatibility con-

ditions for the jumps of the second order derivatives:

$$\sum_{j=1}^{m} \sum_{k,l=1}^{n} a_{ij}^{kl} \left[\frac{\partial^2 u_j}{\partial x_k \, \partial x_l} \right] = 0 . \tag{19}$$

Each function u_j will have its own coefficient of proportionality h_j in the kinematic compatibility conditions:

$$\left[\frac{\partial^2 u_j}{\partial x_k \, \partial x_l} \right] = h_j \frac{\partial \omega_1}{\partial x_k} \frac{\partial \omega_1}{\partial x_l} . \tag{20}$$

On substituting these expressions in condition (19) and using notation (8), we again obtain a system of homogeneous equations for the factors h_j, the determinant of which vanishes, by (9):

$$\sum_{j=1}^{m} \omega'_{ij} h_j = 0. \tag{21}$$

164. The equations of hydrodynamics. Let us apply the theory of characteristics to the equations of hydrodynamics. Let (u_1, u_2, u_3) be the components of the velocity vector, p the pressure, ϱ the density and f_1, f_2, f_3 the components of the external force per unit mass. The independent variables are time t and the spatial coordinates x_1, x_2, x_3. We have the three Euler equations:

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} u_k + \frac{1}{\varrho} \frac{\partial p}{\partial x_i} = f_i \quad (i = 1, 2, 3)$$

and the equation of continuity [II, 114, 115]:

$$\frac{\partial \varrho}{\partial t} + \sum_{k=1}^{3} \frac{\partial \varrho}{\partial x_k} u_k + \varrho \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} = 0.$$

We shall assume that the fluid is compressible and that the equation of state is determined by the dependence of the pressure on the density $p = p(\varrho)$, where $p(\varrho)$ is a given function. We finally have four first order equations for the functions u_1, u_2, u_3, ϱ of the independent variables x_1, x_2, x_3, t:

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} u_k + \frac{1}{\varrho} \frac{dp}{d\varrho} \frac{\partial \varrho}{\partial x_i} = f_i \quad (i = 1, 2, 3)$$

$$\varrho \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} + \frac{\partial \varrho}{\partial t} + \sum_{k=1}^{3} \frac{\partial \varrho}{\partial x_k} u_k = 0.$$

The quantities ω_{ij}, given by (5), become in the present case:

$$\omega_{12} = \omega_{21} = \omega_{13} = \omega_{31} = \omega_{23} = \omega_{32} = 0;$$

$$\omega_{ii} = \frac{d\omega_1}{dt} = \frac{\partial \omega_1}{\partial t} + \sum_{k=1}^{3} \frac{\partial \omega_1}{\partial x_k} u_k \quad (i = 1, 2, 3, 4)$$

$$\omega_{i4} = \frac{1}{\varrho} \frac{dp}{d\varrho} \frac{\partial \omega_1}{\partial x_i}; \quad \omega_{4i} = \varrho \frac{\partial \omega_1}{\partial x_i} \quad (i \neq 4),$$

where, as previously, ω_1 is the left-hand side of the equation of the characteristic surface

$$\omega_1 (x_1, x_2, x_3, t) = 0. \tag{22}$$

We write g^2 as above for the sum

$$g^2 = \sum_{k=1}^{3} \left(\frac{\partial \omega_1}{\partial x_k} \right)^2 .$$

The first order equation (6), which must be satisfied by characteristic surface (22), has the form here:

$$\begin{vmatrix} \frac{d\omega_1}{dt}, & 0, & 0, & \frac{1}{\varrho} \frac{dp}{d\varrho} \frac{\partial \omega_1}{\partial x_1} \\ 0, & \frac{d\omega_1}{dt}, & 0, & \frac{1}{\varrho} \frac{dp}{d\varrho} \frac{\partial \omega_1}{\partial x_2} \\ 0, & 0, & \frac{d\omega_1}{dt}, & \frac{1}{\varrho} \frac{dp}{d\varrho} \frac{\partial \omega_1}{\partial x_3} \\ \varrho \frac{\partial \omega_1}{\partial x_1}, & \varrho \frac{\partial \omega_1}{\partial x_2}, & \varrho \frac{\partial \omega_1}{\partial x_3}, & \frac{d\omega_1}{dt} \end{vmatrix} = 0.$$

$$\left(\frac{d\omega_1}{dt} = \frac{\partial \omega_1}{\partial t} + \frac{\partial \omega_1}{\partial x_1} u_1 + \frac{\partial \omega_1}{\partial x_2} u_2 + \frac{\partial \omega_1}{\partial x_3} u_3 \right).$$

We obtain on expanding the determinant:

$$\left(\frac{d\omega_1}{dt} \right)^2 \left[\left(\frac{d\omega_1}{dt} \right)^2 - g^2 \frac{dp}{d\varrho} \right] = 0. \tag{23}$$

The velocity P of displacement of surface (22) in the direction normal to the surface is given, as we know, by (75) of [141]. At each given instant, surface (22) will pass through certain fluid particles. Let u_n be the velocity component of a fluid particle lying on the surface along the normal at the point in question. Since the direction-cosines of the normal (towards the side where $\omega_1 > 0$) are $\partial \omega_1 / \partial x_k : g$, we have:

$$u_n = \frac{1}{g} \sum_{k=1}^{3} u_k \frac{\partial \omega_1}{\partial x_k} .$$

The difference $P - u_n$, giving the velocity of the surface relative to the particles, is usually called the *velocity of wave propagation*. We have the following expres-

sion for this velocity:

$$V = P - u_n = -\frac{1}{g}\frac{\partial\omega_1}{\partial t} - \frac{1}{g}\sum_{k=1}^{3} u_k \frac{\partial\omega_1}{\partial x_k},$$

or

$$V = -\frac{1}{g}\frac{d\omega_1}{dt}. \tag{24}$$

The differential equation of characteristic surfaces (23) is equivalent to two equations:

$$V^2 = 0; \quad V^2 = \frac{dp}{d\varrho}. \tag{25}$$

The first equation corresponds to the case of a stationary discontinuity and we shall only consider below the second of the equations. The velocity V defined by (25) is the velocity of sound:

$$V = \sqrt{\frac{dp}{d\varrho}}. \tag{26}$$

Let us now establish the nature of the discontinuity by using the kinematic and dynamic compatibility conditions. Let h_k denote the discontinuity coefficients appearing in (17) for functions u_k, and r the corresponding coefficient for the function ϱ. Equation (18) becomes in the present case:

$$\frac{d\omega_1}{dt} h_k + \frac{1}{\varrho}\frac{dp}{d\varrho}\frac{\partial\omega_1}{\partial x_k} r = 0 \qquad (k = 1, 2, 3)$$

or, on taking (24) and (25) into account:

$$-gh_k + \frac{1}{\varrho} V \frac{\partial\omega_1}{\partial x_k} r = 0,$$

i.e.

$$h_k = \frac{rV}{\varrho}\cos\alpha_k, \tag{27}$$

where $\cos\alpha_k$ are the direction-cosines of the normal to the surface of discontinuity. We shall regard (h_1, h_2, h_3) as the components of a vector h (the discontinuity vector for the velocity derivatives). The previous formula can be written in the vector form:

$$\mathbf{h} = \frac{rV}{\varrho}\mathbf{n},$$

where $\mathbf{n}$ is the unit normal to the discontinuity surface. It is thus seen that *the discontinuity vector for the velocity derivatives is directed along the normal to the surface of discontinuity* (a longitudinal wave).

The components w_i of the acceleration vector are given by

$$w_i = \frac{\partial u_i}{\partial t} + \sum_{k=1}^{3}\frac{\partial u_i}{\partial x_k} u_k \quad (i = 1, 2, 3)$$

and have discontinuities on passing through the surface. Suppose we have rest on one side of the surface. Since the velocity is continuous, its boundary

values vanish on both sides of the surface; whilst the velocity derivatives will have values equal to the jump on the surface, since they vanish in front of the surface, where we have rest. The same can be said of the components of the acceleration vector. By (27) and (24), the jumps of these components are given by

$$[w_i] = h_i \frac{\partial \omega_1}{\partial t} + \sum_{k=1}^{3} h_i u_k \frac{\partial \omega_1}{\partial x_k} + h_i \frac{d\omega_1}{dt} = -\frac{rgV^2}{\varrho} \cos \alpha,$$

or in vector form:

$$[\mathbf{w}] = -\frac{rgV^2}{\varrho} \mathbf{n}.$$

With the above-mentioned condition, this formula will give the acceleration vector on the discontinuity surface.

We now consider the so-called stationary case, when the functions u_k and ϱ do not depend on t. Assuming that ω_1 is also independent of t, we have $P = 0$ and $V = -u_n$. Suppose that, in a certain domain, the fluid velocity is less than the velocity of sound (26). All the more, in this case, $| u_n | < \sqrt{dp/d\varrho}$, and the equation $V = -u_n$ is impossible. It is thus seen that we cannot have propagation of discontinuities in the stationary case with infrasonic velocities.

165. Equations of the theory of elasticity. As an example of an application of the theory of characteristics to systems of second order equations, let us take the equations of the theory of elasticity in the elementary case of a homogeneous isotropic medium. Let (u_1, u_2, u_3) be the components of the displacement vector and λ and μ the usual elasticity constants for the medium. The fundamental equations of the theory of elasticity comprise the following system of three second order equations for the functions (u_1, u_2, u_3) of the independent variables (x_1, x_2, x_3, t):

$$(\lambda + \mu) \frac{\partial}{\partial x_i} \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} + \mu \Delta u_i - \varrho \frac{\partial^2 u_i}{\partial t^2} + \ldots = 0,$$

We have in the present case:

$$\omega'_{ij} = (\lambda + \mu) \frac{\partial \omega_1}{\partial x_i} \frac{\partial \omega_1}{\partial x_j} + \delta_{ij} \left[\mu \sum_{k=1}^{3} \left(\frac{\partial \omega_1}{\partial x_k}\right)^2 - \varrho \left(\frac{\partial \omega_1}{\partial t}\right)^2\right] \begin{pmatrix} (i, j = 1, 2, 3) \\ \delta_{ij} = \begin{matrix} 0, i \neq j \\ 1, i = j \end{matrix} \end{pmatrix} \quad (28)$$

Equation (9) here becomes, after expanding the determinant:

$$\left[(\lambda + 2\mu) g^2 - \varrho \left(\frac{\partial \omega_1}{\partial t}\right)^2\right] \left[\mu g^2 - \varrho \left(\frac{\partial \omega_1}{\partial t}\right)^2\right]^2 = 0. \quad (29)$$

By (75) of [141], this equation gives us the following two possible displacement velocities for the discontinuity surface:

$$P_1 = \sqrt{\frac{\lambda + 2\mu}{\varrho}}; \quad P_2 = \sqrt{\frac{\mu}{\varrho}}.$$

The deformations are here assumed small and there is no sense in speaking separately of a propagation velocity, i.e. of a displacement velocity relative to the particles of the material medium.

Let us now consider the nature of the discontinuities. We introduce the coefficients h_j of discontinuity of the second order derivatives of the functions u_j:

$$\left[\frac{\partial^2 u_j}{\partial x_i\, \partial x_k}\right] = h_j \frac{\partial \omega_1\, \partial \omega_1}{\partial x_i\, \partial x_k}\,. \tag{30}$$

By (28), equations (21) become in the present case:

$$\left[\mu g^2 - \varrho\left(\frac{\partial \omega_1}{\partial t}\right)^2\right] h_i + (\lambda + \mu)\frac{\partial \omega_1}{\partial x_i} \sum_{j=1}^{3} \frac{\partial \omega_1}{\partial x_j} h_j = 0 \quad (i = 1, 2, 3).$$

On observing the fact that

$$\frac{\partial \omega_1}{\partial x_k} = g \cos(n, x_k) \qquad (k = 1, 2, 3).$$

where n is the direction of the normal to surface (3), we can rewrite the previous equations as

$$\left[\mu g^2 - \varrho\left(\frac{\partial \omega_1}{\partial t}\right)^2\right] h_i + (\lambda + \mu)\, g^2 \cos(n, x_i) \sum_{j=1}^{3} \cos(n, x_j)\, h_j = 0.$$

We introduce the vector $\mathbf{h}$ with components (h_1, h_2, h_3). The previous equations can be written in the form:

$$\left[\mu g^2 - \varrho\left(\frac{\partial \omega_1}{\partial t}\right)^2\right] h_i + (\lambda + \mu)\, g^2 \cos(n, x_i)\, h_n = 0,$$

where h_n is the projection of the vector $\mathbf{h}$ on the normal n to surface (3), or, in vector form:

$$\left[\mu g^2 - \varrho\left(\frac{\partial \omega_1}{\partial t}\right)^2\right] \mathbf{h} + (\lambda + \mu)\, g^2 h_n\, \mathbf{n} = 0, \tag{31}$$

where $\mathbf{n}$ is the unit normal to surface (3). If we consider the displacement velocity P_2, the coefficient of $\mathbf{h}$ vanishes, and we must have $h_n = 0$, i.e. the vector $\mathbf{h}$ must lie in the tangent plane to surface (3) (a transverse wave). If we consider the velocity P_1, it follows at once from (31) that $\mathbf{h}$ only differs by a numerical factor from $\mathbf{n}$, i.e. $\mathbf{h}$ must be directed along the normal to surface (3) (a longitudinal wave). We remark further that the factor giving the transverse wave velocity is squared in (29). This point will be explained in the next section, where we consider the equations of elasticity for an anisotropic medium.

Let us explain the mechanical significance of the vector $\mathbf{h}$. Suppose that we have rest on one side of the weak discontinuity surface S: $\omega_1(x_1, x_2, x_3, t) = 0$, i.e. the u_j $(j = 1, 2, 3)$ vanish. At points of the surface S the functions u_j and their first order derivatives all vanish. On the side where there is motion, the values of the second derivatives of the u_j will be given on S by (30), since these derivatives vanish identically on the other side of the surface, i.e.

$$\left.\frac{\partial^2 u_j}{\partial x_i\, \partial x_k}\right|_S = h_j \frac{\partial \omega_1}{\partial x_i} \left.\frac{\partial \omega_1}{\partial x_k}\right|_S \qquad (i, k = 0, 1, 2, 3),$$

where we take $x_0 = t$. Let some point M of the surface S be taken as origin in space (x_0, x_1, x_2, x_3). We expand the u_j in a Maclaurin series in the neighbourhood of the point M, the expansions being taken as far as the second degree terms. If we use the previous formulae and the fact that u_j and its first order partial derivatives vanish at M, we obtain the approximate equation:

$$u_j \sim \frac{h_j}{2} \sum_{i,k=0}^{3} \left(\frac{\partial \omega_1}{\partial x_i}\right)_0 \left(\frac{\partial \omega_1}{\partial x_k}\right)_0 x_i x_k,$$

where the zero subscript indicates that the values of the derivatives must be taken at the point M.

In view of the fact that the function ω_1 vanishes at the point M, we obtain the following Maclaurin expansion, carried as far as the first degree terms:

$$\omega_1 \sim \sum_{i=1} \left(\frac{\partial \omega_1}{\partial x_i}\right)_0 x_i$$

and the previous expression can be rewritten as

$$\mathbf{u} \sim \frac{\mathbf{h}}{2} \omega_1^2 (x_0, x_1, x_2, x_3).$$

This approximation for the displacement vector $\mathbf{u}$ will hold close to the discontinuity surface on the side where there is motion.

166. Anisotropic elastic media. We now bring in the components of the deformation tensor, with some modification to the notation of [94]:

$$\varepsilon_i = \frac{\partial u_i}{\partial x_i}; \quad \gamma_1 = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}; \quad \gamma_2 = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1};$$

$$\gamma_3 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \qquad (i = 1, 2, 3).$$

In the case of an anisotropic medium with three mutually perpendicular planes of symmetry, the work done by the deformation forces per unit volume is given in terms of the components of the deformation tensor by the following homogeneous second degree polynomial:

$$A = \frac{1}{2} (a\varepsilon_1^2 + b\varepsilon_2^2 + c\varepsilon_3^2 + 2a' \varepsilon_2 \varepsilon_3 + 2b' \varepsilon_3 \varepsilon_1 + 2c' \varepsilon_1 \varepsilon_2 + a'' \gamma_1^2 + b'' \gamma_2^2 + c'' \gamma_3^2).$$

where the coefficients $a, b, \ldots, c''$ are functions of (x_1, x_2, x_3, t), or constants in the case of a homogeneous medium. The equations of [94] can be written when inertia forces are present as:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial A}{\partial \varepsilon_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial A}{\partial \gamma_3}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial A}{\partial \gamma_2}\right) - \varrho \frac{\partial^2 u_1}{\partial t^2} + X_1 = 0$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial A}{\partial \gamma_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial A}{\partial \varepsilon_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial A}{\partial \gamma_1}\right) - \varrho \frac{\partial^2 u_2}{\partial t^2} + X_2 = 0$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial A}{\partial \gamma_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial A}{\partial \gamma_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial A}{\partial \varepsilon_3}\right) - \varrho \frac{\partial^2 u_3}{\partial t^2} + X_3 = 0.$$

On substituting the expression for A, we obtain the following equations:

$$a\frac{\partial^2 u_1}{\partial x_1^2} + c''\frac{\partial^2 u_1}{\partial x_2^2} \quad b''\frac{\partial^2 u_1}{\partial x_3^2} +$$

$$+ (c'+c'')\frac{\partial^2 u_2}{\partial x_1\,\partial x_2} + (b'+b'')\frac{\partial^2 u_3}{\partial x_1\,\partial x_3} - \varrho\frac{\partial^2 u_1}{\partial t^2} + \ldots = 0$$

$$c''\frac{\partial^2 u_2}{\partial x_1^2} + b\frac{\partial^2 u_2}{\partial x_2^2} + a''\frac{\partial^2 u_2}{\partial x_3^2} +$$

$$+ (c'+c'')\frac{\partial^2 u_1}{\partial x_1\,\partial x_2} + (a'+a'')\frac{\partial^2 u_3}{\partial x_2\,\partial x_3} - \varrho\frac{\partial^2 u_2}{\partial t^2} + \ldots = 0$$

$$b''\frac{\partial^2 u_3}{\partial x_1^2} + a''\frac{\partial^2 u_3}{\partial x_2^2} + c\frac{\partial^2 u_3}{\partial x_3^2} +$$

$$+ (b'+b'')\frac{\partial^2 u_1}{\partial x_1\,\partial x_3} + (a'+a'')\frac{\partial^2 u_2}{\partial x_2\,\partial x_3} - \varrho\frac{\partial^2 u_3}{\partial t^2} + \ldots = 0$$

If we simplify the writing by using the notation

$$p_0 = \frac{\partial\omega_1}{\partial t}; \qquad p_i = \frac{\partial\omega_1}{\partial x_i} \quad (i = 1, 2, 3), \tag{32}$$

the coefficients ω'_{ij} can be written as:

$$\omega'_{11} = ap_1^2 + c''p_2^2 + b''p_3^2 - \varrho p_0^2; \quad \omega'_{12} = (c'+c'')\,p_1 p_2; \quad \omega'_{13} = (b'+b'')\,p_1 p_3$$
$$\omega'_{21} = (c'+c'')\,p_1 p_2; \quad \omega'_{22} = c''p_1^2 + bp_2^2 + a''p_3^2 - \varrho p_0^2; \quad \omega'_{23} = (a'+a'')\,p_2 p_3$$
$$\omega'_{31} = (b'+b'')\,p_3 p_1; \quad \omega'_{32} = (a'+a'')\,p_2 p_3; \quad \omega'_{33} = b''p_1^2 + a''p_2^2 + cp_3^2 - \varrho p_0^2.$$

The first order equation (9), defining the characteristic surface, is easily seen to be the same as the fundamental equation with respect to $\lambda = \varrho p_0^2$ which enables us to reduce the ellipsoid

$$(ap_1^2 + c''p_2^2 + b''p_3^2)\,\xi_1^2 + (c''p_1^2 + bp_2^2 + a''p_3^2)\,\xi_2^2 + (b''p_1^2 + a''p_2^2 + cp_3^2)\,\xi_3^2 +$$
$$+ 2(a'+a'')\,p_2 p_3\,\xi_2\xi_3 + 2(b'+b'')\,p_3 p_1\,\xi_3\xi_1 + 2(c'+c'')\,p_1 p_2\,\xi_1\xi_2 = 1 \tag{33}$$

to the axes of symmetry [III, 32, 33]. We remark that the left-hand side of the last equation can be obtained from the expression for $2A$ if we put $\varepsilon_k = p_k\xi_k$; $\gamma_1 = p_2\xi_3 + p_3\xi_2$; $\gamma_2 = p_3\xi_1 + p_1\xi_3$; $\gamma_3 = p_1\xi_2 + p_2\xi_1$, so that it becomes a positive definite quadratic form in the ξ_s (since $A > 0$), i.e. (33) in fact corresponds to an ellipsoid. On solving the above-mentioned equation for λ, we obtain at each point of the body three positive roots for p_0^2, where p_0^2 is a homogeneous function of the second degree in p_1, p_2, p_3. If both sides of (33) are divided through by g^2, the p_k become $\cos\alpha_k$, where $\cos\alpha_k$ are the direction-cosines of the normal to the wave surface, and a root obtained for p_0^2 becomes P^2. Therefore, given any fixed direction, we obtain three possible wave displacement velocities at every point.

The discontinuity vector components (h_1, h_2, h_3) will be obtained from the homogeneous system from which the directions of the axes of symmetry of ellipsoid (33) are defined. Thus, given a definite direction, we have at every point three mutually perpendicular discontinuity vectors, corresponding to the

three displacement velocities. The necessary and sufficient condition for both longitudinal and transverse waves to be present is that one of the axes of the ellipsoid be directed along the normal to the corresponding wave. If this is the case, we have one longitudinal and two transverse waves, on the assumption that our cubic equation has three distinct roots for the fixed direction. In the case of a homogeneous isotropic medium, one root is double, as we saw. The direction-cosines of the normal to the wave are proportional to P_1, P_2, P_3, so that the above condition is equivalent to the fact that, for some root $\lambda = \varrho p_0^2$, the quantities (h_1, h_2, h_3) must be proportional to (p_1, p_2, p_3) for any choice of p_k, i.e. for any choice of direction. On replacing the h_k by the proportional quantities p_k in the homogeneous system for the h_k, we get:

$$\left.\begin{aligned}
&(ap_1^2 + c'' p_2^2 + b'' p_3^2 - \varrho p_0^2)\, p_1 + (c' + c'')\, p_1 p_2^2 + (b' + b'')\, p_1 p_3^2 = 0\\
&(c' + c'')\, p_1^2 p_2 + (c'' p_1^2 + bp_2^2 + a'' p_3^2 - \varrho p_0^2)\, p_2 + (a' + a'')\, p_2 p_3^2 = 0\\
&(b' + b'')\, p_1^2 p_3 + (a' + a'')\, p_2^2 p_3 + (b'' p_1^2 + a'' p_2^2 + cp_3^2 - \varrho p_0^2)\, p_3 = 0.
\end{aligned}\right\} \quad (34)$$

If we take into account that the same value must be obtained for ϱp_0^2 from the three equations (34) for any choice of p_1, p_2, p_3, we arrive at the following conditions for the coefficients of the elastic potential A:

$$a = b = c = a' + 2a'' = b' + 2b'' = c' + 2c'', \quad (35)$$

and the three equations give us: $\varrho p_0^2 = ag^2$, i.e. we have for the velocity of the longitudinal wave:

$$P = \sqrt{\frac{a}{\varrho}}\,.$$

The two remaining roots, corresponding to the transverse waves, are in general different and depend on the choice of wave direction, i.e. on the choice of p_1. Equations (35) give us five conditions for the nine coefficients appearing in the expression for the elastic potential A.

167. Electromagnetic waves. We first consider the two Maxwell equations for an isotropic medium:

$$c \operatorname{curl} \mathbf{H} = \lambda \mathbf{E} + \varepsilon \mathbf{E}_t, \qquad c \operatorname{curl} \mathbf{E} = -\mu \mathbf{H}_t, \quad (36)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electrical and magnetic field intensities, c the velocity of light, λ the conductivity of the medium, ε the dielectric constant and μ the magnetic permeability. The vectors $\mathbf{E}$ and $\mathbf{H}$ are functions of the independent variables (x_1, x_2, x_3, t). On writing (e_1, e_2, e_3) and (h_1, h_2, h_3) for the components of these vectors, we can rewrite (36) as:

$$\left.\begin{aligned}
&\frac{\varepsilon}{c}\frac{\partial e_1}{\partial t} + \frac{\partial h_2}{\partial x_3} - \frac{\partial h_3}{\partial x_2} + \ldots = 0 \qquad \frac{\mu}{c}\frac{\partial h_1}{\partial t} + \frac{\partial e_3}{\partial x_2} - \frac{\partial e_2}{\partial x_3} = 0\\
&\frac{\varepsilon}{c}\frac{\partial e_2}{\partial t} + \frac{\partial h_3}{\partial x_1} - \frac{\partial h_1}{\partial x_3} + \ldots = 0 \qquad \frac{\mu}{c}\frac{\partial h_2}{\partial t} + \frac{\partial e_1}{\partial x_3} - \frac{\partial e_3}{\partial x_1} = 0\\
&\frac{\varepsilon}{c}\frac{\partial e_3}{\partial t} + \frac{\partial h_1}{\partial x_2} - \frac{\partial h_2}{\partial x_1} + \ldots = 0 \qquad \frac{\mu}{c}\frac{\partial h_3}{\partial t} + \frac{\partial e_2}{\partial x_1} - \frac{\partial e_1}{\partial x_2} = 0,
\end{aligned}\right\} \quad (37)$$

where the absent terms do not contain derivatives of the functions e_k and h_k. We have here a system of six first order equations with six functions. We enumerate these functions in the following order:

$$u_1 = e_1;\quad u_2 = e_2;\quad u_3 = e_3;\quad u_4 = h_1;\quad u_5 = h_2;\quad u_6 = h_3.$$

On forming expression (5) and writing equation (6), we obtain the following first order equation for the characteristic surfaces:

$$\begin{vmatrix} \frac{\varepsilon}{c} p_0, & 0, & 0, & 0, & p_3 & -p_2 \\ 0, & \frac{\varepsilon}{c} p_0, & 0, & -p_3, & 0, & p_1 \\ 0, & 0, & \frac{\varepsilon}{c} p_0, & p_2, & -p_1, & 0 \\ 0, & -p_3, & p_2, & \frac{\mu}{c} p_0, & 0, & 0 \\ p_3, & 0, & -p_1. & 0, & \frac{\mu}{c} p_0, & 0 \\ -p_2, & p_1 & 0. & 0, & 0, & \frac{\mu}{c} p_0 \end{vmatrix} = 0. \tag{38}$$

We multiply the elements of the first three columns of this determinant by $\mu p_0/c$. After this, we add to the elements of the first column the elements of the fifth multiplied by $(-p_3)$, and of the sixth multiplied by p_2; to the elements of the second column we add the elements of the fourth multiplied by p_3, and of the sixth multiplied by $(-p_1)$; to the elements of the third column we add the elements of the fourth multiplied by $(-p_2)$, and the elements of the fifth multiplied by p_1. On then expanding by elements of the sixth, fifth and fourth rows, we arrive at the equation:

$$\begin{vmatrix} q + p_1^2, & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & q + p_2^2, & p_2 q_3 \\ p_3 p_1 & p_3 p_2, & q + q_3^2 \end{vmatrix} = 0, \tag{39}$$

where

$$q = \frac{\varepsilon\mu}{c} p_0^2 - g^2. \tag{40}$$

Expansion of this determinant gives us the equation:

$$q^2 (q + g^2) = 0 \qquad (g^2 = p_1^2 + p_2^2 + p_3^2), \tag{41}$$

which factorizes. If we equate the sum in brackets to zero, we obtain $p_0 = 0$ and have a stationary wave [141]. We dwell on the second case, when $q = 0$, i.e. when

$$\frac{\varepsilon\mu}{c^2} p_0^2 - g^2 = 0 \tag{42}$$

which gives the familiar expression for the wave displacement velocity:

$$V = \frac{c}{\sqrt{\varepsilon\mu}}. \tag{43}$$

We now consider the nature of the discontinuity. We write (a_1, a_2, a_3) for the discontinuity coefficients of the derivatives of the components of vector $\mathbf{E}$ and $(\beta_1, \beta_2, \beta_3)$ for the corresponding quantities for the components of vector $\mathbf{H}$. We introduce as usual the discontinuity vectors $\boldsymbol{\alpha}(a_1, a_2, a_3)$ and $\boldsymbol{\beta}(\beta_1, \beta_2, \beta_3)$. We can write:

$$\left.\begin{array}{l}[\mathbf{E}_{x_k}] = p_k \boldsymbol{\alpha} \\ [\mathbf{H}_{x_k}] = p_k \boldsymbol{\beta}\end{array}\right\} \qquad (k = 0, 1, 2, 3;\ x_0 = t). \tag{44}$$

The first three equations (18) here take the form:

$$\left.\begin{array}{l}\dfrac{\varepsilon}{c} p_0 \alpha_1 + p_3 \beta_2 - p_2 \beta_3 = 0 \\[2mm] \dfrac{\varepsilon}{c} p_0 \alpha_2 + p_1 \beta_3 - p_3 \beta_1 = 0 \\[2mm] \dfrac{\varepsilon}{c} p_0 \alpha_3 + p_2 \beta_1 - p_1 \beta_2 = 0,\end{array}\right\} \tag{45}$$

or, on writing $\mathbf{n}$ for the unit normal to the wave surface $\omega_1 = 0$, directed towards the side where $\omega_1 > 0$, we can write the last equations as:

$$\frac{\varepsilon V}{c} \boldsymbol{\alpha} = \boldsymbol{\beta} \times \mathbf{n}, \tag{46}$$

where the right-hand side contains the vector product of $\boldsymbol{\beta}$ and $\mathbf{n}$. Similarly, the last three of equations (8) can be written as

$$\frac{\mu V}{c} \boldsymbol{\beta} = - \boldsymbol{\alpha} \times \mathbf{n}. \tag{47}$$

It follows at once from these equations that vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lie in the tangent plane to the wave and are mutually perpendicular.

Suppose that we have rest in front of the wave surface, i.e. where $\omega_1 > 0$; in other words, $\mathbf{E}$ and $\mathbf{H}$ are zero. Expressions (44) give the derivatives of vectors $\mathbf{E}$ and $\mathbf{H}$ on the actual wave surface:

$$\mathbf{E}_{x_k} = - p_k \boldsymbol{\alpha}; \quad \mathbf{H}_{x_k} = - p_k \boldsymbol{\beta}. \tag{48}$$

We expand $\mathbf{E}$ and $\mathbf{H}$ close to the wave front in a Taylor series, the expansion being carried as far as the terms containing first order derivatives. On using the fact that $\mathbf{E}$ and $\mathbf{H}$ vanish on the wave surface, we can write the following approximations with the aid of (48):

$$\mathbf{E} \sim - \boldsymbol{\alpha} \sum_{k=0}^{3} p_k \left(x_k - x_k^{(0)}\right); \quad \mathbf{H} \sim - \boldsymbol{\beta} \sum_{k=0}^{3} p_k \left(x_k - x^{(0)}_k\right),$$

where $(x_0^{(0)}, x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ is a point of the wave surface. On applying the Taylor's formula for the function ω_1, we can use the fact that $\omega_1(x_0^{(0)}, x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = 0$ to write:

$$\omega_1 (x_0, x_1, x_2, x_3) \sim \sum_{k=0}^{3} p_k \left(x_k - x^{(0)}_k\right),$$

and the previous expressions can be rewritten as [cf. 164]:

$$\mathbf{E} \sim -\omega_1 (x_0, x_1, x_2, x_3,) \boldsymbol{\alpha}; \quad \mathbf{H} \sim -\omega_1 (x_0, x_1, x_2, x_3) \boldsymbol{\beta}. \tag{49}$$

These approximate formulae will hold close to the wave on the side where the electromagnetic process occurs.

In the case of a homogeneous anisotropic medium, ε must be taken as a symmetric matrix with nine elements, instead of as a number. These elements appear in the formula connecting the electrical displacement vector with the vector $\mathbf{E}$ [II, 118]. We shall assume that μ is numerical, as before. We choose the coordinate axes so that the matrix ε is reduced to the diagonal form, and let $\varepsilon_3 > \varepsilon_2 > \varepsilon_1 > 0$ be its eigenvalues [III$_1$, 32, 33]. The first three of equations (37) now have the form:

$$\frac{\varepsilon_1}{c} \frac{\partial e_1}{\partial t} + \frac{\partial h_2}{\partial x_3} - \frac{\partial h_3}{\partial x_2} + \ldots = 0$$

$$\frac{\varepsilon_2}{c} \frac{\partial e_2}{\partial t} + \frac{\partial h_3}{\partial x_1} - \frac{\partial h_1}{\partial x_3} + \ldots = 0$$

$$\frac{\varepsilon_3}{c} \frac{\partial e_3}{\partial t} + \frac{\partial h_1}{\partial x_2} - \frac{\partial h_2}{\partial x_1} + \ldots = 0,$$

and instead of (39) we have the equation:

$$\begin{vmatrix} q_1 + p_1^2, & p_1 p_2 & p_1 p_3 \\ p_1 p_2, & q_2 + p_2^2, & p_2 p_3 \\ p_1 p_3, & p_2 p_3, & q_3 + p_3^2 \end{vmatrix} = 0, \tag{50}$$

where

$$q_i = \frac{\varepsilon_i \mu}{c^2} p_0^2 - g^2 \qquad (i = 1, 2, 3).$$

On introducing the notation:

$$V_i^2 = \frac{c^2}{\varepsilon_i \mu},$$

we can write:

$$q_i = \frac{p_0^2}{V_i^2} - g^2. \tag{51}$$

On dividing both sides of (50) by g^2, this can be written as

$$q_2 q_3 \cos^2 \alpha_1 + q_3 q_1 \cos^2 \alpha_2 + q_1 q_2 \cos^2 \alpha_3 + \frac{1}{g^2} q_1 q_2 q_3 = 0. \tag{52}$$

The obvious solution of this equation is $q_1 = 0$, $\cos \alpha_1 = 0$. On taking (51) into account, we see that V_1 is the possible wave displacement velocity in any direction parallel to the plane $x_1 = 0$. Similarly, V_2 and V_3 are the possible velocities in directions parallel to the planes $x_2 = 0$ and $x_3 = 0$. In the general case, we can rewrite (52), after multiplying both sides by g^2 and writing $q_1 q_2 q_3 = q_1 q_2 q_3 (\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3)$, in the form [141]:

$$V^2 q_1 q_2 q_3 \sum_{i=1}^{3} \frac{\cos^2 \alpha_i}{V^2 - V_i^2} = 0. \tag{53}$$

Neglecting the solution $V = 0$, which corresponds to a stationary wave we obtain the following quadratic equation for V^2 when the wave direction is characterized by the quantities $\cos \alpha_k$:

$$\sum_{i=1}^{3} \frac{\cos^2 \alpha_i}{V^2 - V_i^2} = 0. \tag{54}$$

It can be shown, as in [II, 137], that this equation has two distinct positive roots for V^2.

If we solve equation (50) or (52) for p_0, we get an equation of the form:

$$p_0 + F(p_1, p_2, p_3) = 0, \tag{55}$$

where F is a homogeneous function of the first degree. Since (55) does not contain x_k, the Cauchy system for this equation leads to constant values for the p_k, and the bicharacteristics are straight lines. Their equations may be written as

$$\frac{dx_k}{dt} = F_{p_k} \qquad (k = 1, 2, 3).$$

We bring in the characteristic conoid with vertex at the origin. It is the wave surface from a point source at the origin at different instants. Its equation is: $x_k = F_{p_k} t$ or, with $t = 1$:

$$x_k = F_{p_k} \qquad (k = 1, 2, 3). \tag{56}$$

Since F_{p_k} is a homogeneous function of zero degree, the right-hand sides of equations (56) contain two parameters, viz, the ratios of two of the p_1, p_2, p_3 to the third. Let S be surface (56), $P(x_1, x_2, x_3)$ a point on S and δ the distance from the origin to the tangent plane to S at P. If $\cos \alpha_i$ are the direction-cosines of the normal to S at P, we obtain on using Euler's formula for homogeneous functions:

$$\delta = \sum_{i=1}^{3} x_i \cos \alpha_i = \sum_{i=1}^{3} F_{p_i} \cos \alpha_i = \pm \frac{1}{g} \sum_{i=1}^{3} p_i F_{p_i} = \pm \frac{F}{g} = \pm \frac{p_0}{g} = \pm V.$$

On taking the (+) sign for definiteness, which has a trivial effect on what follows, we can write the equation of the tangent plane to S in the form:

$$\sum_{i=1}^{3} x \cos \alpha_i - V = 0. \tag{57}$$

This equation contains the four parameters $\cos \alpha_i$ ($i = 1, 2, 3$) and V, which are connected by the two relationships:

$$\sum_{i=1}^{3} \cos^2 \alpha_i = 1; \qquad \sum_{i=1}^{3} \frac{\cos^2 \alpha_i}{V^2 - V_i^2} = 0,$$

so that equation (57) contains two independent parameters, as must in fact be the case. The surface S will be the envelope of the family of planes (57), depending on two parameters. If we continue the working (which is here omitted),

we arrive at the following equation of a surface:

$$\sum_{i=1}^{3} \frac{V_i^2 x_i^2}{V_i^2 - (x_1^2 + x_2^2 + x_3^2)} = 0.$$

If say $V_1 = V_2$, this fourth order surface degenerates into a set of spheres and ellipsoids.

168. Strong discontinuities in the theory of elasticity. We have discussed the question of strong discontinuities for solutions of one equation [142]. Let us now investigate the equations of the theory of elasticity from the point of view of the theory of strong discontinuities.

We confine ourselves here to the plane case. Let (u, v) be the components of the displacement vector on the (x, y) plane, and X, Y the components of the force per unit volume. On writing σ_x, σ_y, τ_{xy} as usual for the components of the stress tensor, we have the following two fundamental equations of the theory of elasticity:

$$\left.\begin{aligned} \varrho \frac{\partial^2 u}{\partial t^2} - \frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} = X, \\ \varrho \frac{\partial^2 v}{\partial t^2} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y} = Y. \end{aligned}\right\} \tag{58}$$

We must add to these equations the further connection between the stress tensor and deformation tensor (Hooke's law):

$$\sigma_x = \lambda (u_x + v_y) + 2\mu u_x; \quad \sigma_y = \lambda (u_x + v_y) + 2\mu v_y, \quad \tau_{xy} = \mu (u_y + v_x). \tag{58$_1$}$$

On substituting the last expressions in (58), we obtain the elasticity equations expressed in terms of the displacement vector $\mathfrak{w}$:

$$\varrho \frac{\partial^2 \mathfrak{w}}{\partial t^2} = (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathfrak{w} + \mu \Delta \mathfrak{w} + \mathbf{F}.$$

We can understand (u, v) in future as any two functions of (x, y, t), having continuous derivatives up to the second order. Equations (58) now give us X and Y, corresponding to the functions taken for (u, v). We also introduce two linear operators, containing the first order derivatives of functions (u, v):

$$\left.\begin{aligned} P_x(u, v) = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) - \sigma u_t \cos(n, t), \\ P_y(u, v) = \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) - \varrho v_t \cos(n, t). \end{aligned}\right\} \tag{59}$$

We consider two pairs of functions (u, v) and (u', v') and let σ_x', σ_y', τ_{xy}', X', Y' be the values of quantities (58$_1$) and X, Y, corresponding to the pair of functions (u', v'). We thus have:

$$\sigma_x' = \lambda (u_x' + v_y') + 2\mu u_x'; \quad \sigma_y' = \lambda (u_x' + v_y') + 2\mu v_y'; \quad \tau_{xy}' = \mu (u_y' + v_x').$$

On using these expressions and applying the usual Ostrogradskii formula, we get the following analogue of Green's formula:

$$-\iiint_{D_1} (uX' + vY' - u' X - v' Y)\, d\tau =$$

$$= \iint_S [u\, P_x (u', v') + v\, P_y (u', v') - u'\, P_x (u, v) - v'\, P_y (u, v)]\, dS, \quad (60)$$

where D, as above, is some domain in space (x, y, t) bounded by a surface S, and n is the direction of the outward normal to S. The above formula was first given by Volterra. We remark that X, Y are understood simply to be the expressions on the right-hand sides of (58) and similarly for X', Y'. When deducing (60) it is naturally assumed that the functions (u, v) and (u', v') have continuous derivatives up to the second order in the domain D.

We now pass to the case when the first order derivatives of functions (u, v) have discontinuities. Let D be split by the surface σ into two parts D_1 and D_2, and let the first order derivatives of functions (u, v) have discontinuities on σ satisfying the kinematic compatibility conditions of [142]. Suppose further that expressions (59) remain continuous on passing through the surface σ. We shall explain later the mechanical significance of these dynamic compatibility conditions. We can say, precisely as in [142], that (60) holds throughout D provided (u, v) satisfy the above-mentioned discontinuity conditions, and (u', v') are any functions with continuous derivatives up to the second order.

The consequences of the above-mentioned conditions are as follows. As in [142], we can say that the vectors grad $u \times \mathbf{n}$ and grad $v \times \mathbf{n}$ must remain continuous on passing through σ. If we write down the components of these vectors, we get six expressions which must remain continuous on passing through σ. On further adding expressions (59), which we transform by substituting in them the expressions of (58_1) for the components of the stress tensor, we obtain the following eight expressions, which must remain continuous on passing through σ:

$$\begin{aligned}
&u_x \cos(n, y) - u_y \cos(n, x) = M_1\\
&u_y \cos(n, t) - u_t \cos(n, y) = M_2\\
&u_t \cos(n, x) - u_x \cos(n, t) = M_3\\
&v_x \cos(n, y) - v_y \cos(n, x) = M_4\\
&v_y \cos(n, t) - v_t \cos(n, y) = M_5\\
&v_t \cos(n, x) - v_x \cos(n, t) = M_6
\end{aligned}$$

$$(\lambda + 2\mu)\, u_x \cos(n, x) + \mu u_y \cos(n, y) - \varrho u_t \cos(n, t) + \\ + \mu v_x \cos(n, y) + \lambda v_y \cos(n, x) = M_7$$

$$\lambda u_x \cos(n, y) + \mu u_y \cos(n, y) + \mu v_x \cos(n, x) + \\ + (\lambda + 2\mu)\, v_y \cos(n, y) - \varrho v_t \cos(n, t) = M_8.$$

We shall regard these as eight equations in the six first order derivatives of functions u and v. If the matrix of the coefficients of these equations were to contain at least one non-zero sixth order determinant, we would be able to express all six first order derivatives of u and v in terms of the continuous

functions M_k and the derivatives would have no discontinuity on σ. We can therefore say that all the sixth order determinants of the matrix must vanish. On striking out the last two rows of the matrix and equating the remaining determinant to zero, we obtain an identity. By considering the remaining cases, we arrive at the single equation:

$$\{\varrho \cos^2 (n, t) - (\lambda + 2\mu) [\cos^2 (n, x) + \cos^2 (n, y)]\}\times$$
$$\times \{\varrho \cos^2 (n, t) - \mu [\cos^2 (n, x) + \cos^2 (n, y)]\} = 0, \tag{61}$$

which expresses the fact that the matrix has a rank less than six. Let the equation of σ be $\psi (x, y, t) = 0$. The above equation splits into two:

$$\varrho\psi_t^2 - (\lambda + 2\mu) (\psi_x^2 + \psi_y^2) = 0 \quad \text{and} \quad \varrho\psi_t^2 - \mu (\psi_x^2 + \psi_y^2) = 0,$$

and hence we see that σ must be a characteristic surface of the elasticity equations [164].

The present case differs essentially from that of a single wave equation. The kinematic compatibility conditions, which lead to the continuity of $M_1, M_2, \ldots, M_6$, together with the fact that σ is a characteristic surface, which leads to equation (61), do not guarantee the further continuity of M_7 and M_8, i.e. do not guarantee the dynamic compatibility conditions. Let us explain the auxiliary conditions in which we obtain continuity of M_7 and M_8.

Let N be a point on σ, and l the intersection of the tangent plane to σ at N with the plane $t =$ const. through N. We choose the straight line l as the y axis. The t axis has a fixed direction at N perpendicular to l. Thus we also determine the x axis. We first take the case when the first factor on the left-hand side of (61) vanishes:

$$\varrho \cos^2 (n, t) - (\lambda + 2\mu) [\cos^2 (n, x) + \cos^2 (n, y)] = 0, \tag{62}$$

which corresponds to the longitudinal wave velocity. In view of the choice of y axis, we have $\cos (n, y) = 0$ at N, and in addition, the derivatives u_y and v_y remain constant on passing through σ at N. We form the expression:

$$(\lambda + 2\mu) u_x \cos (n, x) - \varrho u_t \cos (n, t) = r. \tag{63}$$

We can write by using (62):

$$r \cos (n, x) = - \varrho \cos (n, t) M_3,$$

whence it follows that expression (63) is continuous at N, by virtue of the kinematic compatibility conditions and equation (62). Now, M_7 is also continuous at N, whilst the necessary and sufficient condition for continuity of M_8 is continuity of:

$$\mu v_x \cos (n, x) - \varrho v_t \cos (n, t) = M. \tag{64}$$

Furthermore, we have continuity of the expression:

$$v_x \cos (n, t) - v_t \cos (n, x) = - M_6. \tag{65}$$

The determinant of the system of equations (64) and (65), equal to $\varrho \cos^2(n,t) - \mu \cos^2(n,x)$, by virtue of (62) and $\cos (n,y) = 0$, differs from zero, so that the continuity of expression (64) is equivalent to continuity of the partial derivatives v_x and v_t. Furthermore, we already have continuity of the partial

derivative v_y at the point N. The intersection of the surface σ with the plane a $t = \text{const}$ is a line of discontinuity on the (x, y) plane at a given instant, whilst l is the tangent to this line at N. The quantity v is the projection of the displacement vector on the direction l, tangential to the discontinuity curve. We have shown above that all the first order derivatives of v must be continuous at N, i.e. only the component of the displacement vector in the direction perpendicular to the discontinuity curve can have a strong discontinuity (a longitudinal discontinuity). Thus, if the kinematic compatibility conditions and equation (62) are fulfilled, the necessary and sufficient condition for the dynamic compatibility conditions to be satisfied is that only the component of the displacement vector normal to the discontinuity curve moving in the (x, y) plane have a strong discontinuity. We can similarly consider the equation

$$\varrho \cos^2 (n, t) - \mu [\cos^2 (n, x) + \cos^2 (n, y)] = 0.$$

In this case, only the component of the displacement vector along the tangent to the discontinuity curve can have a strong discontinuity.

Suppose that the displacement field is lamellar:

$$(u, v) = \text{grad}\, \varphi,$$

whence it follows that:

$$u_y = v_x.$$

On choosing the coordinate axes as before, we have continuity of the derivatives u_y, v_y, and v_x at the point N. But it now follows from the continuity of M_6 that v_t is continuous, so that, in the case of a lamellar field, only the component of the displacement vector along the normal to the discontinuity curve can have a discontinuity.

Now let the displacement field be solenoidal, i.e.

$$u_x + v_y = 0.$$

Here, we have continuity of the derivatives u_y, v_y and u_x, so that, by virtue of the continuity of M_3, the derivative u_t is continuous, i.e. in a solenoidal field only the component of the displacement vector along the tangent to the discontinuity curve can have a discontinuity.

Let us now explain the mechanical significance of the above theory. In fact, let us prove that, in the simplest particular cases, the existence of formula (60) shows that the impulse-momentum theorem still holds for a volume containing a discontinuity surface in its interior. Let $u' = 1$ in the formula and $v' = 0$. Now, by (58_1), the components of the stress tensor for (u', v') will vanish, and (60) reduces to:

$$\iiint_D X \, d\tau = - \iint_S P_x (u, v) \, dS. \tag{66}$$

Similarly, if we put $u' = 0$, $v' = 1$, we obtain

$$\iiint_D Y \, d\tau = - \iint_S P_y (u, v) \, dS. \tag{67}$$

We take as the domain D a cylinder whose generators are parallel to the t axis, whilst its bases S_1 and S_2 lie in the planes $t = t_1$ and $t = t_2$. Suppose that

a discontinuity surface σ lies inside this cylinder. We have $\cos(n, x) = \cos(n, y) = 0$ on the upper and lower bases S_2 and S_1 of the cylinder. On the lower base, $\cos(n, t) = -1$ and on the upper, $\cos(n, t) = +1$. On the lateral surface, $\cos(n, t) = 0$. On writing S_t for the variable section of the cylinder by a plane perpendicular to its generators, and l_t for the line of intersection of this plane with the lateral surface, we can rewrite (66) as:

$$\int_{t_1}^{t_2} \Big[\iint_{S_t} X \, dx \, dy\Big] dt = \iint_{S_t} \varrho u_t \, dx \, dy \,\Big|_{t=t_2} - \iint_{S_t} \varrho u_t \, dx \, dy \,\Big|_{t=t_1} = \int_{t_1}^{t_2} \Big[\int_{l_t} \sigma_n \, ds\Big] dt,$$

where

$$\sigma_n = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y),$$

or

$$\int_{t_1}^{t_2} \Big[\iint_{S_t} X \, dx \, dy\Big] dt + \int_{t_1}^{t_2} \Big[\int_{l_t} \sigma_n \, ds\Big] dt = \iint_{S_t} \varrho u_t \, dx \, dy \,\Big|_{t=t_2} - \iint_{S_t} \varrho u_t \, dx \, dy \,\Big|_{t=t_1}.$$

The first term on the left-hand side gives the impulse of the forces per unit volume applied to the area S_t of the (x, y) plane over the time interval $[t_1, t_2]$. The second term gives the impulse of the stress forces, acting on the contour of this area, whilst the difference on the right is the increment in the momentum, measured for the same area; both the impulse and the increment in the momentum are here projected on the x axis. Similarly, (67) gives us an analogous relationship for the projections of the impulses and the increment in the momentum on the y axis. Therefore, we in fact obtain the impulse-momentum theorem for the volume D, containing a discontinuity surface.

169. Characteristics and higher frequencies. There is a connection between the formulae obtained above when discussing the theory of characteristics of systems of equations, and the formulae which are obtained if we try to satisfy approximately a system of differential equations by functions of a special type. Suppose we have the system of second order equations:

$$\sum_{j=1}^{m} \sum_{k, l=1}^{n} a_{ij}^{kl} \frac{\partial^2 u_j}{\partial x_k \partial x_l} + \ldots = 0 \qquad (i = 1, 2, \ldots, m). \tag{68}$$

We shall try to satisfy this system by specifying functions u_j in the form:

$$u_j = X_j e^{i\omega\Phi} \qquad (j = 1, 2, \ldots, m), \tag{69}$$

where X_i and φ are certain required functions of the independent variables and ω is a number. On substituting expressions (69) in equations (68) and retaining only the terms containing the square of ω, we arrive at the following system of equations:

$$\sum_{j=1}^{m} \sum_{k, l=1}^{n} a_{ij}^{kl} X_j \Phi_{x_k} \Phi_{x_l} = 0 \qquad (i = 1, 2, \ldots, m). \tag{70}$$

We shall regard this system as a system of homogeneous equations in the X_j. To obtain a solution different from zero, we must equate the determinant of the system to zero. We thus arrive at an equation of the first order for the required function Φ:

$$|\omega'_{ij}| = 0 \qquad \left(\omega'_{ij} = \sum_{k,\,l=1}^{n} a_{ij}^{kl}\,\Phi_{x_k}\Phi_{x_l}\right),$$

which is the same as the equation for the characteristic surface. By taking any solution of this equation, we can in general define the X_j apart from an arbitrary factor, from system (70). This system is the same as system (21), which we obtained to find the discontinuity coefficient h_j. The equations of this last system must hold only on the wave surface. Equations (70) must hold everywhere. But here, we have only approximately satisfied system (68) by the functions of form (69). In the present case $\Phi = \text{const}$ are equiphase surfaces.

Let us consider in more detail the case of the single wave equation:

$$u_{tt} = a^2\,(u_{xx} + u_{yy} + u_{zz}), \tag{71}$$

and seek its solution as a harmonic vibration of frequency ω with respect to time t:

$$u = A\mathrm{e}^{i\omega(t+\Phi)},$$

where A and φ are required functions of the coordinates (x, y, z) only. It amounts to substituting the expression

$$v = A\mathrm{e}^{i\omega\Phi} \tag{72}$$

in the equation

$$\Delta v + k^2\,v = 0 \qquad \left(k^2 = \frac{\omega^2}{a^2}\right). \tag{73}$$

We have:

$$v_x = (A_x + i\omega\,A\Phi_x)\,\mathrm{e}^{i\omega\Phi}$$
$$v_{xx} = (A_{xx} + i\omega\,A\Phi_{xx} + 2i\omega\,A_x\,\Phi_x - \omega^2\,A\Phi_x^2)\,\mathrm{e}^{i\omega\Phi}.$$

Similar expressions are obtained for the derivatives with respect to y and z. On substituting in equation (73) and equating to zero the coefficient of ω^2, we obtain the equation for Φ:

$$\Phi_x^2 + \Phi_y^2 + \Phi_z^2 = \frac{1}{a^2}. \tag{74}$$

If we also equate to zero the coefficient of ω, we get an equation which contains the amplitude $A(x, y, z)$ of solution (72):

$$A\,\Delta\Phi + 2\,(A_x\,\Phi_x + A_y\,\Phi_y + A_z\,\Phi_z) = 0,$$

or

$$\operatorname{grad}\log A \cdot \operatorname{grad}\Phi = -\frac{1}{2}\Delta\Phi. \tag{75}$$

It is easy to establish the connection between (74) and the equation of the characteristic surfaces. We have the following equation for the characteristic surfaces corresponding to equation (71):

$$a^2\left[\left(\frac{\partial\omega_1}{\partial x}\right)^2+\left(\frac{\partial\omega_1}{\partial y}\right)^2+\left(\frac{\partial\omega_1}{\partial z}\right)^2\right]=\left(\frac{\partial\omega_1}{\partial t}\right)^2,$$

and the substitution $\omega_1 = t + \Phi$ in fact gives us (74). On writing $\mathbf{n}$ for the unit normal at a point M to the equiphase surface $\Phi = \text{const.}$ passing through this point, we have:

$$\operatorname{grad}\Phi = \varphi(x, y, z)\,\mathbf{n},$$

where $\varphi(x, y, z)$ is the length of the vector grad Φ at the point (x, y, z). Equation (75) can now be written as

$$\operatorname{grad}_n \log A = -\frac{1}{2\varphi}\operatorname{div}(\varphi\,\mathbf{n}), \tag{76}$$

where $\operatorname{grad}_n \log A$ is the projection of $\operatorname{grad}\log A$ on the direction n. Equations (74) and (76) must hold throughout the space. But we have established equation (71) only approximately.

Similarly, if we substitute

$$\mathbf{E} = \mathbf{e}e^{i\omega\Phi}; \qquad \mathbf{H} = \mathbf{h}e^{i\omega\Phi} \tag{77}$$

in Maxwell's equation (36), where $\mathbf{e}$ and $\mathbf{h}$ are vectors, Φ is a scalar function depending on (x_1, x_2, x_3, t) and ω is a number, we obtain on collecting terms containing the factor ω:

$$\Phi_t\frac{\varepsilon}{c}\,\mathbf{e} = \operatorname{grad}\Phi\times\mathbf{h}. \tag{78}$$

This equation is the same in essence as (46) of [167]. An equation analogous to (47) may be similarly obtained. Equation (78) must hold elsewhere, apart from on the surface $\Phi = \text{const}$; this latter is not a discontinuity surface, but an equiphase surface for solution (77).

170. The case of two independent variables. Let us take a system of first order equations with two independent variables and let it be soluble for the partial derivatives with respect to x_2. We thus have a system of the form:

$$\frac{\partial u_i}{\partial x_2} = \sum_{j=1}^{m} a_{ij}\frac{\partial u_j}{\partial x_1} + \Phi_j(x_1, x_2, u_j) \qquad (i = 1, \ldots, m), \tag{79}$$

where the a_{ij} may depend on x_1, x_2.

On introducing the vectors $\mathbf{u}$ and $\mathbf{\Phi}$ with components u_i and Φ_i and the matrix A with elements a_{ij}, we can rewrite system (79) as a single vector equation:

$$\frac{\partial\mathbf{u}}{\partial x_2} = A\frac{\partial\mathbf{u}}{\partial x_1} + \mathbf{\Phi}(x_1, x_2, u_j). \tag{80}$$

We introduce a new vector $\mathbf{v}$ in a plane of $\mathbf{u}$ in accordance with the formula:

$$\mathbf{u} = B\mathbf{v}, \tag{81}$$

where B is a matrix with elements b_{ik}, depending on x_1, x_2, having continuous derivatives in some domain D of the (x_1, x_2) plane, and with a non-zero determinant. We have

$$\frac{\partial \mathbf{u}}{\partial x_i} = B\frac{\partial \mathbf{v}}{\partial x_i} + \frac{\partial B}{\partial x_i}\mathbf{v} \qquad (i = 1, 2), \tag{82}$$

where the differentiation of matrix B amounts to differentiation of its elements. On substituting (81) and (82) in (80), we obtain for $\mathbf{v}$:

$$B\frac{\partial \mathbf{v}}{\partial x_2} = AB\frac{\partial \mathbf{v}}{\partial x_1} + \boldsymbol{\Psi},$$

where $\boldsymbol{\Psi}$ is a vector whose components depend on (x_1, x_2, v_j). On multiplying both sides by B^{-1}, we obtain the transformed equation:

$$\frac{\partial \mathbf{v}}{\partial x_2} = B^{-1}AB\frac{\partial \mathbf{v}}{\partial x_1} + \boldsymbol{\Psi}_1. \tag{83}$$

If possible, we now choose B so that $B^{-1}AB$ has the diagonal form. As we know, this is bound up with the solutions of the characteristic equation for matrix A [III_3, 27]:

$$D(A - \lambda) = 0, \tag{84}$$

where the left-hand side is the determinant of the matrix $(A - \lambda)$, or in the expanded form:

$$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & \dots, & a_{1m} \\ a_{21}, & a_{22} - \lambda, & \dots, & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1}, & a_{m2}, & \dots, & a_{mm} - \lambda, \end{vmatrix} = 0. \tag{85}$$

Suppose that the coefficients a_{ik} have continuous derivatives in the neighbourhood of the point $(x_1^{(0)}, x_2^{(0)})$ and that equation (85) has distinct roots $\lambda_k(x_1, x_2)$ $(k = 1, \dots, m)$. This last is essential for what follows. By using the method of [III_3, 27], we can now construct in the neighbourhood in question a matrix B with the above properties such that $B^{-1}AB$ reduces to the purely diagonal form; and now, by writing down all the components, we can write (83) as

$$\frac{\partial v_i}{\partial x_2} - \lambda_i(x_1, x_2)\frac{\partial v_i}{\partial x_1} + \psi_i(x_1, x_2, v_j) = 0 \qquad (i = 1, 2, \dots, m). \tag{86}$$

If all the $\lambda_i(x_1, x_2)$ are real in the neighbourhood, the system is said to be hyperbolic in this neighbourhood.

Using the notation of [161], we have for the system:

$$a_{ij}^{(k)} = 0 \quad \text{when} \quad i \neq j; \qquad a_{ii}^{(2)} = 1; \qquad a_{ii}^{(1)} = -\lambda_i(x_1, x_2), \tag{87}$$

and we obtain for the ω_{ij}, defined by expressions (5):

$$\omega_{ij} = 0 \quad \text{when} \quad i \neq j; \quad \omega_{ii} = \frac{\partial \omega_1}{\partial x_2} - \lambda_i(x_1, x_2) \frac{\partial \omega_1}{\partial x_1}.$$

Equation (6) becomes

$$\left[\frac{\partial \omega_1}{\partial x_2} - \lambda_1(x_1, x_2) \frac{\partial \omega_1}{\partial x_1}\right] \cdots \left[\frac{\partial \omega_1}{\partial x_2} - \lambda_m(x_1, x_2) \frac{\partial \omega_1}{\partial x_1}\right] = 0,$$

and it splits into the m linear equations:

$$\frac{\partial \omega_1}{\partial x_2} - \lambda_i(x_1, x_2) \frac{\partial \omega_1}{\partial x_1} = 0 \qquad (i = 1, \ldots, m). \tag{88}$$

If $\omega_i(x_1, x_2)$ is the solution of one of these equations, the family $\omega_1(x_1, x_2) = C$ is a family of characteristics for system (86). Equations (88) are equivalent to the ordinary equation:

$$dx_1 + \lambda_i(x_1, x_2)\, dx_2 = 0, \text{ i.e. } \frac{\partial x_1}{\partial x_2} = -\lambda_i(x_1, x_2), \tag{89}$$

and m characteristics pass through every point of the plane in the domain where we have functions $\lambda_i(x_1, x_2)$ with continuous first order derivatives.

We now consider points sufficiently close to the axis $x_2 = 0$, and let l_i be the part of the integral curve of equation (89) passing through the point (x_1, x_2), between this point and its intersection $(x_1^{(i)}, 0)$ with the axis $x_2 = 0$. We can take any function $\psi(x_1, x_2)$ as a function of x_2 along a curve l_i, and we have, by (89):

$$\frac{d\psi}{dx_2} = \frac{\partial \psi}{\partial x_2} - \lambda_i(x_1, x_2) \frac{\partial \psi}{\partial x_1} \text{ along } l_i.$$

Therefore, the system of equations (86) is equivalent to the following system of integral equations:

$$v_i(x_1, x_2) = v_i(x_1^{(i)}, 0) - \int_{l_i} \psi_i(x_1, x_2, v_j)\, dx_2 \quad (i = 1, \ldots, m). \tag{90}$$

Assuming that we are given the values of the functions $v_i(x_1, x_2)$ on the $x_2 = 0$ axis, we can take the $v_i(x_1^{(i)}, 0)$ as known and can apply to system (90) the method of successive approximations. This gives an existence and uniqueness theorem for the solution of the Cauchy

problem and the continuity of the dependence of the solution on the initial data. A detailed treatment of this subject, and a discussion of the case when equation (85) has multiple roots, may be found in the above-mentioned book by Petrovskii.

171. Examples. 1. Let us take the system of equations which define the real and imaginary parts of an analytic function [III_2, 2]:

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0; \quad \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = 0. \tag{91}$$

We have:

$$a_{11}^{(1)} = a_{21}^{(2)} = a_{22}^{(1)} = 1; \quad a_{12}^{(2)} = -1.$$

and the remaining $a_{ij}^{(k)}$ vanish. After replacing the $\partial\omega_1/\partial x_k$ by a_k, the left-hand side of equation (6) becomes $a_1^2 + a_2^2$, so that system (91) is of the elliptic type. By using the connection mentioned of this system with analytic functions, we can say that every solution with continuous first order derivatives is an analytic function of x_1 and x_2.

2. Let us take the system (Perron, *Math. Zeitschr.*, Bd. 27, H. 4, 1927):

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0; \quad \frac{\partial u_2}{\partial x_1} - a\frac{\partial u_1}{\partial x_2} + F(x_2) = 0, \tag{92}$$

where a is a constant.

The left-hand side of equation (6) becomes, after replacing $\partial\omega_1/\partial x_k$ by a_k: $a_1^2 - a a_2^2$, so that the system is elliptic for $a < 0$ and parabolic for $a = 0$. Equation (85) becomes, on treating (92) as a system in which the unknowns are the partial derivatives with respect to x_1:

$$\begin{vmatrix} -\lambda, & 1 \\ a, & -\lambda \end{vmatrix} = 0, \quad \text{i.e.} \quad \lambda^2 - a = 0,$$

and it has real distinct roots for $a > 0$, i.e. the system is hyperbolic for $a > 0$.

Suppose first that $a > 0$. Proceeding as in [170], we introduce new functions in place of u_1, u_2:

$$v_1 = \sqrt{a}u_1 + u_2; \quad v_2 = \sqrt{a}u_1 - u_2 \tag{93}$$

and obtain two separate equations for v_1 and v_2:

$$\frac{\partial v_1}{\partial x_1} - \sqrt{a}\frac{\partial v_1}{\partial x_2} + F(x_2) = 0; \quad \frac{\partial v_2}{\partial x_1} + \sqrt{a}\frac{\partial v_2}{\partial x_2} - F(x_2) = 0. \tag{94}$$

After introducing the new independent variables:

$$2\xi = \sqrt{a}x_1 + x_2; \quad 2\eta = -\sqrt{a}x_1 + x_2$$

the system can be rewritten as

$$-\sqrt{a}\frac{\partial v_1}{\partial \eta} + F(\xi+\eta) = 0, \quad \sqrt{a}\frac{\partial v_2}{\partial \xi} - F(\xi + \eta) = 0. \tag{95}$$

Let us find the solution of system (95) which satisfies the initial condition

$$v_1 |_{x_1=0} = v_2 |_{x_1=0} = 0; \quad \text{i.e.} \quad v_1 |_{\eta=\xi} = 0; \quad v_2 |_{\eta=\xi} = 0.$$

On using (95), we get:

$$v_1 = \frac{1}{\sqrt{a}} \int\limits_{2\xi}^{\xi+\eta} F(t)\, dt; \quad v_2 = \frac{1}{\sqrt{a}} \int\limits_{2\eta}^{\xi+\eta} F(t)\, dt.$$

In the original independent variables:

$$v_1 = \frac{1}{\sqrt{a}} \int\limits_{\sqrt{a}x_1 + x_2}^{x_2} F(t)\, dt; \quad v_2 = \frac{1}{\sqrt{a}} \int\limits_{-\sqrt{a}x_1 + x_2}^{x_2} F(t)\, dt,$$

and we can determine u_1 and u_2 from (93), such that they are solutions of system (92) and satisfy the initial conditions:

$$u_1 |_{x_1=0} = u_2 |_{x_1=0} = 0. \tag{96}$$

This solution is obviously unique.

When $a = 0$, system (92) becomes:

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0; \quad \frac{\partial u_2}{\partial x_1} + F(x_2) = 0,$$

and we obtain a unique solution satisfying conditions (96):

$$u_1 = -\frac{x_1^2}{2} F'(x_2); \quad u_2 = -x_1 F(x_2),$$

where it has to be assumed that $F(x_2)$ has a continuous second order derivative.

Let us finally take the case when $a = -b^2 < 0$. On putting

$$bx_1 = x; \quad x_2 = y; \quad v_1 = bu_1 + \frac{1}{b}\int\limits_c^y F(t)\, dt; \quad v_2 = u_2, \tag{97}$$

we can rewrite system (92) as

$$\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} = 0; \quad \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} = 0.$$

It is clear from this that $v_1 + v_2 i$ must be a regular function of $z = x + yi$, and, by (96) and (97), this function must tend to the real function

$$\frac{1}{b}\int\limits_c^y F(t)\, dt. \tag{98}$$

as $x \to 0$. We can say that this regular function must be capable of analytic continuation through the straight line $x = 0$, so that it must be an analytic function on $x = 0$ itself [III_2, 24]. Thus function (98), and hence $F(y)$ also, must be analytic functions for real y. On expanding function (98) in powers of $(y - y_0)$, where y_0 is any real number:

$$-\frac{1}{b}\int_c^y F(t)\,\mathrm{d}t = \sum_{k=0}^{\infty} a_k (y - y_0)^k,$$

we get:

$$v_1 + v_2 i = \sum_{k=0}^{\infty} (-i)^k a_k (z - iy_0)^k \qquad (z = x + yi)$$

for z close to iy_0. Knowing v_1 and v_2, we can find u_1 and u_2 from (97).

CHAPTER IV

BOUNDARY VALUE PROBLEMS

§ 1. Boundary value problems for an ordinary differential equation

172. Green's function for a linear second order equation. The present chapter will be devoted to a study of boundary value problems both for ordinary differential equations and for partial differential equations. We have already encountered a number of these problems. The aim of the present chapter is to give a systematic exposition of the subject.

The application of Fourier's method to the solution of boundary value problems of mathematical physics has led us several times to the following boundary value problem for an ordinary differential equation of the second order containing a parameter: to find the values of the parameter λ for which there exists in the finite interval $[a, b]$ a non-zero solution of the homogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[p(x)y'] + [\lambda r(x) - q(x)]y = 0, \tag{1}$$

satisfying at the ends of the interval the homogeneous boundary conditions:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0; \qquad \beta_1 y(b) + \beta_2 y'(b) = 0, \tag{2}$$

where α_k and β_k are given numbers. It is naturally assumed here that at least one of the numbers α_2 and α_1, as also β_2 and β_1, differs from zero. We shall assume that $p(x)$, $v(x)$ and $r(x)$ are continuous functions in the closed interval $[a, b]$, where $p(x)$ does not vanish in the interval and has a continuous derivative. We introduce a special notation for the sum of those terms of the left-hand side of (1) that do not contain λ:

$$L(y) = \frac{\mathrm{d}}{\mathrm{d}x}[p(x)y'] - q(x)y.$$

As usual, we shall describe as *eigenvalues* the values of λ for which our homogeneous problem has no zero solution, the solutions themselves being *eigenfunctions.* They are evidently defined up to a constant factor. It is easily seen that there can only be one eigenfunction corresponding to each eigenvalue. For suppose, on the contrary, that, for a given λ, there are two linearly independent solutions (1), satisfying boundary conditions (2). In this case, the general solution of (1) would also satisfy boundary conditions (2). But this is impossible, since a solution can be found of equation (1) with initial data for $y(a)$ and $y'(a)$ which do not satisfy the first of the boundary conditions (2). By applying elementary transformations, such as have been used several times [III_2, 102, 145, 157], it can be shown that the eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ corresponding to different eigenvalues have the property of orthogonality, viz:

$$\int_a^b r(x)\,\varphi_1(x)\,\varphi_2(x)\,\mathrm{d}x = 0.$$

We now introduce for the operator $L(y)$ a function analogous to the statical bending of a string under the action of a concentrated force, which we discussed in [1]. In this latter case, the role of operator $L(y)$ was played by the operator y''. In order to arrive in a natural way at an explanation of the properties of this function, we consider the non-homogeneous equation

$$L(y) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,y'\right] - q(x)\,y = -f(x) \tag{3}$$

and assume that $f(x)$ vanishes throughout the interval $[a, b]$, except for a small interval $[\xi - \varepsilon, \xi + \varepsilon]$, where ξ is a fixed point inside $[a, b]$, where the condition is fulfilled:

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} f(x)\,\mathrm{d}x = 1. \tag{4}$$

As ε tends to zero, we obtain in the limit an analogue of the force concentrated at the point $x = \xi$. Given this assumption regarding $f(x)$, we consider the solution $y_\varepsilon(x)$ of equation (1) satisfying boundary conditions (2), supposing that such a solution exists. On integrating both sides of (3) with respect to x and taking (4) into account, we have:

$$p(x)\,y_\varepsilon'(x)\Big|_{x=\xi-\varepsilon}^{x=\xi+\varepsilon} - \int_{\xi-\varepsilon}^{\xi+\varepsilon} q(x)\,y_\varepsilon(x)\,\mathrm{d}x = -1,$$

or in the limit, as $\varepsilon \to 0$:

$$y'(\xi + 0) - y'(\xi - 0) = -\frac{1}{p(\xi)},$$

i.e. the derivative $y'(x)$ of our solution must have a discontinuity of the first kind at the point $x = \xi$ with a jump equal to $-1/p(\xi)$. This solution will naturally depend on the actual point of $[a, b]$ that we choose as ξ, so that it will be a function of the two variables (x, ξ); we shall write it in future as $G(x, \xi)$ and call it Green's function for the operator $L(y)$ with boundary conditions (2). The above discussion leads us to the following strict definition of Green's function: *Green's function of operator $L(y)$ with boundary conditions* (2) *is the function $G(x, \xi)$ satisfying the following conditions*: (1) *it is defined and continuous in the square k_0 given by $a \leqslant x, \xi \leqslant b$*; (2) *as a function of the variable x, it has continuous derivatives up to the second order for $a \leqslant x < \xi$ and $\xi < x \leqslant b$ and satisfies the homogeneous equation $L(y) = 0$;* (3) *as a function of x, it satisfies boundary conditions* (2); (4) *on the diagonal of the square*, i.e. *for $x = \xi$, its derivative with respect to x, which we write as $G'(x, \xi)$, has a discontinuity of the first kind, whilst the two conditions are satisfied:*

$$\left.\begin{aligned} G'(\xi + 0, \xi) - G'(\xi - 0, \xi) &= -\frac{1}{p(\xi)}, \\ G'(\xi, \xi + 0) - G'(\xi, \xi - 0) &= \frac{1}{p(\xi)}. \end{aligned}\right\} \tag{5}$$

These latter conditions amount to a single requirement: the derivative $G'(x, \xi)$ must have definite values on approaching any point $x = \xi$ on the diagonal, both from above, i.e. from the domain $\xi > x$, and from below, i.e. from the domain $\xi < x$; and the difference between these limiting values must be equal to $1/p(\xi)$. Since $L(G) = 0$, the second derivative with respect to x is given in each of these two domains by:

$$p(x) G''(x, \xi) = -p'(x) G'(x, \xi) + q(x) G(x, \xi),$$

so that this second derivative will also have definite limiting values on approaching points of the diagonal from either side.

We now show that there exists a unique Green's function satisfying all the conditions stated above. We shall assume here that $\lambda = 0$ *is not an eigenvalue*, i.e. that the equation $L(y) = 0$ has no solutions, not identically zero, satisfying conditions (2). We shall see later how the definition of Green's function must be modified in the case when $\lambda = 0$ is an eigenvalue. We construct the solution $y_1(x)$ of the homo-

geneous equation $L(y) = 0$, taking as the initial values $y_1(a)$ and $y_1'(a)$ certain numbers satisfying the first of conditions (2). This solution $y_1(x)$, and in general all the solutions $c_1 y_1(x)$, where c_1 is an arbitrary constant, will satisfy the first of the boundary conditions. It is easy to see that these exhaust all the solutions satisfying the first of conditions (2). For, if some solution $y(x)$ satisfies this condition, we have two homogeneous equations in α_1 and α_2:

$$\alpha_1 y_1(a) + \alpha_2 y_1'(a) = 0; \quad \alpha_1 y(a) + \alpha_2 y'(a) = 0,$$

and since we are obviously assuming that at least one of these numbers is non-zero, the determinant of the system written must vanish, i.e. the Wronskian of solutions $y(x)$ and $y_1(x)$ vanishes for $x = a$, so that $y(x)$ and $y_1(x)$ are linearly dependent, i.e. $y(x) = cy_1(x)$ [II, 24].

Similarly, let $c_2 y_2(x)$, where c_2 is an arbitrary constant, be solutions of the equation $L(y) = 0$ satisfying the second of conditions (2). By the existence and uniqueness theorem, both the solutions $y_1(x)$ and $y_2(x)$ are defined throughout the interval $[a, b]$ and are linearly independent. For, if they were linearly dependent, $y_1(x)$ would satisfy both the boundary conditions (2), and $\lambda = 0$ would be an eigenvalue, which contradicts our hypothesis. When $x \leqslant \xi$, the function $G(x, \xi)$ must be of the form $c_1 y_1(x)$, whilst with $x \geqslant \xi$, it must be of the form $c_2 y_2(x)$. It remains to choose the constants c_1 and c_2 so that the function is continuous at $x = \xi$, whilst the derivative has the required jump. This leads us to the following two equations for c_1 and c_2:

$$\left.\begin{aligned} c_1 y_1(\xi) - c_2 y_2(\xi) &= 0 \\ c_1 y_1'(\xi) - c_2 y_2'(\xi) &= \frac{1}{p(\xi)}\,. \end{aligned}\right\} \tag{6}$$

The determinant of this system $[y_2(\xi)y_1'(\xi) - y_1(\xi)y_2'(\xi)]$ differs from zero, since our solutions are linearly independent, and we therefore obtain definite values for constants c_1 and c_2. The Wronskian of our two solutions is easily seen [II, 24] to be given by

$$y_1(x)\,y_2'(x) - y_2(x)\,y_1'(x) = \frac{c}{p(x)}\,,$$

where c is a non-zero constant. On adding a constant factor, say to the solution $y_1(x)$, we can assume that our solutions satisfy the relationship:

$$p(x)\,[y_1(x)\,y_2'(x) - y_2(x)\,y_1'(x)] = 1\,.$$

It follows at once from this that system (6) has the solution: $c_1 = y_2(\xi)$ and $c_2 = y_1(\xi)$, and Green's function $G(x, \xi)$ is given by

$$G(x, \xi) = \begin{cases} y_1(x)\, y_2(\xi) & (x \leqslant \xi) \\ y_2(x)\, y_1(\xi) & (x \geqslant \xi). \end{cases} \tag{7}$$

It can easily be shown directly that it satisfies all four conditions. Its uniqueness is a direct consequence of the above discussion.

173. Reduction to an integral equation. We consider the non-homogeneous equation

$$L(y) = \frac{d}{dx}[p(x)\, y'] - q(x)\, y = -f(x), \tag{8}$$

where $f(x)$ is a given function continuous in the interval $[a, b]$. We shall seek the solution of (8) satisfying boundary conditions (2). Such a solution must be unique, since if there were two, their difference would satisfy the homogeneous equation $L(y) = 0$ and boundary conditions (2), i.e. $\lambda = 0$ would be an eigenvalue. Let us show that the unique solution of (8), satisfying boundary conditions (2), is given by

$$y(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi. \tag{9}$$

The analogue of this formula, mentioned in [1], had the simple mechanical meaning that, given the statical bending due to a concentrated force, we could obtain the statical bending due to a continuously distributed force by means of integration.

We turn to the proof that the function given by (9) satisfies (8) and conditions (2). In view of the above-mentioned discontinuity of Green's function, we divide the interval of integration into two parts:

$$y = \int_a^x G(x, \xi)\, f(\xi)\, d\xi + \int_x^b G(x, \xi)\, f(\xi)\, d\xi.$$

Differentiation with respect to x gives:

$$y' = \int_a^x G'(x, \xi)\, f(\xi)\, dx + G(x, x-0)\, f(x) + \\ + \int_x^b G'(x, \xi)\, f(\xi)\, d\xi - G(x, x+0)\, f(x)$$

or, by virtue of the continuity of Green's function, i.e. since $G(x, x + 0) = G(x, x - 0)$:

$$y' = \int_a^x G'(x, \xi) f(\xi) \, \mathrm{d}\xi + \int_x^b G'(x, \xi) f(\xi) \, \mathrm{d}\xi =$$

$$= \int_a^b G'(x, \xi) f(\xi) \, \mathrm{d}\xi. \tag{10}$$

It follows at once from (9) and (10) and the fact that $G(x, \xi)$ satisfies boundary conditions (2), that function (9) satisfies these boundary conditions. To verify equation (8), we differentiate y' again with respect to x. We obtain after simple transformations:

$$y'' = \int_a^b G''(x, \xi) f(\xi) \, \mathrm{d}\xi + \left[G'(x, x - 0) - G'(x, x + 0)\right] f(x),$$

and it follows from (5) that

$$y'' = \int_a^b G''(x, \xi) f(\xi) \, \mathrm{d}\xi - \frac{f(x)}{p(x)}. \tag{11}$$

On substituting in the left-hand side of (8) expressions (9), (10) and (11) for y, y' and y'' respectively, we obtain

$$\int_a^b L(G) f(\xi) \, \mathrm{d}\xi - f(x) = - f(x),$$

i.e. equation (8) is satisfied, since $G(x, \xi)$ is a solution of the homogeneous equation $L(y) = 0$. We remark further that it follows immediately from the above formulae that the function y given by (9) has continuous derivatives up to the second order throughout the interval. We thus arrive at the following proposition: *If $\lambda = 0$ is not an eigenvalue of differential equation* (8), *this equation has a solution satisfying boundary conditions* (2) *for any function $f(x)$ continuous in* $[a, b]$, *and this solution is given by* (9). We can say alternatively that: *Given any continuous function $f(x)$, function* (9) *has continuous derivatives up to the second order, and satisfies equation* (8) *and boundary conditions* (2).

We remark that, if $y(x)$ is any function having continuous derivatives up to the second order in the interval $[a, b]$ and satisfying boundary conditions (2), by substituting this function in the left-hand side of (8) we can construct the corresponding continuous function $f(x)$, and by what has been said, the function $y(x)$ will now be expressed in terms of $f(x)$ by (9).

Hence, formulae (8) and (9) establish a one-to-one correspondence between functions of two classes: the functions $y(x)$, having continuous derivatives up to the second order in $[a, b]$ and satisfying conditions (2), belong to the first class, and the functions $f(x)$, continuous in $[a, b]$, belong to the second. Transition from $y(x)$ to $f(x)$ is accomplished with the aid of formula (8), and from $f(x)$ to $y(x)$ with the aid of (9).

The above discussion leads directly to the possibility of reducing the boundary value problem stated at the beginning of this section to an integral equation. For, after writing equation (1) as:

$$L(y) = -\lambda r(x) y,$$

we find at once from the results obtained above that this equation, with boundary conditions (2), is equivalent to the integral equation:

$$y(x) = \lambda \int_a^b G(x, \xi) r(\xi) y(\xi) \, d\xi. \tag{12}$$

Similarly, the non-homogeneous equation:

$$\frac{d}{dx}[p(x) y'] + [\lambda r(x) - q(x)] y = F(x) \tag{12_1}$$

with boundary conditions (2) is equivalent to the integral equation:

$$y(x) = F_1(x) + \lambda \int_a^b G(x, \xi) r(\xi) y(\xi) \, d\xi, \tag{12_2}$$

where

$$F_1(x) = -\int_a^b G(x, \xi) F(\xi) \, d\xi$$

and a continuous solution $y(x)$ is to be sought in both integral equations.

174. Symmetry of Green's function. Formula (7) defines Green's functions at the ends $x = a$ and $x = b$ of the interval as well as for $a < x < b$, i.e. throughout the closed square k_0: $a \leqslant x, \ \xi \leqslant b$, and it follows at once from this formula that Green's function has the symmetry property:

$$G(x, \xi) = G(\xi, x) \tag{13}$$

throughout the square.

Another proof may be given of the symmetry of Green's function, based on an idea applicable in more general cases. The following identity is easily verified:

$$uL(v) - vL(u) = \frac{\mathrm{d}}{\mathrm{d}x}[p(x)(uv' - vu')]. \tag{14}$$

In this identity $u(x)$ and $v(x)$ are any two functions with continuous derivatives up to the second order. We substitute in (14): $u = G(x, \xi_1)$ and $v = G(x, \xi_2)$, where we assume for definiteness that $\xi_1 < \xi_2$. On integrating over the intervals $[a, \xi_1]$, $[\xi_1, \xi_2]$ and $[\xi_2, b]$ and using the fact that Green's function satisfies the homogeneous equation $L(y) = 0$, we get:

$$[p(x)(G(x, \xi_1)G'(x, \xi_2) - G(x, \xi_2)G'(x, \xi_1))]_{x=a}^{x=\xi_1} = 0$$
$$[p(x)(G(x, \xi_1)G'(x, \xi_2) - G(x, \xi_2)G'(x, \xi_1))]_{x=\xi_1}^{x=\xi_2} = 0$$
$$[p(x)(G(x, \xi_1)G'(x, \xi_2) - G(x, \xi_2)G'(x, \xi_1))]_{x=\xi_2}^{x=b} = 0.$$

On adding these three equations and using the continuity of Green's function itself and the discontinuity of its first derivative, we arrive at the following relationship:

$$G(\xi_1, \xi_2) - G(\xi_2, \xi_1) =$$
$$= [p(x)(G(x, \xi_1)G'(x, \xi_2) - G(x, \xi_2)G'(x, \xi_1))]_{x=a}^{x=b}. \tag{15}$$

It is easily seen that the difference on the right-hand side vanishes for $x = a$ and $x = b$. For, Green's function satisfies the first of boundary conditions (2), i.e.

$$\alpha_1 G(a, \xi_1) + \alpha_2 G'(a, \xi_1) = 0$$
$$\alpha_1 G(a, \xi_2) + \alpha_2 G'(a, \xi_2) = 0,$$

and since we naturally assume that the given constants α_1 and α_2 cannot vanish simultaneously, the determinant of this last homogeneous system must vanish, i.e. the difference in question in fact vanishes for $x = a$. It can similarly be shown that it vanishes for $x = b$ also, so that (15) in fact shows that Green's function is symmetrical.

Boundary conditions of a more general type than (2) can be considered, in which the values of the function and of its derivative at both ends of the interval appear in both conditions:

$$\alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'(b) = 0$$
$$\beta_1 y(a) + \beta_2 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = 0.$$

All the previous discussion, except for the proof of the symmetry of Green's function, retains its force, whilst the necessary and sufficient condition for the proof of the symmetry of Green's function to remain valid is that

$$p(b)\begin{vmatrix}\alpha_1, & \alpha_2\\ \beta_1, & \beta_2\end{vmatrix} = p(a)\begin{vmatrix}\alpha_3, & \alpha_4\\ \beta_3, & \beta_4\end{vmatrix}.$$

We shall not dwell on the proof of this assertion. It can easily be verified directly that the symmetry of Green's function is retained with purely periodic boundary conditions $y(a) = y(b)$; $y'(a) = y'(b)$, if $p(a) = p(b)$, i.e. if the function $p(x)$ is also periodic. We remark that, if the remaining coefficients $q(x)$ and $r(x)$ are periodic, the boundary value problem with the above-mentioned periodic boundary conditions amounts to seeking the values of the parameter λ for which equation (1) has a periodic solution.

175. The eigenvalues and eigenfunctions of a boundary value problem. Since we have reduced our boundary value problem to an integral equation, we can use the results of the general theory of integral equations and hence establish a number of propositions regarding the eigenvalues and eigenfunctions of the problem. We first take the case $r(x) \equiv 1$, when equation (1) becomes:

$$\frac{d}{dx}[p(x)y'] + (\lambda - q(x))y = 0, \tag{16}$$

the boundary conditions being assumed to be such that Green's function is symmetrical. Integral equation (12) will be an equation with a symmetrical kernel. It will have real eigenvalues and its eigenfunctions, corresponding to different eigenvalues, will be orthogonal. In the present case, as we saw in [172], for every eigenvalue there is a corresponding unique eigenfunction. We have proved this for boundary conditions of the form (2). In the case of periodic boundary conditions, two, but not more than two, eigenfunctions can correspond to an eigenvalue, since (16) has only two linearly independent solutions. We show further that the kernel $G(x, \xi)$ of equation (16) is a complete kernel, i.e. there exists no continuous function $f(x)$, orthogonal to the kernel and not identically zero. Suppose, on the contrary, that such a function exists:

$$\int_a^b G(x, \xi) f(\xi)\, d\xi = 0.$$

We now find that function (9) must vanish identically on the one hand, whilst on the other hand, by what has been proved above, it must satisfy non-homogeneous equation (8), which is impossible. As we know from [25], the fact that the kernel is complete implies the existence of an infinite set of eigenvalues. Let λ_n ($n = 1, 2, \ldots$) be the eigenvalues of (16), i.e. of our boundary value problem, and let $\varphi_n(x)$ be the corresponding eigenfunctions, forming an orthonormal system. Suppose that $f(x)$ satisfies the boundary conditions and has continuous derivatives up to the second order. On putting $L(f) = -h(x)$, we obtain an expression for this $f(x)$ in terms of the kernel:

$$f(x) = \int_a^b G(x, \xi)\, h(\xi)\, d\xi.$$

Thus every function satisfying the boundary conditions and having continuous derivatives up to the second order in the interval $[a, b]$, *can be expanded in this interval in a regularly convergent Fourier series in the eigenfunctions* $\varphi_n(x)$ [38]. The following theorem may also be proved easily:

THEOREM. *If the Fourier series of the continuous function* $f(x)$*:*

$$\sum_{n=1}^{\infty} c_n \varphi_n(x); \quad c_n = \int_a^b f(x)\, \varphi_n(x)\, dx \tag{17}$$

is uniformly convergent in the interval $[a, b]$, *its sum is equal to* $f(x)$.

We use *reductio ad absurdum.* Let $f_1(x)$ be the sum of series (17), and let $f_1(x)$ not be identically equal to $f(x)$ in $[a, b]$. The difference $f_1(x) - f(x)$, which is not identically zero, must now be orthogonal to all the $\varphi_n(x)$, and hence orthogonal to the kernel, which contradicts the completeness of the kernel. We shall use this theorem later.

It can be shown, not only that the kernel $G(x, \xi)$ is complete, but also that the *eigenfunctions* $\varphi_n(x)$ *form a closed system.* A direct consequence of this is the theorem just proved.

We shall give a proof below, when considering the n-dimensional case, that the closure equation holds for any continuous function. This proof will also be suitable for the one-dimensional case.

We now take the case when $r(x)$ differs from unity, but is positive. It may be seen by using the result of [32] that, in this case also, the boundary value problem for equation (1) leads to an integral equation with symmetric kernel. In particular, every function satisfying the boundary conditions and having continuous derivatives up to the second

order in $[a, b]$ can be expanded in a regularly convergent Fourier series in the eigenfunctions of the problem:

$$f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x), \tag{18}$$

the coefficients of which are given by

$$c_n = \int_a^b r(x) f(x) \varphi_n(x) \, dx. \tag{19}$$

To prove this assertion, we observe that, by what was said in [173]:

$$f(x) = -\int_a^b G(x, \xi) L[f(\xi)] \, d\xi.$$

But we can obviously write $L[f(\xi)] = -\sqrt{r(\xi)}\, h(\xi)$, where, since $r(\xi) > 0$, the function $h(\xi)$ is continuous in $[a, b]$. We therefore have an expression for the function $\sqrt{r(x)}\, f(x)$ in terms of the kernel of a symmetric integral equation:

$$\sqrt{r(x)}\, f(x) = \int_a^b G(x, \xi) \sqrt{r(x)\, r(\xi)}\, h(\xi) \, d\xi \tag{20}$$

and the discussion of [32] gives us at once the expansion theorem stated above. The closure of the kernel can be proved as above, and hence the existence of an infinite set of eigenvalues. On repeating the discussion of [172] for the case when $f(x)$ has a continuous first derivative and a piecewise continuous second derivative, and recalling that Theorem II of [22] still holds when the function is expressed in terms of a kernel with the aid of a piecewise continuous function $h(x)$, we can show that our expansion theorem still holds when the function $f(x)$ satisfying the boundary conditions has a continuous first derivative and a piecewise continuous second derivative. We shall indicate later the cases when piecewise continuity of the first derivative is also permissible in the statement of the expansion theorem.

176. The signs of the eigenvalues. We shall investigate the signs of the eigenvalues on the assumption that $r(x) \equiv 1$; this is for the sake of simplicity in later formulae. The whole of the discussion is easily extended to the general case. We first of all give an expression for the eigenvalues in terms of the corresponding eigenfunctions. As above,

let λ_n be the eigenvalues, and $\varphi_n(x)$ the eigenfunctions, forming an orthonormal system. We have:

$$L(\varphi_n) = -\lambda_n \varphi_n(x).$$

On multiplying both sides by $\varphi_n(x)$, integrating and taking into account the normalization of the eigenfunctions, we get:

$$\lambda_n = -\int_a^b L(\varphi_n)\varphi_n(x)\,\mathrm{d}x = -\int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}(p(x)\varphi_n') - q(x)\varphi_n\right]\varphi_n(x)\,\mathrm{d}x,$$

whence, by integrating the first term by parts, we arrive at the expression:

$$\lambda_n = \int_a^b [p(x)\varphi_n'^2(x) + q(x)\varphi_n^2(x)]\,\mathrm{d}x - [p(x)\varphi_n(x)\varphi_n'(x)]_{x=a}^{x=b}. \qquad (21)$$

We suppose that the term outside the integral vanishes here. This will be the case if say the boundary conditions are: $\varphi_n(a) = \varphi_n(b) = 0$. Expression (21) can now be rewritten as:

$$\lambda_n = \int_a^b [p(x)\varphi_n'^2(x) + q(x)\varphi_n^2(x)]\,\mathrm{d}x. \qquad (22)$$

Let $p(x) > 0$. If, in addition, we assume $q(x) \geqslant 0$ in $[a, b]$, it follows at once from this last equation that all the eigenvalues are positive. Now let $q(x)$ be an arbitrary continuous function, and m its minimum in the interval, i.e. $q(x) \geqslant m$ in $[a, b]$. It follows at once from (22) that:

$$\lambda_n \geqslant \int_a^b p(x)\varphi_n'^2(x)\,\mathrm{d}x + m \geqslant m.$$

Thus there can only be a finite number of negative eigenvalues in the present case. Now let the boundary conditions be:

$$y'(a) - h_1 y(a) = 0; \quad y'(b) + h_2 y(b) = 0, \qquad (23)$$

where h_1 and h_2 are positive constants. The term outside the integral in (21) is now positive, and it may be seen, as above, that all the eigenvalues are positive for boundary conditions (23) and $q(x) \geqslant 0$.

If all the eigenvalues are positive or if there exists only a finite number of negative eigenvalues, Mercer's theorem must hold, and we can write the expansion of the kernel in an absolutely and uniformly convergent series:

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}. \qquad (24)$$

This equation gives us a simple means of extending the theorem proved in [175] for an expansion in eigenfunctions to a more general class of functions. Suppose, in fact, that $f(x)$ is continuous, has a continuous derivative throughout the interval, except at the point $x = c$, at which it has a discontinuity of the first kind:

$$f'(c+0) - f'(c-0) = k,$$

and that a piecewise continuous second order derivative exists. Moreover, we assume as usual that $f(x)$ satisfies the boundary conditions. We form the difference:

$$f(x) - \frac{k}{p(c)} G(x, c),$$

which has a continuous derivative throughout the interval with no exceptions. The theorem on expansion in eigenfunctions holds for this difference. On the other hand, by (24), the subtrahend $G(x, c)$ can be expanded in eigenfunctions, so that the initial function $f(x)$ can be expanded in an absolutely and uniformly convergent Fourier series in eigenfunctions. These arguments evidently still hold when the derivative $f'(x)$ has a finite number of discontinuities of the first kind in $[a, b]$. They are similar to those which we used earlier when improving the convergence of Fourier series [II, 158].

177. Examples. 1. Let us take the equation

$$y'' + \lambda y = 0$$

and the boundary conditions $y(0) = y(1) = 0$. Here, $L(y) = y''$ and Green's function becomes:

$$G(x, \xi) = \begin{cases} (1-\xi)x & (x < \xi) \\ (1-x)\xi & (\xi < x). \end{cases}$$

The eigenvalues and eigenfunctions are given in the finite form:

$$\lambda_n = n^2\pi^2; \quad \varphi_n(x) = \sqrt{2}\sin n\pi x \qquad (n = 1, 2, \ldots),$$

and any functions satisfying the conditions indicated in the previous section can be expanded in these eigenfunctions. The conditions for the expansion theorem to hold can be substantially widened, but we shall not dwell on this.

2. We retain the previous differential equation and take new boundary conditions $y(0) = y'(1) = 0$. Here:

$$G(x, \xi) = \begin{cases} x & (x \leqslant \xi) \\ \xi & (\xi \leqslant x), \end{cases}$$

and the eigenvalues and eigenfunctions have the form:

$$\lambda_n = (2n+1)^2 \frac{\pi^2}{4}; \quad \varphi_n(x) = \sqrt{2} \sin (2n+1) \frac{\pi}{2} x.$$

3. We now consider the same equation with boundary conditions of the form $y(0) = 0$; $y(1) + hy'(1) = 0$.

We form the corresponding Green's function. We construct two solutions of the equation $y'' = 0$, one of which satisfies the first boundary condition, and the other the second: $y_1(x) = x$; $y_2(x) = (1+h) - x$. On arguing as in [172], we arrive at the following expression for Green's function:

$$G(x, \xi) = \begin{cases} \dfrac{1+h-\xi}{1+h} x & (x \leqslant \xi) \\[2ex] \dfrac{1+h-x}{1+h} \xi & (\xi \leqslant x). \end{cases}$$

Here, all the eigenvalues are positive, and on putting $\lambda = \mu^2$, it may easily be seen that the μ are given by the equation $\tan \mu + h\mu = 0$, whilst the eigenfunctions are $c_n \sin \mu_n x$, where the constants c_n have to be found from the condition for these functions to be normalized.

4. When investigating the vibrations of a circular membrane, fixed at the ends, we arrived at the following boundary value problem. To find the values of the parameter λ for which the equation

$$y'' + \frac{1}{x} y' + \left(\lambda - \frac{n^2}{x^2}\right) y = 0 \tag{25}$$

has a solution which is finite at the end $x = 0$ and zero at $x = l$. The letter n denotes a non-negative integer. This boundary value problem has a special feature as compared with those so far discussed, in that the coefficients of the equation have a pole at the end $x = 0$, and at this end, instead of a definite boundary condition, we establish merely the condition for the solution to be bounded in the neighbourhood of $x = 0$. This leads to a definite finite value for the solution of (25) at $x = 0$.

We have already encountered special boundary value problems of this type a number of times. On multiplying both sides of (25) by x, we can write the equation in the ordinary form:

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy') + \left(\lambda x - \frac{n^2}{x}\right) y = 0, \tag{26}$$

where we assume that the given n is positive.

The definition of Green's function remains as before, except that, instead of a boundary condition at the end $x = 0$, we require that Green's function be finite at $x = 0$. The equation $L(y) = 0$ becomes Euler's equation [II, 42] and has linearly independent solutions x^n and x^{-n}.

On taking into account the condition for finiteness at the end $x = 0$, we must take $c_1 x^n$ for Green's function in the interval $0 \leqslant x \leqslant \xi$, whilst in the interval $\xi \leqslant x \leqslant 1$ we have to form a linear combination of the above-mentioned solutions that vanishes for $x = 1$, i.e. we have to take $c_2(x^n - x^{-n})$ as Green's

function in this interval. The constants c_1 and c_2 are determined, as usual, from the conditions for continuity of Green's function and the jump of its first derivative at $x = \xi$. This gives us the following expression:

$$G(x, \xi) = \begin{cases} \frac{1}{2n}\left[\left(\frac{x}{\xi}\right)^n - (x\xi)^n\right] & (x < \xi) \\ \frac{1}{2n}\left[\left(\frac{\xi}{x}\right)^n - (x\xi)^n\right] & (\xi < x). \end{cases}$$

Precisely as above, the non-homogeneous equation $L(y) = -f(x)$ has a unique solution satisfying the above-mentioned boundary conditions, and this solution is given by

$$y(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi.$$

The arguments of [176] show that all the eigenvalues are positive. On putting $\lambda = \mu^2$, we obtain the transcendental equation $J_n(\mu) = 1$ for the eigenvalues, whilst the eigenfunctions are $\varphi_n(x) = c_n J_n(\mu_n, x)$. In the case $n = 0$, the equation $L(y) = 0$ has the linearly independent solutions $y_1(x) = 1$ and $y_2(x) = \log x$, whilst Green's function is given by

$$G(x, \xi) = \begin{cases} -\log \xi & (x < \xi) \\ -\log x & (\xi < x). \end{cases} \tag{27}$$

We remark that (9) gives us in the present case:

$$y(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi = -\log x \int_0^x f(\xi)\, d\xi - \int_x^1 f(\xi) \log \xi\, d\xi,$$

and it may be verified directly that this function satisfies the equation $L(y) = -f(x)$ and the boundary conditions. It follows at once from the form of equation (26) that we have $r(x) = x$ in the present case. On reducing our boundary value problem to an integral equation, we obtain the kernel $G(x, \xi)\sqrt{\xi x}$, which will be continuous throughout the square, including its corner $x = \xi = 0$.

The eigenfunctions of this integral equation are $\varphi_n(x) = c_n \sqrt{x} J_0(\mu_n x)$, and we have the expansion in an absolutely and uniformly convergent Fourier series:

$$G(x, \xi)\sqrt{\xi x} = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}.$$

We obtain after dividing by $\sqrt{\xi x}$:

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{c_n^2 J_0(\mu_n x) J_0(\mu_n \xi)}{\lambda_n}$$

and we can assert the uniform convergence of this series only in the interval $[\varepsilon, 1]$, where ε is any given positive number. The constants c_n are given, in view of the normalization condition, by the formula [III_2, 145]:

$$c_n^2 = \frac{2}{J_1(\mu_n)}.$$

We remark here that the function $G(x, \xi)$ defined by (27) tends to infinity when the point (x, ξ) tends to the corner (0, 0) of the square. In the present case, apart from the above-mentioned peculiarities, we have the feature that the function $r(x) = x$ vanishes at the end $x = 0$.

We shall discuss in detail in Vol. V boundary value problems for equations having singular points at the ends of the interval, and for equations in an infinite interval.

178. The generalized Green's function. We now turn to an investigation of the case when equation (1), with boundary conditions (2), has an eigenvalue $\lambda = 0$, i.e. the homogeneous equation $L(y) = 0$ has a solution $y = \varphi_0(x)$, satisfying the boundary conditions (2). This solution can be assumed normalized, and we shall make this assumption in what follows. In this case, we cannot construct Green's function satisfying all four conditions of [171], and we have to modify the definition. On retaining as before the conditions regarding continuity of the function itself, the discontinuity of its first derivative at $x = \xi$ and the satisfaction of the boundary conditions, we require that *the function $G(x, \xi)$ satisfy at every point of the intervals $[a, \xi]$ and $[\xi, b]$ an equation with a right-hand side instead of the homogeneous equation* $L(y) = 0$:

$$L[G(x, \xi)] = \varphi_0(\xi)\, \varphi_0(x). \tag{28}$$

If $y(x)$ is a solution of this equation satisfying the boundary conditions, the sum $y(x) + c\varphi_0(x)$, where c is an arbitrary constant, will also satisfy the equation, since $\varphi_0(x)$ satisfies the homogeneous equation and the boundary conditions. In order to determine the constant c, we further introduce a new auxiliary condition, that the functions $G(x, \xi)$ and $\varphi_0(x)$ be orthogonal:

$$\int_a^b [G(x, \xi)]\, \varphi_0(x)\, dx = 0. \tag{29}$$

The presence of a right-hand side in equation (28) has a simple physical interpretation. If $\lambda = 0$ is an eigenvalue of the problem, we have resonance at a frequency equal to zero, and a finite statical deviation cannot be obtained in the presence of a concentrated force. In order to obtain such a deviation, we must have a continuously distributed force in addition to the concentrated force, the former being characterized by the addition of the right-hand side in (28).

We shall construct the generalized Green's function along the same lines as in [172]. Let $\omega(x)$ be any solution of the non-homogeneous equation

$$L(\omega) = \varphi_0(\xi)\,\varphi_0(x) \tag{30}$$

and $\varphi_1(x)$ the solution of the corresponding homogeneous equation, linearly independent of $\varphi_0(x)$ and such that

$$p(x)\,[\varphi_0(x)\,\varphi_1'(x) - \varphi_1(x)\,\varphi_0'(x)] = 1\,. \tag{31}$$

On recalling that the general solution of a non-homogeneous equation is the sum of its solution $\omega(x)$ and the general solution of the homogeneous equation, we must put:

$$\begin{aligned} G(x;\xi) &= \omega(x) + c_1\varphi_0(x) + c_2\varphi_1(x) \qquad (x \leqslant \xi)\\ G(x;\xi) &= \omega(x) + c_3\varphi_0(x) + c_4\varphi_1(x) \qquad (x \geqslant \xi)\,. \end{aligned} \tag{32}$$

This function must satisfy boundary conditions (2). In view of the fact that $\varphi_0(x)$ satisfies these conditions, we get the two equations:

$$\begin{aligned} \alpha_1\omega(a) + \alpha_2\omega'(a) + c_2[\alpha_1\varphi_1(a) + \alpha_2\varphi_1'(a)] &= 0\\ \beta_1\omega(b) + \beta_2\omega'(b) + c_4[\beta_1\varphi_1(b) + \beta_2\varphi_1'(b)] &= 0, \end{aligned} \tag{33}$$

from which c_2 and c_4 are determined. The coefficients of c_2 and c_4 differ from zero, since $\varphi_1(x)$ is linearly independent of $\varphi_0(x)$ and cannot satisfy either of conditions (2) [172]. The condition for continuity at the point $x = \xi$ and the discontinuity of the derivative at this point lead to the following two equations:

$$\begin{aligned} (c_1 - c_3)\,\varphi_0(\xi) + (c_2 - c_4)\,\varphi_1(\xi) &= 0\\ (c_1 - c_3)\,\varphi_0'(\xi) + (c_2 - c_4)\varphi_1'(\xi) &= 1 : p(\xi), \end{aligned}$$

which, by (31), can be rewritten as

$$c_1 - c_3 = -\varphi_1(\xi)\,; \quad c_2 - c_4 = \varphi_0(\xi)\,. \tag{34}$$

It still remains to satisfy condition (29). The constants c_2 and c_4 are already determined by (33). The first of equations (34) gives: $c_1 = c_3 - \varphi_1(\xi)$. On substituting in the first of equations (32), we can find c_3 from condition (29) and c_1 is given by the formula just written. All the constants are now determined: but we have not taken into account the second of equations (34), and it remains for us to verify that c_2 and c_4, determined by (33), satisfy this equation.

For this, we write down formula (14):

$$\varphi_0(x) L(\omega) - \omega(x) L(\varphi_0) = \frac{\mathrm{d}}{\mathrm{d}x}[p(x)(\varphi_0 \omega' - \omega \varphi_0')].$$

We integrate both sides of this equation over the fundamental interval $[a, b]$. Using the fact that $L(\varphi_0) = 0$, equation (30), and the normalization of the function $\varphi_0(x)$, we obtain:

$$\varphi_0(\xi) = [p(x)(\varphi_0 \omega' - \omega \varphi_0')]_{x=a}^{x=b}. \tag{35}$$

The second of equations (34), which we have to verify, can be written, by (33), in the form:

$$\frac{\beta_1 \omega(b) + \beta_2 \omega'(b)}{\beta_1 \varphi_1(b) + \beta_2 \varphi_1'(b)} - \frac{\alpha_1 \omega(a) + \alpha_2 \omega'(a)}{\alpha_1 \varphi_1(a) + \alpha_2 \varphi_1'(a)} = \varphi_0(\xi). \tag{36}$$

We have the boundary conditions for $\varphi_0(x)$:

$$\alpha_1 \varphi_0(a) + \alpha_2 \varphi_0'(a) = 0; \quad \beta_1 \varphi_0(b) + \beta_2 \varphi_0'(b) = 0. \tag{37}$$

On writing (31) for $x = a$ and $x = b$, we can find $\varphi_0(a)$, $\varphi_0'(a)$, $\varphi_0(b)$ and $\varphi_0'(b)$ from the equations obtained and equations (37). On substituting the expressions obtained in equation (35), which is already proved, we arrive at (36). We carry out the working for the boundary conditions: $y(a) = y(b) = 0$, i.e. for the case when $\alpha_2 = \beta_2 = 0$. Equation (35) can now be rewritten as

$$\varphi_0(\xi) = p(a)\omega(a)\varphi_0'(a) - p(b)\omega(b)\varphi_0'(b).$$

With $x = a$ and $x = b$, (31) gives:

$$p(a)\varphi_1(a)\varphi_0'(a) = p(b)\varphi_1(b)\varphi_0'(b) = -1,$$

i.e.

$$p(a)\varphi_0'(a) = -\frac{1}{\varphi_1(a)}; \quad p(b)\varphi_0'(b) = -\frac{1}{\varphi_1(b)},$$

and substitution in the previous equations yields

$$\frac{\omega(b)}{\varphi_1(b)} - \frac{\omega(a)}{\varphi_1(a)} = \varphi_0(\xi),$$

which is in fact equation (36) in the case $\alpha_2 = \beta_2 = 0$.

The symmetry of the generalized Green's function can be proved by writing down the two equations:

$$L[G(x;\xi_1)] = \varphi_0(\xi_1)\varphi_0(x); \quad L[G(x;\xi_2)] = \varphi_0(\xi_2)\varphi_0(x).$$

We multiply the first by $G(x; \xi_2)$, the second by $G(x; \xi_1)$, subtract term by term and integrate over the basic interval. By using Green's

formula, the boundary conditions and condition (29), we arrive at the equation:

$$\left[p(x)\left(G(x;\xi_2)\,G'(x;\xi_1) - G(x;\xi_1)\,G'(x;\xi_2)\right)\right]_{x=\xi_1+0}^{x=\xi_1-0} + \left[\quad\right]_{x=\xi_2+0}^{x=\xi_2-0} = 0,$$

whence it follows at once that, as before, $G(\xi_1, \xi_2) = G(\xi_2; \xi_1)$. We remark that the interval has to be split into three parts when integrating over it, as was the case in [174].

We now return to a discussion of the non-homogeneous equation

$$L(y) = -f(x), \tag{38}$$

where $f(x)$ is a given continuous function, orthogonal to $\varphi_0(x)$. Equation (38) can only have one solution satisfying the boundary conditions and orthogonal to $\varphi_0(x)$. For, if there were two, their difference would have to satisfy the homogeneous equation and the boundary conditions, i.e. would have to have the form $c\,\varphi_0(x)$, i.e. could not be orthogonal to $\varphi_0(x)$. We now show that this unique solution, orthogonal to $\varphi_0(x)$, is given by

$$y(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi. \tag{39}$$

In fact, on splitting the interval of integration into $[a, x]$ and $[x, b]$, we can show as in [173] that

$$L(y) = \int_a^b L[G(x, \xi)]\, f(\xi)\, d\xi - f(x).$$

Using (28), we obtain from this:

$$L(y) = \varphi_0(x) \int_a^b \varphi_0(\xi)\, f(\xi)\, d\xi - f(x),$$

and (38) follows at once, since $f(x)$ is orthogonal to $\varphi_0(x)$ by hypothesis. Thus, if $f(x)$ is orthogonal to $\varphi_0(x)$, equation (38) has a unique solution satisfying boundary conditions (2) and orthogonal to $\varphi_0(x)$, and this solution is given by formula (39).

If $F(x)$ is any function orthogonal to $\varphi_0(x)$, satisfying the boundary conditions and having continuous derivatives up to the second order, on putting $f(x) = -L(F)$, we can express $F(x)$ by (39). This assertion can be proved simply by showing that the $f(x)$ constructed is orthogonal to $\varphi_0(x)$. For this, we write down Green's formula (14) for $u = \varphi_0(x)$ and $v = F(x)$. On recalling that $L(\varphi_0) = 0$ and using the boundary conditions for $\varphi_0(x)$ and $F(x)$, we can prove the orthogonality of $\varphi_0(x)$ and $f(x)$ by integrating Green's formula over the basic interval.

We remark further that (39) yields a function orthogonal to $\varphi_0(x)$ for any choice of continuous function $f(x)$, since the kernel $G(x, \xi)$ possesses this property.

We now return to the boundary value problem for the equation

$$L(y) = \frac{d}{dx}[p(x)y'] - q(x)y = -\lambda y \tag{40}$$

with boundary conditions (2). Every eigenfunction of this problem different from $\varphi_0(x)$, i.e. corresponding to a non-zero eigenvalue, must be orthogonal to $\varphi_0(x)$ and it may be seen, on taking into account everything that has been said above, that our boundary value problem (excluding the function $\varphi_0(x)$) is equivalent to the integral equation:

$$y(x) = \lambda \int_a^b G(x, \xi)\, y(\xi)\, d\xi. \tag{41}$$

We now turn to the theorem on an expansion in eigenfunctions for the equation written. We have to discuss the question of expressing a function in terms of a kernel. We saw above that every function having continuous derivatives up to the second order, satisfying the boundary conditions and orthogonal to $\varphi_0(x)$, can be written in terms of a kernel. Consequently, for every such function we have an absolutely and uniformly convergent Fourier expansion in eigenfunctions of equation (41). We remark that the extra condition of orthogonality of the required function to $\varphi_0(x)$ is necessary, since all the eigenfunctions of (41) are orthogonal to $\varphi_0(x)$. A direct consequence of this last fact is that the kernel of (41) is not complete. As usual, continuity of the second derivative can be replaced by piecewise continuity in the above-mentioned expansion theorem.

A second, more elementary approach may be mentioned to the case when $\lambda = 0$ is an eigenvalue. Equation (41) will have an eigenvalue with a minimum absolute value, say m. There will be the unique eigenvalue $\lambda = 0$ of our boundary problem inside the interval $[-m, +m]$. We take any non-zero value λ' inside this interval and replace λ in equation (40) by a new parameter μ, by putting $\lambda = \lambda' + \mu$. With the new choice of parameter, (16) takes the form:

$$\frac{d}{dx}[p(x)y'] + [\lambda' - q(x)]\,y = -\mu y,$$

where $\mu = 0$ will not be an eigenvalue, from what has been said. In view of this, our entire theory for ordinary Green's functions will hold. In particular, the eigenfunctions of the problem will form a closed system. Hence it follows at once that, if we associate

$\varphi_0(x)$ with the eigenfunctions of (41), a closed system will be obtained. As we shall see in a later example, the introduction of the new parameter may be accomplished by integration of the equation for the ordinary Green's function. We shall use the generalized Green's function in the next section to discuss the boundary value problem leading to Legendre polynomials. In this case the function $p(x)$ vanishes at both ends of the interval, and the role of boundary conditions is played by the requirement that the solution be finite at the ends of the interval. Everything said still holds in this case.

As we have seen, there can only be one eigenfunction corresponding to the eigenvalue $\lambda = 0$ for equation (1) with boundary conditions (2). For boundary conditions of a periodic type, say $y(a) = y(b)$ and $y'(a) = y'(b)$, there can in fact be two eigenfunctions. For equations of order higher than the second, which will be discussed below, there can also be more than one. A Green's function analogous to the above can be constructed in these cases. We must here have on the right-hand side of (28) a sum distributed over all the eigenfunctions corresponding to the eigenvalue $\lambda = 0$, these functions being assumed mutually orthogonal and normalized.

179. Legendre polynomials. Let us find the values of the parameter λ for which the equation

$$\frac{d}{dx}[(1-x^2)\,y'] + \lambda y = 0 \tag{42}$$

has a solution bounded at both ends of the interval $[-1, 1]$. We already know that the eigenvalues of this problem are $\lambda_n = n(n+1)$ [III_2, 102], whilst the orthogonal and normalized eigenfunctions are

$$\varphi_n(x) = \sqrt{\frac{2n+1}{2}}\,P_n(x) \qquad (n = 0, 1, \ldots), \tag{43}$$

where $P_n(x)$ are Legendre polynomials. It may easily be seen that there can be no other eigenvalues and eigenfunctions. If other eigenfunctions were to exist, we should have an eigenfunction orthogonal to all the functions (43), and we can show that there is no such function simply by showing that functions (43) form a closed system. Let us prove this. Let $f(x)$ be any given function continuous in $[-1, 1]$. By Weierstrass's theorem [II, 154], given any positive ε, a polynomial $Q(x)$ can be found such that $|f(x) - Q(x)| < \varepsilon$ throughout $[-1, 1]$, from which it follows immediately that

$$\int_{-1}^{1} [f(x) - Q(x)]^2\,dx < 2\varepsilon^2.$$

Let m be the degree of polynomial $Q(x)$. Since the function $\varphi_n(x)$ is a polynomial of degree precisely equal to n, we can write $Q(x)$ as a linear combination of

polynomials $\varphi_0(x), \ldots, \varphi_m(x)$, and the previous inequality can be written as

$$\int_{-1}^{1}\left[f(x)-\sum_{k=0}^{m} a_k \varphi_k(x)\right]^2 dx < 2\varepsilon^2.$$

If, instead of the coefficients a_k, we take the Fourier coefficients of $f(x)$ with respect to the system of functions (43), the last inequality is all the more satisfied.

On taking into account the arbitrary smallness of ε, we can say that the mean square error in representing $f(x)$ by a segment of its Fourier series in functions (43) tends to zero, i.e. functions (43) in fact form a closed system.

We return to equation (42). We have in this case:

$$L(y) = \frac{d}{dx}\left[(1-x^2)\, y'\right],$$

and it is immediately obvious that the first of functions (43), i.e. the constant $\varphi_0(x) = 1/\sqrt{2}$, satisfies the homogeneous equation $L(y) = 0$ and the boundary conditions, i.e. is bounded at the ends of the interval. In other words, $\lambda = 0$ is an eigenvalue, as also follows from the formula $\lambda_n = n(n+1)$ with $n = 0$. To construct Green's function, we write non-homogeneous equation (28), which here takes the form

$$\frac{d}{dx}\left[(1-x^2)\, y'\right] = \frac{1}{2}\,.$$

A particular solution of this equation is $y = -(1/4)\log(1-x^2)$, whilst the general solution of the corresponding homogeneous equation is $c_1 + c_2 \log[(1+x)/(1-x)]$. The solutions which remain finite at the ends $x = \pm 1$ have the respective forms:

$$y_1(x) = -\frac{1}{4}\log(1-x^2) + \frac{1}{4}\log\frac{1+x}{1-x} + \alpha = -\frac{1}{2}\log(1-x) + \alpha$$

$$y_2(x) = -\frac{1}{4}\log(1-x^2) - \frac{1}{4}\log\frac{1+x}{1-x} + \beta = -\frac{1}{2}\log(1+x) + \beta,$$

where α and β are constants. We choose these constants so that our solution is continuous at $x = \xi$ and is orthogonal to $\varphi_0(x) = 1/\sqrt{2}$. The first of these conditions gives:

$$-\frac{1}{2}\log(1-\xi) + \alpha = -\frac{1}{2}\log(1+\xi) + \beta,$$

and we can put

$$\alpha = -\frac{1}{2}\log(1+\xi) + \gamma; \qquad \beta = -\frac{1}{2}\log(1-\xi) + \gamma,$$

where γ is a constant which must be defined from the condition that Green's function $G(x, \xi)$ is orthogonal to $\varphi_0(x)$. We have:

$$G(x,\xi) = \begin{cases} -\dfrac{1}{2}\log[(1-x)(1+\xi)] + \gamma & (x \leqslant \xi), \\[2ex] -\dfrac{1}{2}\log[(1+x)(1-\xi)] + \gamma & (\xi \leqslant x). \end{cases}$$

The orthogonality condition:

$$\int_{-1}^{1} G(x, \xi)\,\varphi_0(x)\,\mathrm{d}x = 0, \quad \text{or simply} \quad \int_{-1}^{1} G(x, \xi)\,\mathrm{d}x = 0,$$

gives us the following value for the constant γ: $\gamma = 1/2 - \log 2$; hence the generalized Green's function is finally given by:

$$G(x, \xi) = \begin{cases} -\dfrac{1}{2}\log\left[(1-x)(1+\xi)\right] - \log 2 + \dfrac{1}{2} & (x \leqslant \xi) \\ -\dfrac{1}{2}\log\left[(1+x)(1-\xi)\right] - \log 2 + \dfrac{1}{2} & (x \geqslant \xi) \end{cases} \tag{44}$$

Kernel (44) becomes unbounded in the neighbourhood of the corners $x = \xi = -1$ and $x = \xi = 1$ of the basic square. It is easily shown that every function expressible in terms of the kernel:

$$\int_{-1}^{1} G(x, \xi)\,g(\xi)\,\mathrm{d}\xi \tag{45}$$

is continuous if $g(x)$ is continuous, and we have the same theorem as in [174] regarding the expansion of such functions in an absolutely and uniformly convergent Fourier series in functions $\varphi_n(x)$ $(n = 1, 2, \ldots)$. Every function $f(x)$ having continuous derivatives up to the second order in $[-1, 1]$ and satisfying the condition

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 0, \tag{46}$$

which expresses the orthogonality of $f(x)$ and $\varphi_0(x)$, can be expressed in terms of a kernel in accordance with (45) and can be expanded in an absolutely and uniformly convergent Fourier series in functions $\varphi_n(x)$ $(n = 1, 2, \ldots)$, i.e. in Legendre polynomials $P_n(x)$ $(n = 1, 2, \ldots)$. If $f(x)$ does not satisfy condition (46), it is sufficient to apply the general expansion theorem to the function

$$f_1(x) = f(x) - \tfrac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x,$$

which satisfies the required condition. We obtain for the original function $f(x)$ an expansion in all the Legendre polynomials, including $P_0(x) = \text{const}$.

The Fourier series for the kernel becomes in the present case

$$\sum_{n=1}^{\infty} \frac{(2n+1)\,P_n(x)\,P_n(\xi)}{2n\,(n+1)}\,. \tag{47}$$

It cannot be uniformly convergent throughout the square k_0, since the kernel is unbounded. We use the asymptotic expression for the Legendre polynomials with large values of n, [III_2, 163]:

$$P_n(\cos t) = \sqrt{\frac{2}{n\pi \sin t}}\left\{\cos\left[\left(n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + \delta_n\right\},$$

where $\delta_n \to 0$ uniformly with respect to t, if t belongs to the interval $[\varepsilon, \pi - \varepsilon]$, where ε is any given positive number. We choose some value of ξ inside $[-1, 1]$. For $P_n(\xi)$ we have an asymptotic inequality of the form $|P_n(\xi)| \leqslant m_n/\sqrt{n}$, where m_n remains bounded as n increases. We have $|P_n(x)| \leqslant 1$ for any x satisfying the condition $-1 \leqslant x \leqslant 1$, [III, 132]. Hence it is clear that, given a fixed ξ, series (47) is absolutely and uniformly convergent with respect to x in $[-1, 1]$. Function (44) is orthogonal to $\varphi_0(x)$, so that series (47) is its Fourier series with respect to the closed system of functions (43). It follows from the uniform convergence of the series that its sum is equal to kernel (44) [3]. It also follows at once from the previous discussion that series (47) is absolutely and uniformly convergent in the square k_0 if we exclude from the square its corners $(-1, -1)$ and $(1, 1)$ by small circles with centres at these corners and as small a radius as desired.

We now use the other approach to the boundary value problem for equation (42), mentioned in the previous section. We replace λ by the new parameter μ in accordance with the formula $\lambda = \mu + p(p + 1)$, where p is a fixed non-integral number. Equation (42) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\, y'\right] + p\,(p + 1)\, y + \mu y = 0.$$

The value $\mu = 0$ will no longer be an eigenvalue, whilst we must put

$$L\,(y) = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\, y'\right] + p\,(p + 1)\, y.$$

If we replace x by the new variable $t = (1 + x)/2$, the equation $L(y) = 0$ becomes Gauss's equation [III_2, 100, 101] with parameters $\alpha = -p$, $\beta = p + 1$, $\gamma = 1$. We have two solutions of this equation:

$$y_1\,(x) = F\left(-p,\, p + 1,\, 1;\; \frac{1 + x}{2}\right); \quad y_2\,(x) = cF\left(-p,\, p + 1,\, 1;\; \frac{1 - x}{2}\right),$$

the first of which is regular for $x = -1$, and the second for $x = 1$. A constant C can be chosen so that

$$y_1'\,(x)\, y_2\,(x) - y_2'\,(x)\, y_1(x) = \frac{1}{1 - x^2}\,.$$

It can be shown that this gives $c = \pi/(4 \sin p\pi)$, so that the ordinary Green's function is defined by

$$G_1\,(x, \xi) = \frac{\pi F\left(-p,\, p + 1,\, 1;\; \dfrac{1 + x}{2}\right) F\left(-p,\, p + 1,\, 1;\; \dfrac{1 - \xi}{2}\right)}{4 \sin p\pi}\ (x \leqslant \xi). \quad (48)$$

When $\xi \leqslant x$, the letters x and ξ must be interchanged. Due to the change in parameter, the eigenvalues will be given by $\mu_n = n(n + 1) - p(p + 1)$, whilst the eigenfunctions will be $\varphi_n(x)$ as before. The Fourier series for kernel (48) becomes in the present case:

$$G_1\,(x, \xi) = \sum_{n=0}^{\infty} \frac{(2n + 1)\, P_n\,(x)\, P_n\,(\xi)}{2\,[n\,(n + 1) - p\,(p + 1)]}\,,$$

and it will give Green's function (48), as above, for any fixed ξ inside $[-1, 1]$ and any x of this interval. We remark that kernel (48) is here unbounded.

180. Hermite and Laguerre functions. A Green's function can also be constructed for boundary value problems leading to Hermite and Laguerre functions.

The Hermite functions $\psi_n(x)$ [III_2, 156] are the eigenfunctions of the equation

$$y'' + (\lambda - x^2)\, y = 0$$

with the basic interval $(-\infty, +\infty)$ and the condition that $y \to 0$ as $x \to -\infty$ and $x \to +\infty$. The eigenvalues are: $\lambda_n = 2n + 1$ $(n = 0, 1, \ldots)$. On replacing λ by $(\lambda - 1)$, we can rewrite the equation as

$$L(y) + \lambda y = 0, \qquad \text{where} \qquad L(y) = y'' - (1 + x^2)\, y,$$

the eigenvalues being now given by: $\lambda_n = 2n + 2$ $(n = 0, 1, \ldots)$. The equation

$$L(y) = y'' - (1 + x^2)\, y = 0$$

has the solution $y = e^{x^2/2}$, and if we replace y by the new unknown w given by $y = we^{x^2/2}$, the general solution can be obtained at once as

$$y = C_1\, e^{\frac{x^2}{2}} \int_{C_2}^{x} e^{-v^2}\, dv\,,$$

where C_1 and C_2 are arbitrary constants. When $x \leqslant \xi$, we have to take the solution that vanishes with $x = -\infty$:

$$y_1 = a e^{\frac{x^2}{2}} \int_{-\infty}^{x} e^{-v^2}\, dv,$$

where a is a constant. When $x \geqslant \xi$, we similarly take the solution

$$y_2 = b e^{\frac{x^2}{2}} \int_{x}^{+\infty} e^{-v^2}\, dv\,,$$

where b is a new constant. These constants are determined from the condition that $G(x, \xi)$ be continuous at $x = \xi$, and from the jump of the derivative $G'(x, \xi)$ at $x = \xi$. We finally obtain:

$$G(x, \xi) = \begin{cases} \dfrac{1}{\sqrt{\pi}}\, e^{\frac{x^2+\xi^2}{2}} \displaystyle\int_{-\infty}^{x} e^{-v^2}\, dv \int_{\xi}^{+\infty} e^{-t^2}\, dt & (x \leqslant \xi) \\[2ex] \dfrac{1}{\sqrt{\pi}}\, e^{\frac{x^2+\xi^2}{2}} \displaystyle\int_{-\infty}^{\xi} e^{-v^2}\, dv \int_{x}^{+\infty} e^{-t^2}\, dt & (x \geqslant \xi). \end{cases}$$

The Laguerre functions $\omega_n(x)$ (cf. [III_2, 160] with $s = 0$) are the eigenfunctions of the equation

$$xy'' + y' + \left(\lambda - \frac{x}{4}\right) y = 0$$

with the basic interval $(0, +\infty)$ and the condition that the solution be bounded in the neighbourhood of $x = 0$ and vanish at $x = +\infty$. The eigenvalues are $\lambda_n = n + 1/2$. On replacing λ by $(\lambda - 1/2)$, we can rewrite the equation as

$$L(y) + \lambda y = 0, \quad \text{where} \quad L(y) = xy'' + y' - \left(\frac{1}{2} + \frac{x}{4}\right) y\,;$$

the eigenvalues become $\lambda_n = n + 1$ $(n = 0, 1, \ldots)$. The equation

$$L(y) = xy'' + y' - \left(\frac{1}{2} + \frac{x}{4}\right) y = 0$$

has the solution $y = e^{x/2}$, and on substituting for the unknown $y = we^{x/2}$, we can find the general solution as

$$v = C_1 e^{\frac{x}{2}} \left(\int\limits_{+\infty}^{x} \frac{e^{-v}}{v} dv + C_2 \right).$$

When $x \leqslant \xi$, we have to take the solution which is regular at $x = 0$:

$$y_1 = ae^{\frac{x}{2}},$$

and when $x \geqslant \xi$, the solution which vanishes at $x = +\infty$:

$$y_2 = be^{\frac{x}{2}} \int\limits_{x}^{+\infty} \frac{e^{-v}}{v} dv.$$

On determining a and b as above, we finally obtain:

$$G(x, \xi) = \begin{cases} e^{\frac{x+\xi}{2}} \displaystyle\int\limits_{\xi}^{\infty} \frac{e^{-v}}{v} dv & (x < \xi) \\ e^{\frac{x+\xi}{2}} \displaystyle\int\limits_{x}^{\infty} \frac{e^{-v}}{v} dv & (x > \xi). \end{cases}$$

181. Equations of the fourth order. The concept of Green's functions and of the reduction of a problem to an integral equation can be extended on the same lines to higher order equations. When considering the vibrations of a rod, we obtained the following boundary value problem: to find the values of the parameter λ for which the equation

$$y^{(\mathrm{iv})} - \lambda y = 0 \tag{49}$$

has a non-zero solution in the case of four homogeneous boundary conditions. If, say, the rod is clamped at the end $x = 0$ and free at the end $x = l$, we obtain the boundary conditions:

$$y|_{x=0} = y'|_{x=0} = 0; \quad y''|_{x=l} = y'''|_{x=l} = 0. \tag{50}$$

The equation for a non-homogeneous rod is

$$y^{(\mathrm{iv})} - \lambda r(x)\, y = 0. \tag{51}$$

Green's function $G(x, \xi)$ will correspond to the statical bending of the rod under the action of a concentrated force. It is defined by the following conditions: (1) it is continuous together with its first two derivatives with respect to x in the square k_0; (2) for $0 < x < \xi$ and $\xi < x < l$ it has continuous derivatives up to the fourth order and satisfies the homogeneous equation $G^{(\mathrm{iv})}(x, \xi) = 0$; (3) for any values of ξ in the interval $[0, l]$ it satisfies the boundary conditions; (4) on the diagonal of the rectangle its third derivative has a jump defined by the condition

$$G'''(\xi + 0, \xi) - G'''(\xi - 0, \xi) = -1. \tag{52}$$

If $y(x)$ is a function with continuous derivatives up to the fourth order, where the fourth order derivative may only be piecewise continuous, and $y(x)$ satisfies boundary conditions (50), it follows from $y^{(\mathrm{iv})} = -f(x)$ that

$$y(x) = \int_0^l G(x, \xi)\, f(\xi)\, d\xi, \tag{53}$$

and conversely, the function given by this last equation has continuous derivatives up to the fourth order and satisfies the boundary conditions and the equation $y^{(\mathrm{iv})} = -f(x)$. Thus the boundary value problem for equation (49) leads to the integral equation

$$y(x) = -\lambda \int_0^l G(x, \xi)\, y(\xi)\, d\xi,$$

and for equation (51) to the integral equation

$$y(x) = -\lambda \int_0^l G(x, \xi)\, r(\xi)\, y(\xi)\, d\xi.$$

The eigenfunctions form a closed system in this case, as previously, and any function satisfying the boundary conditions and having continuous derivatives up to the fourth order can be expanded in an absolutely and uniformly convergent Fourier series in the eigen functions. Precisely as in [176], it can be shown that all the eigenvalues are positive, so that, by Mercer's theorem, we have an expansion of the kernel itself in eigenfunctions.

Suppose we in fact construct Green's function for the case of a rod clamped at both ends, i.e. with the boundary conditions $y(0) = y'(0) = y(1) = y'(1) = 0$, where we assume $r(x) = 1$ and $l = 1$. The general solution of the equation $y^{(iv)} = 0$ is a third degree polynomial with arbitrary coefficients. We can at once write down solutions satisfying the boundary conditions only at the left-hand end and only at the right-hand end. These solutions are:

$$y_1(x) = x^2 (a_1 + a_2 x); \quad y_2(x) = (x-1)^2 (b_1 + b_2 x).$$

The arbitrary constants are determined from four conditions, namely from the continuity of the function and of its derivatives up to the second order at $x = \xi$, and from the discontinuity (52) of the third derivative. We arrive by elementary working at the following expression for Green's function in the present case:

$$G(x, \xi) = \frac{x^2 (\xi - 1)^2}{6} (2x\xi + x - 3\xi) \quad (x \leqslant \xi).$$

When $\xi \leqslant x$ we have to interchange x and ξ.

182. Steklov's stricter expansion theorems.

We obtained in [175] a theorem on the expansion in eigenfunctions $\varphi_n(x)$ of equation (16). The boundary conditions are taken as

$$y(a) = y(b) = 0. \tag{54}$$

Steklov has published theorems on expansions in functions $\varphi_n(x)$ with very wide conditions, independently of the theory of integral equations. The results of relevance here are collected in his book *Fundamental Problems of Mathematical Physics* (Osnovnye zadachi matematicheskoi fiziki), vol. I (1922). We shall quote certain of his results.

Let us consider equation (16); in addition to the assumptions mentioned above, we further assume that $q(x) \geqslant 0$. Also, *let $f(x)$ be a continuous function having a continuous derivative in the interval $[a, b]$ and satisfying boundary conditions* (54). We shall not assume the existence of the second order derivative. As a preliminary, we show that

$$\int_a^b [p(x)\varphi_k'(x)\varphi_l'(x) + q(x)\varphi_k(x)\varphi_l(x)]\,dx = 0 \quad \text{for} \quad k \neq l. \tag{55}$$

For, on integrating by parts and using the equation which is satisfied by the eigenfunctions:

$$q(x)\varphi_k(x) - \frac{d}{dx}[p(x)\varphi_k'(x)] = \lambda_k \varphi_k(x), \tag{56}$$

we get

$$\int_a^b [p(x)\varphi_k'(x)\varphi_l'(x) + q(x)\varphi_k(x)\varphi_l(x)]\,dx =$$
$$= p(x)\varphi_k'(x)\varphi_l(x)\Big|_{x=a}^{x=b} + \lambda_k \int_a^b \varphi_k(x)\varphi_l(x)\,dx.$$

But the term outside the integral vanishes, since $\varphi_l(a) = \varphi_l(b) = 0$, and the last integral vanishes by virtue of the orthogonality of the eigenfunctions. We now consider the functional:

$$J(y) = \int_a^b [p(x)y'^2 + q(x)y^2]\,dx \tag{57}$$

and substitute in it:

$$y = r_n(x) = f(x) - \sum_{k=1}^{n-1} c_k\varphi_k(x), \tag{58}$$

where the c_k are the Fourier coefficients of $f(x)$:

$$c_k = \int_a^b f(x)\varphi_k(x)\,dx. \tag{59}$$

On removing the brackets and taking (22) and (55) into account, we get

$$J\left[f(x) - \sum_{k=1}^{n-1} c_k\varphi_k(x)\right] = \int_a^b [p(x)f'^2(x) + q(x)f^2(x)]\,dx + \sum_{k=1}^{n-1}\lambda_k c_k^2 -$$
$$- 2\sum_{k=1}^{n-1} c_k \int_a^b [p(x)f'(x)\varphi_k'(x) + q(x)f(x)\varphi_k(x)]\,dx.$$

On integrating by parts in the last integral and using (56) and the fact that $f(a) = f(b) = 0$ by hypothesis, we obtain

$$J\left[f(x) - \sum_{k=1}^{n-1} c_k\varphi_k(x)\right] =$$
$$= \int_a^b [p(x)f'^2(x) + q(x)f^2(x)]\,dx - \sum_{k=1}^{n-1}\lambda_k c_k^2. \tag{60}$$

If it is assumed that $q(x) \geqslant 0$ as well as $p(x) > 0$, an inequality analogous to Bessel's inequality follows at once from (60):

$$\sum_{k=1}^{\infty}\lambda_k c_k^2 \leqslant \int_a^b [p(x)f'^2(x) + q(x)f^2(x)]\,dx, \tag{61}$$

and the convergence of the series on the left follows. All the terms of this series are positive, since $\lambda_k > 0$ for $q(x) \geqslant 0$.

We remark that the proof of (61) is entirely unchanged if we assume that the continuous function $f(x)$ has a derivative $f'(x)$ everywhere in $[a, b]$, except at a finite number of points $a_1, a_2, \ldots, a_m$, the derivative being continuous everywhere except at these points, at which it has finite limits from the left and right (discontinuities of the first kind). When integrating by parts, it is sufficient to integrate over the intervals in which $f'(x)$ is continuous and then add all these integrals.

We now show that, with the assumptions made above regarding $f(x)$, the Fourier series of the function

$$\sum_{k=1}^{\infty} c_k \varphi_k(x) \tag{62}$$

is regularly convergent in $[a, b]$, i.e. the series

$$\sum_{k=1}^{\infty} | c_k \varphi_k(x) | \tag{63}$$

is uniformly convergent in this interval. On using the integral equation

$$\varphi_k(x) = \lambda_k \int_a^b G(x, \xi) \varphi_k(\xi) \, d\xi, \tag{64}$$

we can rewrite (63) as

$$\sum_{k=1}^{\infty} \lambda_k | c_k \psi_k(x) |, \tag{65}$$

where

$$\psi_k(x) = \int_a^b G(x, \xi) \varphi_k(\xi) \, d\xi \tag{66}$$

can be regarded as the Fourier coefficients of $G(x, \xi)$ as a function of ξ. On using inequality (61), we can write:

$$\sum_{k=1}^{\infty} \lambda_k \psi_k^2(x) \leqslant \int_a^b [p(\xi) G_\xi^2(x, \xi) + q(\xi) G^2(x, \xi)]^2 \, d\xi, \tag{67}$$

where $G_\xi(x, \xi)$ is the derivative of $G(x, \xi)$ with respect to ξ. All the functions appearing under the integral sign are bounded, and it follows from (67) that

$$\sum_{k=1}^{\infty} \lambda_k \psi_k^2(x) \leqslant M, \tag{68}$$

where M is a constant. On replacing λ_k by $\sqrt{\lambda_k}\sqrt{\lambda_k}$, we apply Cauchy's inequality to the segment of series (65):

$$\sum_{k=m}^{m+p} \lambda_k \left| c_k \psi_k(x) \right| \leqslant \sqrt{\sum_{k=m}^{m+p} \lambda_k c_k^2} \sqrt{\sum_{k=m}^{m+p} \lambda_k \psi_k^2(x)}$$

or

$$\sum_{k=m}^{m+p} \lambda_k \left| c_k \psi_k(x) \right| \leqslant \sqrt{\sum_{k=m}^{m+p} \lambda_k c_k^2} \sqrt{M},$$

and this inequality, together with the convergence of the series with terms $\lambda_k c_k^2$, at once implies that series (65) is uniformly convergent in the interval $[a, b]$, i.e. series (62) is regularly convergent. This implies at once that its sum is equal to $f(x)$ [3].

A proof may also be given of the expansion theorem, with the above assumptions regarding $f(x)$, but without the assumption that $q(x) \geqslant 0$. It is also due to Steklov. Using the notation of (58), we show first of all that there exists a constant C (independent of n) such that

$$\sigma_n = \int_a^b p(x)\, r_n'^2(x)\, dx \leqslant C. \tag{69}$$

We have:

$$\sigma_n = \int_a^b p(x) \left[f'(x) - \sum_{k=1}^{n-1} a_k \varphi_k'(x) \right]^2 dx =$$

$$= \int_a^b p(x) \left[f'(x) - \sum_{k=1}^{n-1} a_k \varphi_k'(x) \right] r_n'(x)\, dx =$$

$$= \int_a^b p(x) f'(x) r_n'(x)\, dx - \sum_{k=1}^{n-1} a_k \int_a^b p(x) \varphi_k'(x) r_n'(x)\, dx.$$

On integrating the last integral by parts and taking into account (56) and the orthogonality of the $r_n(x)$ to the functions $\varphi_k(x)$ $(k = 1, 2, \ldots, n-1)$, we obtain:

$$\sigma_n = \int_a^b p(x) f'(x) r_n'(x)\, dx + \int_a^b q(x) r_n(x) f(x)\, dx - \int_a^b q(x) r_n^2(x)\, dx,$$

whence, on writing q_0 for the maximum of $|q(x)|$ in $[a, b]$ and using Bunyakovskii's inequality, with $p(x)$ replaced in the first integral by

$\sqrt{p(x)}\sqrt{p(x)}$, we get:

$$\sigma_n \leqslant \sqrt{\int_a^b p(x) f'^2(x)\,dx}\sqrt{\sigma_n} + \\ + q_0\sqrt{\int_a^b f^2(x)\,dx}\sqrt{\int_a^b r_n^2(x)\,dx} + q_0\int_a^b r_n^2(x)\,dx.$$

We have by the closure equation:

$$\lim_{n\to\infty}\int_a^b r_n^2(x)\,dx = 0,$$

so that we obtain for σ_n an inequality of the form:

$$\sigma_n \leqslant c_1\sqrt{\sigma_n} + c_2,$$

where c_1 and c_2 are positive constants. It is clear from this inequality that σ_n remains bounded as n increases, and we obtain (69).

Further, it follows from

$$\int_\xi^x \frac{d}{dt} r_n^2(t)\,dt = r_n^2(x) - r_n^2(\xi)$$

that

$$r_n^2(x) = r_n^2(\xi) + 2\int_\xi^x r_n(t)\,r_n'(t)\,dt,$$

whence, on applying Buniakowski's inequality and assuming $\xi < x$, we obtain:

$$r_n^2(x) \leqslant r_n^2(\xi) + 2\sqrt{\int_\xi^x r_n^2(t)\,dt}\sqrt{\int_\xi^x r_n'^2(t)\,dt} \leqslant \\ \leqslant r_n^2(\xi) + 2\sqrt{\int_a^b r_n^2(t)\,dt}\cdot\sqrt{\int_a^b r_n'^2(t)\,dt}.$$

When $x < \xi$, we have to interchange the limits of integration ξ and x. On integrating both sides with respect to ξ over $[a, b]$, we get:

$$(b-a)\,r_n^2(x) \leqslant \int_a^b r_n^2(\xi)\,d\xi + 2(b-a)\sqrt{\int_a^b r_n^2(t)\,dt}\cdot\sqrt{\int_a^b r_n'^2(t)\,dt}.$$

On writing p_0 for the minimum value of the positive function $p(x)$ in $[a, b]$, we can write, by (69):

$$\int_a^b r_n'^2(t)\,\mathrm{d}t \leqslant \frac{1}{p_0}\int_a^b p(t)\,r_n'^2(t)\,\mathrm{d}t \leqslant \frac{C}{p_0},$$

and the previous inequality gives:

$$r_n^2(x) \leqslant \frac{1}{b-a}\int_a^b r_n^2(t)\,\mathrm{d}t + 2\sqrt{\frac{C}{p_0}}\sqrt{\int_a^b r_n^2(t)\,\mathrm{d}t}.$$

The right-hand side does not depend on x and tends to zero as n increases indefinitely, whence it follows that $r_n(x) \to 0$ uniformly in $[a, b]$, i.e. series (62) is uniformly convergent in this interval and its sum is equal to $f(x)$. Series (62) can be shown to be regularly convergent without the assumption that $q(x) \geqslant 0$.

183. The justification of Fourier's method for the equation of heat conduction. Let us consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] - q(x)\,u, \tag{70}$$

which corresponds to the propagation of heat in a non-homogeneous rod, taking into account the radiation from the rod surface. We take $a \leqslant x \leqslant b$ and seek the solution of (70) with the initial condition

$$u|_{t=0} = f(x) \qquad (a \leqslant x \leqslant b) \tag{71}$$

and the boundary conditions:

$$u|_{x=a} = 0; \quad u|_{x=b} = 0. \tag{72}$$

On using Fourier's method, we obtain the solution of the problem as

$$u(x, t) = \sum_{k=1}^{\infty} c_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(x), \tag{73}$$

where λ_k and $\varphi_k(x)$ are the eigenvalues and eigenfunctions of the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[p(x)\,y'] + [\lambda - q(x)]\,y = 0 \tag{74}$$

with the boundary conditions

$$y(a) = y(b) = 0, \tag{75}$$

and c_k are the Fourier coefficients (59) of $f(x)$. We shall assume that $q(x) \geqslant 0$ and that $f(x)$ has a continuous derivative in $[a, b]$ and satisfies boundary conditions (72). We remark that all the λ_k are positive, since $q(x) \geqslant 0$. We show that function (73) satisfies all the conditions of the problem, i.e. satisfies (71), (72), and equation (70) with $t > 0$.

We have shown that series (73) is regularly convergent in $[a, b]$. On using the fact that the $\lambda_k > 0$, we can assert that series (73) is absolutely and uniformly convergent for $t \geqslant 0$ and $a \leqslant x \leqslant b$. Its sum is therefore a continuous function for these values of the arguments, i.e.

$$\lim_{t \to +0} u(x, t) = u(x, 0) = \sum_{k=1}^{\infty} c_k \varphi_k(x) = f(x).$$

This proves that the initial condition (71) is fulfilled. The boundary conditions (72) are fulfilled by virtue of the fact that all the functions $\varphi_k(x)$ satisfy conditions (72). It remains to verify equation (70) with $t > 0$. Each term of series (73) satisfies equation (70) by virtue of its construction, and we only have to show that series (73) can be differentiated term by term once with respect to t and twice with respect to x, i.e. we only need to show that the series

$$\sum_{k=1}^{\infty} \lambda_k c_k e^{-\lambda_k t} \varphi_k(x); \qquad (76_1) \qquad\qquad \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} \varphi_k'(x); \qquad (76_2)$$

$$\sum_{k=1}^{\infty} c_k e^{-\lambda_k t} \varphi_k''(x) \qquad (76_3)$$

are uniformly convergent for $t \geqslant \alpha$, where α is any positive number, and for $a \leqslant x \leqslant b$. Since $\lambda_k \to +\infty$ as $k \to +\infty$, we have $\lambda_k e^{-\lambda_k \alpha} \to 0$ and $\lambda_k e^{-\lambda_k t} \leqslant \lambda_k e^{-\lambda_k \alpha}$ for $t \geqslant \alpha$, i.e. there exists an N (independent of t) such that $\lambda_k e^{-\lambda_k t} < 1$ for $t \geqslant \alpha$ and $k \geqslant N$. Hence, on taking into account the uniform convergence of series (63), we find that series (76_1) is uniformly convergent for $t \geqslant \alpha$ and $a \leqslant x \leqslant b$.

We can similarly prove that series (73) can be differentiated term by term with respect to t any number of times for $t > 0$. We investigate the next series by writing down expression (64) for the $\varphi_k(x)$, using (7):

$$\varphi_k(x) = \lambda_k y_1(x) \int_a^x y_2(\xi) \varphi_k(\xi) \, d\xi + \lambda_k y_2(x) \int_x^b y_1(\xi) \varphi_k(\xi) \, d\xi,$$

whence

$$\varphi_k'(x) = \lambda_k y_1'(x) \int_a^x y_2(\xi)\,\varphi_k(\xi)\,\mathrm{d}\xi + \lambda_k y_2'(x) \int_x^b y_1(\xi)\,\varphi_k(\xi)\,\mathrm{d}\xi,$$

and

$$c_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k'(x) = y_1'(x) \int_a^x y_2(\xi)\,\lambda_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(\xi)\,\mathrm{d}\xi + \\ + y_2'(x) \int_x^b y_1(\xi)\,\lambda_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(\xi)\,\mathrm{d}\xi. \tag{77}$$

On taking into account the uniformity with respect to x of the convergence of series (76_1) in $[a, b]$ for $t > 0$, we can say that the series

$$\sum_{k=1}^{\infty} y_2(\xi)\,\lambda_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(\xi) \qquad \text{and} \qquad \sum_{k=1}^{\infty} y_1(\xi)\,\lambda_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(\xi)$$

are uniformly convergent in $[a, b]$, whence, by (77), the uniform convergence of series (76_2) follows. It remains to investigate series (76_3). We make use here of equation (56) for the eigenfunctions. It follows from this that

$$c_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k''(x) = \frac{1}{p(x)}\left[-\,p'(x)\,c_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k'(x) + \right. \\ \left. + q(x)\,c_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(x) - \lambda_k c_k\,\mathrm{e}^{-\lambda_k t}\,\varphi_k(x)\right], \tag{78}$$

and hence, in view of the uniform convergence of series (73), (76_1) and (76_2) in $[a, b]$ for any $t > 0$, it follows that series (76_3) is also uniformly convergent. This proves that the function $u(x, t)$ defined by (73) has the relevant partial derivatives and satisfies equation (70) for $t > 0$. We thus obtain the theorem:

THEOREM. *If the function $f(x)$ appearing in the initial condition has a continuous derivative in the interval $[a, b]$ and satisfies boundary conditions* (72), *the function $u(x, t)$ defined by* (73) *satisfies the initial condition* (71), *the boundary conditions* (72), *and also equation* (70) *with $t > 0$. It is possible to differentiate series* (73) *term by term any number of times with respect to t and twice with respect to x for $t > 0$.*

184. The justification of Fourier's method for the equations of vibrations. We now consider, instead of (70), the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] - q(x)\,u. \tag{79}$$

Here, in addition to boundary conditions (72), we have the two initial conditions:

$$u\Big|_{t=0} = f(x); \quad \frac{\partial u}{\partial t}\Big|_{t=0} = f_1(x), \tag{80}$$

and application of Fourier's method gives the solution of the problem as

$$u(x,t) = \sum_{k=1}^{\infty} (a_k \cos \sqrt{\lambda_k}t + b_k \sin \sqrt{\lambda_k}t)\, \varphi_k(x), \tag{81}$$

where λ_k and $\varphi_k(x)$ have their previous meaning and

$$a_k = \int_a^b f(x)\, \varphi_k(x)\, dx; \quad b_k = \frac{1}{\sqrt{\lambda_k}} \int_a^b f_1(x)\, \varphi_k(x)\, dx. \tag{82}$$

As in [183], it is sufficient for us to show that series (81), and the series which are obtained from it on twice differentiating with respect to t and x, are uniformly convergent in the interval $[a, b]$ for any t.

We split (81) into two series and first consider

$$\sum_{k=1}^{\infty} a_k \cosh \sqrt{\lambda_k}\, t\, \varphi_k(x). \tag{83}$$

In view of the fact that $\lambda_k > 1$ for all sufficiently large k, we can say that $\sqrt{\lambda_k} < \lambda_k$ for all sufficiently large k.

If, with certain conditions imposed on $p(x)$, $q(x)$ and $f(x)$, we can prove that the series

$$\sum_{k=1}^{\infty} \lambda_k \,|\, a_k \varphi_k(x) \,| \tag{84}$$

is uniformly convergent in $[a, b]$, we can prove all our previous assertions regarding the term by term differentiation of series (81) by repeating word for word the arguments of [183].

For, this is obvious for series (83) itself, since $\lambda_k \to +\infty$, and for the series obtained by differentiation with respect to t it follows from the fact that $\sqrt{\lambda_k} < \lambda_k$ for all sufficiently large k. In the case of a single differentiation with respect to x, we only need to show that the series

$$\sum_{k=1}^{\infty} |\, a_k \varphi_k'(x) \,|$$

is uniformly convergent. This follows at once from the uniform convergence of series (84) by virtue of the equation

$$a_k\varphi_k'(x) = y_1'(x) \int_a^x y_2(\xi)\,\lambda_k a_k \varphi_k(\xi)\,\mathrm{d}\xi + y_2'(x) \int_x^b y_1(\xi)\,\lambda_k a_k\varphi_k(\xi)\,\mathrm{d}\xi,$$

analogous to (77). To prove the uniform convergence of the series

$$\sum_{k=1}^{\infty} |\, a_k\varphi_k''(x)\,|$$

we only need to use an equation analogous to (78), after striking out the factor $\mathrm{e}^{-\lambda_k t}$ as above. Hence, the whole problem reduces to proving the uniform convergence of series (84).

We obtain with the aid of equation (56):

$$\lambda_k a_k = \lambda_k \int_a^b f(x)\,\varphi_k(x)\,\mathrm{d}x = \int_a^b f(x) \left\{q(x)\,\varphi_k(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,\varphi_k'(x)\right]\right\}\mathrm{d}x.$$

Integrating by parts, and assuming that $f(x)$ has continuous derivatives up to the second order and satisfies conditions (72), we obtain:

$$\lambda_k a_k = \int_a^b \left\{q(x)\,f(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,f'(x)\right]\right\}\varphi_k(x)\,\mathrm{d}x.$$

If we assume that the expression under the integral sign in the braces has a continuous derivative and satisfies boundary conditions (72), it will follow from this that series (84) is uniformly convergent in $[a, b]$. The requirement indicated reduces to the following: that $f(x)$ has continuous derivatives up to the third order, $p(x)$ has continuous derivatives up to the second order, $q(x)$ has a continuous derivative and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,f'(x)\right] - q(x)\,f(x) = 0 \quad \text{for} \quad x = a \text{ and } x = b. \qquad (85)$$

By virtue of the fact that $f(x)$ must also satisfy conditions (72), we can write (85) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,f'(x)\right] = 0 \quad \text{for} \quad x = a \text{ and } x = b. \qquad (86)$$

We now consider the series:

$$\sum_{k=1}^{\infty} b_k \sin\sqrt{\lambda_k}t\,\varphi_k(x), \qquad (87)$$

where the b_k are given by the second of equations (82). As above, it is sufficient to prove the uniform convergence of the series

$$\sum_{k=1}^{\infty} \lambda_k \,|\, b_k \varphi_k(x) \,|\,,$$

i.e. of the series

$$\sum_{k=1}^{\infty} \sqrt{\lambda_k}\,|\, b'_k \varphi_k(x) \,|\,, \tag{88}$$

where

$$b'_k = \int_a^b f_1(x)\,\varphi_k(x)\,\mathrm{d}x\,.$$

Assuming that $f_1(x)$ has continuous derivatives up to the second order and satisfies conditions (72), we obtain as above:

$$\lambda_k b'_k \quad \int_a^b \left\{ q(x)\,f_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,f'_1(x)\right] \right\} \varphi_k(x)\,\mathrm{d}x = b''_k,$$

where the b''_k are the Fourier coefficients of the continuous function in the braces. On substituting further $\varphi_k(x) = \lambda_k \psi_k(x)$, we obtain

$$\sqrt{\lambda_k}\,|\, b'_k \varphi_k(x) \,| = \sqrt{\lambda_k}\,|\, b''_k \psi_k(x) \,|\,,$$

whence, by Cauchy's inequality:

$$\sum_{k=m}^{m+p} \sqrt{\lambda_k}\,|\, b'_k \varphi_k(x) \,| \leqslant \sqrt{\sum_{k=m}^{m+p} b''^2_k} \cdot \sqrt{\sum_{k=m}^{m+p} \lambda_k \psi_k^2(x)}$$

or, on taking (68) into account:

$$\sum_{k=m}^{m+p} \sqrt{\lambda_k}\,|\, b'_k \varphi_k(x) \,| \leqslant \sqrt{\sum_{k=m}^{m+p} b''^2_k} \cdot \sqrt{M}\,.$$

But the series consisting of the terms b''^2_k is convergent, and it follows at once from the last inequality that series (88) is uniformly convergent. We thus arrive at the following theorem:

THEOREM. *If $p(x)$ has continuous derivatives up to the second order, $q(x) \geqslant 0$ and has a continuous derivative, $f(x)$ has continuous derivatives up to the third order, and satisfies conditions* (72) *and condition* (85), *while $f_1(x)$ has continuous derivatives up to the second order and satisfies conditions* (72), *the function $u(x, t)$ given by* (81) *satisfies the initial conditions* (80), *the boundary conditions* (72), *and equation* (79). *It is*

now possible to differentiate series (81) *term by term with respect to* t *and* x *twice, and the series obtained are uniformly convergent in the interval* $[a, b]$ *for any* t.

185. Uniqueness theorem. We have established the existence of solutions of equations (70) and (79) for the relevant boundary and initial conditions. We shall now prove the uniqueness of these solutions.

We start with equation (70) with $q(x) \geqslant 0$, and assume that the solutions are continuous for $t \geqslant 0$ and $a \leqslant x \leqslant b$ and that, given any $t > 0$, the solution has a continuous derivative with respect to t and derivatives with respect to x up to the second order, continuous in $[a, b]$. We in fact constructed the solution with these properties in [183].

To say that the solution is unique is equivalent to saying that the solution $u_0(x, t)$ of equation (70) with the above-mentioned properties, satisfying the homogeneous initial condition:

$$u_0 \big|_{t=0} = 0 \qquad (a \leqslant x \leqslant b) \tag{89}$$

and boundary conditions (72), is identically zero for $t > 0$.

We write down equation (70) for $u_0(x, t)$, multiply both sides by $u_0(x, t)$ and integrate with respect to x. It is assumed here that $t > 0$. We thus obtain the equation:

$$\frac{1}{2} \frac{\partial}{\partial t} \int\limits_a^b u_0^2 \, dx = \int\limits_a^b u_0 \frac{\partial}{\partial x} \left[p(x) \frac{\partial u_0}{\partial x} \right] dx - \int\limits_a^b q(x) u_0^2 \, dx .$$

All the operations can be carried out by virtue of the properties of $u_0(x, t)$. We integrate by parts in the first integral on the right-hand side and take the boundary conditions into account. We thus obtain:

$$\frac{1}{2} \frac{\partial}{\partial t} \int\limits_a^b u_0^2 \, dx = - \int\limits_a^b p(x) \left(\frac{\partial u_0}{\partial x} \right)^2 dx - \int\limits_u^b q(x) u_0^2 \, dx \leqslant 0 .$$

The non-negative function of t:

$$\int\limits_a^b u_0^2 \, dx, \tag{90}$$

which is continuous for $t \geqslant 0$ and vanishes, by (89), for $t = 0$, thus has a non-positive derivative for $t > 0$. Hence it follows that function (90) vanishes identically for $t > 0$. But now, $u(x, t) = 0$ for $t > 0$, which is what we wished to prove.

We now turn to the proof of the uniqueness theorem for equation (79) with $q(x) \geqslant 0$. We shall assume that the solutions themselves and their derivatives u_t, u_{tt}, u_x, u_{xx} are continuous in the interval $[a, b]$ for any t. The solution with these properties was in fact obtained by us in [184]. To say that the solution is unique is equivalent to saying that the solution $u_0(x, t)$ of equation (79) with the above-mentioned properties, satisfying the homogeneous boundary conditions:

$$u_0\Big|_{t=0} = \frac{\partial u_0}{\partial t}\Big|_{t=0} = 0 \tag{91}$$

and initial conditions (72), is identically zero.

We introduce the function:

$$v(x, t) = \int_0^t u_0(x, \tau)\,\mathrm{d}\tau. \tag{92}$$

It has continuous derivatives: v_x, v_t, v_{xx}, v_{xt}, v_{tt}, for the values of the variables indicated. We write equation (79) for $u_0(x, \tau)$ and integrate with respect to τ over the interval from $\tau = 0$ to $\tau = t$. On taking (91) and (92) into account, we obtain

$$\frac{\partial^2 v(x, t)}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial v(x, t)}{\partial x}\right] - q(x)\,v(x, t)\,.$$

We replace t by τ in this equation, multiply both sides by $v(x, \tau)$ and integrate with respect to τ from $\tau = 0$ to $\tau = t$. On taking (91) and (92) into account we obtain

$$\frac{1}{2} v_t^2(x, t) = \int_0^t v_\tau(x, \tau)\frac{\partial}{\partial x}\left[p(x)\,v_x(x, \tau)\right]\mathrm{d}\tau - \frac{1}{2} q(x)\,v^2(x, t).$$

We integrate both sides with respect to x over $[a, b]$ and change the order of integration in the iterated integral:

$$\frac{1}{2}\int_a^b v_t^2(x, t)\,\mathrm{d}x = \int_0^t \left\{\int_a^b v_\tau(x, \tau)\frac{\partial}{\partial x}\left[p(x)\,v_x(x, \tau)\right]\mathrm{d}x\right\}\mathrm{d}\tau - \\ - \frac{1}{2}\int_a^b q(x)\,v^2(x, t)\,\mathrm{d}x.$$

We integrate by parts in the inner integral and observe that the term outside the integral vanishes by (91) and (92):

$$\frac{1}{2}\int_a^b v_t^2(x, t)\,\mathrm{d}x = -\int_0^t \left\{\int_a^b p(x)\,v_{\tau x}(x, \tau)\,v_x(x, \tau)\,\mathrm{d}x\right\}\mathrm{d}\tau - \\ - \frac{1}{2}\int_a^b q(x)\,v^2(x, t)\,\mathrm{d}x.$$

On again changing the order of integration, performing the integration with respect to τ and observing that $v_x(x, 0) = 0$, we get:

$$\frac{1}{2}\int_a^b v_t^2(x,t)\,dx = -\frac{1}{2}\int_a^b p(x)\,v_x^2(x,\tau)\,d\tau - \frac{1}{2}\int_a^b q(x)\,v^2(x,t)\,dx,$$

whence it follows that

$$\int_a^b v_t^2(x,t)\,dx \leqslant 0,$$

so that $v_t(x, t) \equiv 0$ for $a \leqslant x \leqslant b$ and $-\infty < t < +\infty$. On taking (92) into account, we find that $u_0(x, t) \equiv 0$, which is what we wished to prove.

186. Extremal properties of the eigenvalues and eigenfunctions. We return to the boundary value problem for the equation

$$\frac{d}{dx}[p(x)\,y'] + [\lambda - q(x)]\,y = 0 \tag{93}$$

or

$$L(y) = -\lambda y, \text{ where } L(y) = \frac{d}{dx}[p(x)\,y'] - q(x)\,y.$$

In the general case of equation (1), we can reduce it to the form (93) by replacing x by the new independent variable

$$t = \int_a^x r(x)\,dx. \tag{94}$$

Equation (1) can now be written as

$$r(x)\frac{d}{dt}\left[r(x)\,p(x)\frac{dy}{dt}\right] + \bigl(\lambda r(x) - q(x)\bigr)\,y = 0,$$

and on dividing both sides by $r(x)$, we obtain an equation of the form (93). It is essential to assume with this transformation that $r(x)$ does not vanish in the closed interval $[a, b]$. We assume that $p(x) > 0$ in $[a, b]$, and suppose that the boundary conditions have the form:

$$y(a) = y(b) = 0. \tag{95}$$

Now, as we saw in [176], the eigenvalues are expressible in terms of the corresponding eigenfunctions by the formula:

$$\lambda_n = \int_a^b [p(x)\,\varphi_n'^2(x) + q(x)\,\varphi_n^2(x)]\,dx, \tag{96}$$

and only a finite number of negative eigenvalues can exist, so that it can be assumed that the eigenvalues are arranged in increasing order, i.e. $\lambda_1 < \lambda_2 < \dots$

Our boundary value problem is equivalent to the integral equation

$$\varphi(x) = \lambda \int_a^b G(x, \xi)\, \varphi(\xi)\, \mathrm{d}\xi,$$

where $G(x, \xi)$ is Green's function for the operator $L(y)$ with boundary conditions (95). We know from [26] that the first eigenvalue λ_1 is equal to the minimum of the integral

$$\int_a^b \int_a^b G(x, \xi)\, \omega(x)\, \omega(\xi)\, \mathrm{d}x\, \mathrm{d}\xi \tag{97}$$

in the class of continuous functions $\omega(x)$ satisfying the condition

$$\int_a^b \Big[\int_a^b G(x, \xi)\, \omega(\xi)\, \mathrm{d}\xi \Big]^2 \mathrm{d}x = 1. \tag{98}$$

But the integral

$$y(x) = \int_a^b G(x, \xi)\, \omega(\xi)\, \mathrm{d}\xi \tag{99}$$

is a function of x with continuous derivatives up to the second order satisfying boundary conditions (95), whatever the choice of the continuous function $\omega(\xi)$. Conversely, every function $y(x)$ with the properties just mentioned is expressible by integral (99), with a suitable choice of the continuous function $\omega(x) = -L(y)$.

We can therefore say, in accordance with (97), (98) and (99), that λ_1 is the minimum of the integral

$$-\int_a^b L(y)\, y\, \mathrm{d}x \tag{100}$$

subject to the condition

$$\int_a^b y^2(x)\, \mathrm{d}x = 1 \tag{101}$$

in the class of functions $y(x)$ having continuous derivatives up to the second order and satisfying boundary conditions (95).

On integrating by parts in integral (100), we see that λ_1 is the minimum of the integral

$$\int_a^b \left[p(x)\, y'^2 + q(x)\, y^2 \right] \mathrm{d}x \tag{102}$$

subject to condition (101) in the class of functions $y(x)$ just mentioned. Now, by (96), the first eigenfunction $y = \varphi_1(x)$ gives integral (102) its least value λ_1. We turn to the second eigenvalue λ_2. We know that

this is the minimum of integral (97) if we add to (98) the further condition:

$$\int_a^b \omega(\xi)\,\varphi_1(\xi)\,d\xi = 0. \tag{103}$$

If we define $y(x)$ from (99), then [22]:

$$\int_a^b y(x)\,\varphi_1(x)\,dx = \frac{1}{\lambda_1}\int_a^b \omega(\xi)\,\varphi_1(\xi)\,d\xi,$$

and hence condition (103) is equivalent to the condition

$$\int_a^b y(x)\,\varphi_1(x)\,dx = 0. \tag{104}$$

Therefore λ_2 is the minimum value of integral (102) in the class of functions $y(x)$ having continuous derivatives up to the second order and satisfying conditions (95), with the auxiliary conditions (101) and (104).

In general, the eigenvalue λ_n is the least value of integral (102) in the class of functions $y(x)$, having continuous derivatives up to the second order, satisfying boundary conditions (95), and the following subsidiary conditions:

$$\int_a^b y^2\,dx = 1; \quad \int_a^b \varphi_k(x)\,y(x)\,dx = 0 \quad (k = 1, 2, \ldots, n-1). \tag{105}$$

We now show that equation (93) is Euler's equation, expressing the necessary condition for an extremum of integral (102) subject to the auxiliary condition (101). In fact [68], we have to form the function

$$F = p(x)\,y'^2 + q(x)\,y^2 - \lambda y^2$$

and write Euler's equation for this:

$$\frac{d}{dx}F_{y'} - F_y = 0,$$

which is actually the same as equation (93). We now consider the extremum of integral (102) subject to the two subsidiary conditions (101) and (104). In the present case, we have to form the auxiliary function

$$F = p(x)\,y'^2 + q(x)\,y^2 - \lambda y^2 - \mu\varphi_1(x)\,y,$$

and Euler's equation for this function will have the form:

$$\frac{d}{dx}\left[p(x)\,y'\right] + (\lambda - q(x))\,y + \frac{\mu}{2}\varphi_1(x) = 0. \tag{106}$$

We show that the constant μ must vanish, i.e. that we again arrive at (93). For this, we write (93) for the first eigenfunction:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,\varphi_1'(x)\right] + \left(\lambda_1 - q(x)\right)\varphi_1(x) = 0.$$

We multiply this last equation by y, equation (106) by $\varphi_1(x)$, subtract term by term the equations obtained and integrate the resulting equation over the basic interval. On taking into account the orthogonality condition (104) and the normalization of the first eigenfunction, we arrive at the following equation:

$$\frac{\mu}{2} = \int_a^b \left\{y\,\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,\varphi_1'(x)\right] - \varphi_1(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,y'\right]\right\}\mathrm{d}x.$$

On integrating by parts and using the boundary conditions, we find without difficulty that the integral is zero, whence it immediately follows that $\mu = 0$, which is what we wished to show. In general, if we write Euler's equation, expressing the necessary condition for an extremum of integral (102) subject to auxiliary conditions (105), we arrive as above at equation (93).

We have so far considered the case $r(x) \equiv 1$. Precisely similar results are obtained in the general case, on the assumption that $r(x) > 0$. In this general case, the auxiliary conditions (105) must be written in the form:

$$\begin{aligned} &\int_a^b r(x)\,y^2(x)\,\mathrm{d}x = 1; \\ &\int_a^b r(x)\,\varphi_k(x)\,y(x)\,\mathrm{d}x = 0 \qquad (k = 1,\ 2,\ \ldots,\ n-1). \end{aligned} \tag{107}$$

In order to prove this, we only need to carry out the change of independent variable (94) in the general equation (1). We now obtain an equation of form (93), for which the result has already been proved. On returning to the previous independent variable, we obtain integral (102) and auxiliary conditions (107).

We remark further that the whole of the above discussion also holds for boundary conditions (2).

When finding the successive minima of integral (102), the problem can be posed in the class of functions having only one continuous derivative in $[a, b]$ instead of two. It can be shown that, in this wider statement, the successive minima are yielded as before by the functions $\varphi_n(x)$.

Let us take the equation of the vibrations of a string:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(a = \sqrt{\frac{T_0}{\varrho}}\right),$$

where ϱ is the linear density and T_0 the tension. We have the following expressions for the kinetic and potential energy:

$$T = \frac{1}{2} \int_0^l \varrho u_t^2 \, \mathrm{d}x; \qquad -U = \frac{1}{2} \int_0^l T_0 u_x^2 \, \mathrm{d}x.$$

In the case of sinusoidal vibration of the form $u = y(x) \sin \omega t$, we obtain for $y(x)$:

$$y'' + \lambda y = 0, \qquad \left(\lambda = \frac{\omega^2}{a^2}\right)$$

with the boundary conditions $y(0) = y(l) = 0$, provided the string is fixed at its ends, whilst the kinetic and potential energy are given by

$$T = \frac{\varrho\omega^2}{2} \cos^2 \omega t \int_0^l y^2 \, \mathrm{d}x; \qquad -U = \frac{T_0}{2} \sin^2 \omega t \int_0^l y'^2 \, \mathrm{d}x.$$

The first eigenvalue of this problem is obtained simply by seeking the minimum of the integral

$$\int_0^l y'^2 \, \mathrm{d}x \text{ subject to the condition } \int_0^l y^2 \, \mathrm{d}x = 1.$$

187. Courant's theorem. It follows from the arguments of the previous section that the minimum of integral (102) subject to conditions (105) is attained by the eigenfunctions $\varphi_n(x)$ and is equal to λ_n. When determining λ_n and $\varphi_n(x)$ in this way, we have to know all the preceding eigenfunctions. This fact makes it difficult to apply the extremal principle described. We shall now prove a theorem which enables us to determine λ_n and $\varphi_n(x)$ without using the preceding eigenfunctions. Let $z_1(x), \ldots, z_{n-1}(x)$ be any given functions, continuous in $[a, b]$. We pose the problem of finding the minimum of the integral

$$\int_a^b \left[p(x) y'^2 + q(x) y^2\right] \mathrm{d}x \tag{108}$$

subject to the auxiliary conditions

$$\int_a^b r(x) y^2 \, \mathrm{d}x = 1; \qquad \int_a^b r(x) z_k(x) y \, \mathrm{d}x = 0 \tag{109}$$

$$(k = 1, 2, \ldots, n-1)$$

in the class of functions $y(x)$ satisfying the boundary conditions and having continuous derivatives up to the second order. We do not know in advance whether integral (108) with the conditions imposed attains a minimum, but we can always speak of the strict lower bound of values of the integral. This strict lower bound will evidently depend on the choice of functions $z_k(x)$. We shall denote it by $m(z_1, \ldots, z_{n-1})$. We now prove Courant's theorem: *whatever the choice of continuous functions $z_k(x)$, the number $m(z_1, \ldots, z_{n-1})$ does not exceed the eigenvalue λ_n.* If, given any choice of functions z_k, we can construct a function $y(x)$, satisfying conditions (109) and all the remaining requirements, such that the value of integral (108) corresponding to it is not greater than λ_n, the number $m(z_1, \ldots, z_n)$ will certainly be not greater than λ_n, and the theorem will be proved. We shall seek the function $y(x)$ in the form:

$$y = c_1\varphi_1(x) + \ldots + c_n\varphi_n(x), \tag{110}$$

where $\varphi_k(x)$ are the eigenfunctions of the boundary value problem and the c_k are constants, which we shall now determine. The first of conditions (109), leads us by virtue of the normalization of the functions $\varphi_k(x)$, to the equation:

$$c_1^2 + c_2^2 + \ldots + c_n^2 = 1. \tag{111}$$

The remaining $(n - 1)$ conditions give a system of $(n - 1)$ homogeneous equations with n unknowns $c_1, \ldots, c_n$. This system has a non-zero solution, as we know from [III_1, 10]. Every such solution can be multiplied by an arbitrary constant factor, which can be chosen so that equation (111) is fulfilled. Therefore we have constructed with the aid of (110) a function having continuous derivatives up to the second order, satisfying the boundary conditions and all the auxiliary conditions (109). It only remains for us to substitute expression (110) in integral (108) and to show that the value of this integral is $\leqslant \lambda_n$. After the substitution, we have terms containing the squares $\varphi_k^2(x)$ and $\varphi_k'(x)$ under the integral sign, as also terms in the products $\varphi_k(x)\varphi_l(x)$ and $\varphi_k'(x)\varphi_c'(x)$. But the following formula can be proved, precisely as in [182], even in the case when $r(x)$ differs from unity:

$$\int_a^b [p(x)\varphi_k'(x)\varphi_l'(x) + q(x)\varphi_k(x)\varphi_l(x)]\,dx = 0 \qquad (k \neq l).$$

On further taking (22) into account, it may be seen that the substitution of expression (110) in integral (108) leads to the expression:

$$c_1^2\lambda_1 + \ldots + c_n^2\lambda_n.$$

On using the fact that $\lambda_1 < \ldots < \lambda_n$, and formula (111), we get:

$$c_1^2\lambda_1 + \ldots + c_n^2\lambda_n \leqslant \lambda_n,$$

which finally yields the proof of Courant's theorem.

COROLLARY. If we take $z_1 = \varphi_1(x), \ldots, z_{n-1} = \varphi_{n-1}(x)$, as we saw above, the minimum of integral (108) subject to conditions (109) will vanish, discounting λ_n, and will be attained with $y = \varphi_n(x)$. We can therefore assert that λ_n *is the maximum of all the possible lower bounds* $m_n(z_1, \ldots, z_{n-1})$ *of the values of integral* (108) *subject to auxiliary conditions* (109) *in the class of functions* $y(x)$ *satisfying the boundary conditions and having continuous derivatives up to the second order, this maximum of the strict lower bounds being attained for* $z_k = \varphi_k(x)$ *and* $y = \varphi_n(x)$. This maximo-minimal property of the eigenvalues λ_n also holds for a wider class of partial differential equations and plays a fundamental role in investigating eigenvalues.

188. Asymptotic expression for the eigenvalues. We replace $p(x)$ and $q(x)$ in equation (1) by the new functions $p_1(x)$ and $q_1(x)$, which are not less than the previous functions throughout the interval:

$$p_1(x) \geqslant p(x); \quad q_1(x) \geqslant q(x) \quad (a \leqslant x \leqslant b)$$
$$(p(x) > 0; \quad r(x) > 0). \tag{112}$$

The function $r(x)$ is left as before. We write λ_n' for the characteristic values of the changed equation and show that $\lambda_n' \geqslant \lambda_n$. We use for this purpose the property of the eigenvalues just proved.

When $p(x)$ and $q(x)$ are changed as described, the supplementary conditions (109) remain as before, whilst integral (108) can only increase when the function y is fixed. Since the set of functions y remains as before when the coefficients are changed as described, the strict lower bound $m(z_1, \ldots, z_{n-1})$ of values of integral (108) can never decrease, i.e. the greatest of the numbers $m(z_1, \ldots, z_{n-1})$, that is, λ_n, cannot decrease. This is what we wished to prove.

We now leave $p(x)$ and $q(x)$ unchanged, and replace $r(x)$ by $r_1(x)$, where $r_1(x) \geqslant r(x)$ for $a \leqslant x \leqslant b$. In this case it is no longer possible to speak of preserving the class of functions y, since if y satisfies the first of conditions (109), we have after substituting $r_1(x)$ for $r(x)$:

$$\int_a^b r_1(x)\, y^2\, \mathrm{d}x \geqslant 1\,.$$

However, an admissible function for the new problem is readily obtained from function y. All we need is to select the number θ satisfying the condition $0 < \theta \leqslant 1$ such that

$$\int_a^b r_1(x)\,\theta^2 y^2\,\mathrm{d}x = 1\,.$$

It is easily seen that the function θy also satisfies the remaining conditions (109), though admittedly for different functions $z_k(x)$. For, since θ is constant, it follows from (109) that

$$\int_a^b r_1(x)\,\frac{z_k(x)\,r(x)}{r_1(x)}\,\theta y\,\mathrm{d}x = 0 \qquad (k = 1,\ 2,\ \ldots,\ n-1)\,.$$

But this is in fact again a condition of the form (109) for the modified equation; instead of the functions $z_k(x)$, we have to take here

$$\tilde{z}_k(x) = \frac{z_k(x)\,r(x)}{r_1(x)}\,.$$

For every system of functions $z_k(x)$ there will be a corresponding system of functions $\tilde{z}_k(x)$ and vice versa. The inverse passage from the function θy for the transformed equation to the similar function for the original equation is carried out by division of θy by θ. When y is replaced by θy, the value of integral (108) cannot increase. Consequently, the strict lower bound of these values cannot increase, so that the number λ_n, which is the maximum of these strict lower bounds, can likewise not increase. We thus arrive at the following general proposition: *if the changed coefficients $p_1(x)$ and $q_1(x)$ satisfy condition* (112), *the eigenvalues λ_n cannot decrease, whilst if the changed coefficient $r_1(x)$ satisfies the condition $r_1(x) \geqslant r(x)$, λ_n cannot increase.*

Let us apply this proposition to an asymptotic inequality for the eigenvalues λ_n for large n. Let (p, P), (q, Q), (r, R) be the minimum and maximum values of the functions $p(x)$, $q(x)$, $r(x)$ in the interval $[a, b]$. We replace $p(x)$ by P, $q(x)$ by Q and $r(x)$ by r in equation (1). The new equation with constant coefficients:

$$Py''\,(\lambda r - Q)\,y = 0 \tag{113}$$

will have eigenvalues λ_n', which are never less than the eigenvalues λ_n of the original equation. But the λ_n' can easily be found. We do this by first of all observing that (113) can have a solution satisfying boundary conditions (95) only when $(\lambda r - Q)/P > 0$. On taking this

into account, we can write the general solution of (113) as:

$$y = C_1 \cos \sqrt{\frac{\lambda r - Q}{P}}\, x + C_2 \sin \sqrt{\frac{\lambda r - Q}{P}}\, x.$$

For the sake of simplicity in further working we shall take $[0, l]$ as the basic interval $[a, b]$. It follows from the boundary condition $y(0) = 0$ that $C_1 = 0$, and the second boundary condition $y(l) = 0$ gives us an equation for λ, namely

$$\sqrt{\frac{\lambda r - Q}{P}}\, l = n\pi,$$

whence

$$\lambda'_n = \frac{n^2 \frac{\pi^2}{l^2} P + Q}{r}, \quad \text{so that} \quad \lambda_n \leqslant \frac{n^2 \frac{\pi^2}{l^2} P + Q}{r}.$$

Similarly, on replacing $p(x)$, $q(x)$, $r(x)$ by p, q, R respectively, we can show that

$$\lambda_n \geqslant \frac{n^2 \frac{\pi^2}{l^2} p + q}{r},$$

and hence obtain the following bounds for the eigenvalues:

$$\frac{n^2 \frac{\pi^2}{l^2} P + Q}{r} \geqslant \lambda_n \geqslant \frac{n^2 \frac{\pi^2}{l^2} p + q}{R}.$$

It follows from this that λ_n is of order n^2 for large n and the series

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$$

is convergent. On using the maximo-minimal property of λ_n, a closer inequality can be obtained by first transforming the initial equation. Let $p(x)$ and $r(x)$ have continuous derivatives up to the second order and let (1) be transformed by replacing x by the new independent variable t:

$$t = \int_a^x \sqrt{\frac{r(x)}{p(x)}}\, \mathrm{d}x \tag{114}$$

and y by the new required function u:

$$u = \sqrt[4]{p(x)\, r(x)}\, y. \tag{115}$$

The interval $[a, b]$ of variation of x is transformed to the interval $[0, l]$ for t, where

$$l = \int_a^b \sqrt{\frac{r(x)}{p(x)}}\, \mathrm{d}x .$$

The equation for $u(t)$ will have the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \big(\lambda - s(t)\big)\, u = 0, \tag{116}$$

where $s(t)$ is a continuous function which is easily found from the given coefficients of equation (1). It follows from $y(a) = y(b) = 0$ that $u(0) = u(l) = 0$ and vice versa, so that the eigenfunctions of the initial equation will be obtained from the eigenfunctions of the transformed equation in accordance with (115), and vice versa, whilst the eigenvalues remain as before. When finding the eigenvalues of equation (116), we have to pose the minimal problem for the integral

$$\int_0^l \left[u'^2 + s(t)\, u^2\right] \mathrm{d}t . \tag{117}$$

Let σ be the maximum of $|\, s(t)\,|$ in $[0, l]$, so that

$$-\sigma \leqslant s(t) \leqslant \sigma \qquad (0 \leqslant t \leqslant l) .$$

If we pose the minimal problem for the integrals

$$\int_a^b (u'^2 + \sigma u^2)\, \mathrm{d}t \tag{118_1}$$

and

$$\int_a^b (u'^2 - \sigma u^2)\, \mathrm{d}t \tag{118_2}$$

instead of for integral (117), and write λ_n', λ_n'' for the respective eigenvalues, we get

$$\lambda_n' \geqslant \lambda_n \geqslant \lambda_n'' . \tag{119}$$

But the numbers λ_n' and λ_n'' can be calculated by elementary means from the solutions of the equations

$$u'' + (\lambda - \sigma)\, u = 0 \quad \text{and} \quad u'' + (\lambda + \sigma)\, u = 0$$

with the boundary conditions $u(0) = u(l) = 0$, and we have:

$$\lambda_n' = \frac{n^2 \pi^2}{l^2} + \sigma; \quad \lambda_n'' = \frac{n^2 \pi^2}{l^2} - \sigma .$$

We have from (119):

$$\lambda_n = \frac{n^2\pi^2}{l^2} + A_n \qquad (|A_n| \leqslant \sigma) \tag{120}$$

or

$$\lambda_n = \frac{n^2\pi^2}{l^2} + O(1), \tag{121}$$

where $O(1)$ denotes as usual a magnitude whose absolute value remains bounded for all n. We obtain, on returning to the old variables:

$$\lambda_n = n^2\pi^2\left[\int\limits_a^b \sqrt{\frac{r(x)}{p(x)}}\,\mathrm{d}x\right]^{-2} + O(1), \tag{122}$$

so that

$$\lim_{n\to\infty} \frac{n^2}{\lambda_n} = \frac{1}{\pi^2}\left[\int\limits_a^b \sqrt{\frac{r(x)}{p(x)}}\,\mathrm{d}x\right]^2. \tag{123}$$

An asymptotic expression can similarly be obtained for the eigenvalues with different boundary conditions. This may be seen at once by considering the equation $u'' + \mu u = 0$ for different boundary conditions.

189. Asymptotic expression for the eigenfunctions. Having found an asymptotic expression for the eigenvalues, we can obtain one for the eigenfunctions by using the method employed earlier when deducing the asymptotic expressions for Hermite and Legendre polynomials [III_2, 162, 163].

With the aid of the above transformation of the variables, we can reduce our equation to the form (116):

$$u''(t) + \big(\lambda - s(t)\big)u(t) = 0.$$

The eigenvalues λ_n will be positive for large n, as we know from [176], and we assume in future that n is large enough for $\lambda_n > 0$. Let $u_n(t)$ be the eigenfunctions corresponding to the λ_n. We can write:

$$u_n''(t) + \lambda_n u_n(t) = s(t)\,u_n(t),$$

and we find:

$$u_n(t) = a_n \sin\sqrt{\lambda_n}\,t + b_n \cos\sqrt{\lambda_n}\,t + \\ + \frac{1}{\sqrt{\lambda_n}}\int\limits_0^t s(\tau)\,u_n(\tau)\sin\sqrt{\lambda_n}(t-\tau)\,\mathrm{d}\tau. \tag{124}$$

We apply Buniakowski's inequality to the integral on the right:

$$\left[\int_0^t s(\tau)\, u_n(\tau) \sin \sqrt{\lambda_n}\,(t-\tau)\,\mathrm{d}t\right]^2 \leqslant \int_0^t u_n^2(\tau)\,\mathrm{d}\tau \int_0^t s^2(\tau)\, \sin^2 \sqrt{\lambda_n}\,(t-\tau)\,\mathrm{d}\tau,$$

whence it follows that, for any t of the interval $[0, l]$:

$$\left[\int_0^t s(\tau)\, u_n(\tau)\, \sin \sqrt{\lambda_n}\,(t-\tau)\,\mathrm{d}\tau\right]^2 \leqslant \int_0^l s^2(\tau)\,\mathrm{d}\tau, \tag{125}$$

where the normalization of functions $u_n(t)$ has been taken into account.

Let $\varphi_n(x)$ be the eigenfunction of the initial equation (1) obtained from $u_n(t)$ with the aid of transformations (114) and (115). It follows at once from these transformations that

$$\int_a^b r(x)\, \varphi_n^2(x)\,\mathrm{d}x = \int_0^l u_n^2(t)\,\mathrm{d}t = 1\,, \tag{126}$$

i.e. the ordinary normalization of $u_n(t)$ is equivalent to normalization of $\varphi_n(x)$ with weight $r(x)$. The boundary condition $u(0) = 0$ gives us $b_n = 0$, and we can rewrite (124) as follows:

$$u_n(t) = a_n \sin \sqrt{\lambda_n}\, t + \frac{m_n(t)}{\sqrt{\lambda_n}}\,, \tag{127}$$

where, by virtue of inequality (125), the function $m_n(t)$ remains bounded for all positive integers n and all t of $[0, l]$, i.e. there exists a positive number A such that

$$|m_n(t)| \leqslant A. \tag{128}$$

On squaring both sides of (127), integrating over the basic interval and taking into account the normalization of functions $u_n(t)$, we can write:

$$1 = a_n^2 \int_0^l \sin^2 \sqrt{\lambda_n}\, t\,\mathrm{d}t + \frac{2a_n}{\sqrt{\lambda_n}} \int_0^l m_n(t)\, \sin \sqrt{\lambda_n}\, t\,\mathrm{d}t + \frac{1}{\lambda_n} \int_0^l m_n^2(t)\,\mathrm{d}t.$$

The first of these integrals can be worked out, whilst the other two are bounded in absolute value for all n by virtue of condition (128). We thus obtain:

$$1 = \frac{l}{2} a_n^2 - \frac{\sin 2\sqrt{\lambda_n}\, l}{4\sqrt{\lambda_n}}\, a_n^2 + \frac{a_n}{\sqrt{\lambda_n}}\, p_n + \frac{1}{\lambda_n}\, q_n, \tag{129}$$

where p_n and q_n remain bounded in absolute value for increasing n. We take a_n^2 outside and write:

$$1 = a_n^2 \left(\frac{l}{2} - \frac{\sin 2\sqrt{\lambda_n}}{4\sqrt{\lambda_n}} + \frac{1}{a_n \sqrt{\lambda_n}} p_n + \frac{1}{a_n^2 \lambda_n} q_n \right).$$

If we were to encounter arbitrarily large values of a_n^2 with increasing n, the expression inside the brackets would tend to the non-zero limit $l/2$ for these values of n, and the right-hand side of the last equation could not be equal to unity. We can thus conclude that a_n remains bounded with increasing n. On taking this into account, we can rewrite (129) as:

$$1 = \frac{l}{2} a_n^2 + O\left(\frac{1}{\sqrt{\lambda_n}}\right), \tag{130}$$

where, as usual, $O(1/x_n)$ denotes a magnitude such that the product $x_n \cdot O(1/x_n)$ remains bounded as n increases indefinity. We can rewrite the last formula as follows:

$$a_n^2 = \frac{2}{l} + O\left(\frac{1}{\sqrt{\lambda_n}}\right),$$

whence we have:

$$a_n = \sqrt{\frac{2}{l}} + O\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

Substitution of this in (127) gives us

$$u_n(t) = \sqrt{\frac{2}{l}} \sin \sqrt{\lambda_n}\, t + O\left(\frac{1}{\sqrt{\lambda_n}}\right), \tag{131}$$

where

$$O\left(\frac{1}{\sqrt{\lambda_n}}\right) = \frac{p_n(t)}{\sqrt{\lambda_n}}$$

and $p_n(t)$ is bounded in absolute value for all n and all t of the interval $0 \leqslant t \leqslant l$.

It follows from (121) that:

$$\lambda_n = \frac{n^2\pi^2}{l^2}\left[1 + O\left(\frac{1}{n^2}\right)\right] \text{ or } \sqrt{\lambda_n} = \frac{n\pi}{l} + O\left(\frac{1}{n}\right),$$

whence

$$\sin \sqrt{\lambda_n}\, t = \sin \frac{n\pi}{l} + O\left(\frac{1}{n}\right),$$

where $O(1/n) = q_n(t)/n$, and $q_n(t)$ is bounded in absolute value for all n and for all t of $[0, l]$. On substituting this in (131), we obtain the

following asymptotic expression for the normalized functions $u_n(t)$:

$$u_n(t) = \sqrt{\frac{2}{l}} \sin \frac{n\pi}{l} t + O\left(\frac{1}{n}\right). \tag{132}$$

On returning to the old variables in accordance with (114) and (115), we obtain the following asymptotic formula:

$$\varphi_n(x) = \frac{\sqrt{2}}{\sqrt{l}\,\sqrt[4]{p(x)\,r(x)}} \sin\left[\frac{n\pi}{l} \int_a^x \sqrt{\frac{r(x)}{p(x)}}\,\mathrm{d}x\right] + O\left(\frac{1}{n}\right), \tag{133}$$

where the eigenfunctions $\varphi_n(x)$ are normalized in accordance with (126) and $O(1/n) = r_n(x)/n$, where $r_n(x)$ is bounded in absolute value for all n and all for x of $[a, b]$.

190. Ritz's method. The equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[p(x)\,y'] + [\lambda r(x) - p(x)]\,y = 0 \tag{134}$$

is Euler's equation for the integral

$$\int_a^b [p(x)'^2 + q(x)\,y^2]\,\mathrm{d}x \tag{135}$$

subject to the auxiliary condition

$$\int_a^b r(x)\,y^2(x)\,\mathrm{d}x = 1,$$

and, as we have seen, obtaining the successive eigenvalues and eigenfunctions amounts to an extremal problem for integral (135). This leads us to a practical method for finding approximately the eigenvalues and eigenfunctions. This method (Ritz's method) has already been described as applied to finding the absolute extremum of an integral.

We take a sequence of linearly independent functions $v_1(x), v_2(x), \ldots$ satisfying the boundary conditions, form the linear combination:

$$y = \sum_{k=1}^{n} a_k^{(n)} v_k(x) \tag{136}$$

and substitute it in the integral

$$J(y) = \int_a^b \left\{p(x)\,y'^2 + [q(x) - \lambda r(x)]\,y^2\right\} \mathrm{d}x.$$

We obtain as a result a quadratic form in the $a_k^{(n)}$. On equating to zero its partial derivatives with respect to $a_k^{(n)}$, we arrive at a system of n homogeneous equations with n unknowns $a_k^{(n)}$. On putting the determinant of this system equal to zero, we obtain an equation of the nth degree in λ. The roots of this equation

$\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ can be taken as the approximate values of the first n eigenvalues of the problem. The homogeneous system yields for each of them a set of numbers $a_k^{(n)}$, and the corresponding function y can be constructed for these $a_k^{(n)}$ with the aid of (136). This function y can be taken as the corresponding eigenfunction. The convergence of this process depends essentially on the choice of coordinate functions $v_k(x)$. We shall only quote in this connection certain results obtained by N. M. Krylov *(Memorial des Sciences Math. fasc.* XLIX, 1931).

Suppose that the equation has the form:

$$y'' + \lambda r(x) y = 0 \qquad (r(x) > 0). \tag{137}$$

The boundary conditions will be taken in the elementary form: $y(0) = y(1) = 0$. If we put $v_n(x) = \sqrt{2}\sin n\pi x$, the difference between the true value of λ_m and the nth approximation to it can be estimated as follows:

$$|\lambda_m - \lambda_m^{(n)}| \leqslant \frac{2\lambda_m^2 \max r^{3/2}(x)}{(n+1)^2\pi^2 \min\sqrt{r(x)} - 2\lambda_m \max r^{3/2}(x)},$$

or

$$\left|\frac{\lambda_m^{(n)} - \lambda_m}{\lambda_m}\right| \leqslant \frac{\lambda_m A}{(n+1)^2\pi^2 - \lambda_m B}, \tag{138}$$

where

$$A = [\max r(x) - \min r(x)]\sqrt{\frac{\max r(x)}{\min r(x)}}\,; \quad B = 2\max r(x).$$

Polynomials are often used instead of trigonometric functions in practical computations. Suppose that we first have equation (137) with boundary conditions $y(-1) = y(1) = 0$, and that we take $v_n(x) = (1 - x^2)\,x^{n-1}$ [the factor $(1 - x^2)$ guarantees the satisfaction of the boundary conditions]. With this choice of $v_n(x)$, the following inequality holds:

$$\left|\frac{\lambda_m^{(n)} - \lambda_m}{\lambda_m}\right| < \frac{\lambda_m^{(n)} \max r(x)}{(n+1)(n+2)}. \tag{139}$$

This inequality holds if we merely assume the continuity of the function $r(x)$. If this function also has a continuous derivative, a closer inequality can be obtained, namely

$$\left|\frac{\lambda_m^{(n)} - \lambda_m}{\lambda_m}\right| < \frac{N\lambda_m^{(n)}}{(n+1)^2(n+2)},$$

where

$$N = \left\{\max\left|\frac{r'(x)}{\sqrt{r(x)}}\right| + \sqrt{\lambda_m^{(n)}}\sqrt[4]{\frac{\max r^5(x)}{\min r(x)}}\right\}^2.$$

A still closer inequality is obtained on the assumption that $r(x)$ has a continuous second derivative.

191. Ritz's example. The following is an example of the approximate calculation of eigenvalues and eigenfunctions; these can be obtained accurately in the finite form in this example, so that we are in a position to see the speed of convergence of the process. The example is to be found in an article by Ritz (*J. für die Reine und Angew. Mathem.* Bd. 135, 1909). We take the equation

$$y'' + k^2 y = 0$$

with the boundary conditions $y(-1) = y(1) = 0$, where k^2 plays the role of the parameter λ. The problem of the vibrations of a string fixed at its ends leads to this type of boundary value problem. The fundamental tone of the string is given by the solution:

$$y_1 = \cos\frac{\pi x}{2}, \quad k_1 = \frac{\pi}{2};$$

the first overtone is:

$$y_1 = \sin \pi x, \qquad k_2 = \pi;$$

the second overtone:

$$y_3 = \cos\frac{3\pi x}{2}, \; k_3 = \frac{3\pi}{2} \text{ and so on.}$$

We shall seek the even solutions approximately as polynomials containing even powers of x. The general form of such a polynomial satisfying the boundary conditions will be

$$y = (1 - x^2)(a_0 + a_1 x^2 + \dots + a_n x^{2n}).$$

On confining ourselves to the first two terms:

$$y = (1 - x^2)(a_0 + a_1 x^2)$$

and on substituting in the integral

$$J(y) = \int_{-1}^{1} (y'^2 - k^2 y^2)\, dx,$$

we get:

$$J(y) = \frac{8}{315}\left[(105 - 42k^2)\, a_0^2 + (42 - 12k^2)\, a_0 a_1 + (33 - 2k^2)\, a_1^2\right].$$

On equating to zero the partial derivatives with respect to a_0 and a_1, we arrive at the system:

$$(35 - 14k^2)\, a_0 + (7 - 2k^2)\, a_1 = 0$$

$$(21 - 6k^2)\, a_0 + (33 - 2k^2)\, a_2 = 0,$$

and equating the determinant to zero gives us:

$$k^4 - 28k^2 + 63 = 0,$$

the roots of which are:

$$k_1^2 = 2.46744; \; k_3^2 = 25.6.$$

We obtain from the exact solutions quoted above:

$$k_1^2 = \frac{\pi^2}{4} = 2.467401100\ldots; \quad k_3^2 = \frac{9\pi^2}{4} = 22.207.$$

As a second approximation:

$$y = (1 - x^2)(a_0 + a_1x^2 + a_2x^4).$$

We obtain the equation for k^2:

$$4k^6 - 450k^4 + 8910k^2 - 19305 = 0,$$

from which we find that

$$k_1^2 = 2.467401108\ldots; \quad k_3^2 = 23.301\ldots$$

On substituting this approximate value for k_1^2 in the coefficients of the system for a_0, a_1, a_2, we can determine these coefficients up to a constant factor which can be arranged so that the solution obtained satisfies the condition:

$$\int_{-1}^{1} y^2\, dx = 1,$$

which is satisfied by the exact solution $y = \cos(\pi x/2)$. Hence we arrive at the following approximate solution:

$$y = (1 - x^2)(1 - 0.233430x^2 + 0.018962x^4).$$

The following table shows how little y differs from $\cos(\pi x/2)$; the table gives the mantissae of the common logarithms of these functions:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\log\cos\frac{\pi x}{2}$	994620	978206	949881	907958	849485	769219	657047	489982	194332
$\log y$	994621	978212	949889	907952	849493	769221	657043	489978	194345

The eigenvalues and eigenfunctions which are odd functions of x can be sought approximately in the form

$$y = (1 - x^2)(a_0x + a_1x^3 \ldots + a_nx^{2n+1}).$$

§ 2. Equations of the elliptic type

192. The Newtonian potential. We now turn to a discussion of boundary value problems for partial differential equations. Let us start with Laplace's equation. We have already solved Dirichlet's problem for this equation in the case of a circle and sphere. Other equations of the elliptic type, apart from Laplace's equation, will be considered in this section. Problems can be posed for these equations that are analo-

gous to the Dirichlet and Neumann problems for Laplace's equation. These elliptic equations are usually obtained in physics when considering statical problems or steady states. It may be recalled that Laplace's equation is itself obtained when considering, say, an electrostatic field or a steady heat flow.

The Newtonian potential is of great importance when discussing boundary value problems for Laplace's equation. We shall recall the basic definitions relating to the Newtonian potential, prior to introducingcertain new concepts.

Let D be a bounded domain of three-dimensional space, $\mu(N)$ a continuous function of a point in this domain and r the distance from the point M to the variable point N of D. The *potential of a three-dimensional mass distribution* is defined by the familiar expression

$$v(M) = \iiint_D \frac{\mu(N)}{r} \, dv. \tag{1}$$

Similarly, *the potential of a simple layer* distributed over an area S with density $\mu(N)$ is given by

$$u(M) = \iint_S \frac{\mu(N)}{r} \, ds. \tag{2}$$

As we know from [II, 87, 200], outside the distribution the functions $u(M)$ and $v(M)$ have derivatives of all orders and satisfy Laplace's equation. It is important for our future exposition that we first of all indicate the limitations to be imposed on the surface S, which will be assumed closed in future. The first precise statement on these lines was given by Lyapunov in his work *On Certain Questions Relating to Dirichlet's Problem* (O nekotorich voprosakh, svyazannykh s zadachei Dirikhle) (1898). This work played a decisive part in the development of potential theory and the study of boundary value problems for Laplace's equation. We shall follow Lyapunov's treatment in this and the following sections.

The following requirements are laid down for the surface S.

1. A tangent plane exists at every point of S.

2. There exists a $d > 0$ such that, if N_0 is any point of S, every sphere with centre N_0 of radius less than or equal to d cuts S into two parts, one of which lies inside and the other outside the sphere, and the straight lines parallel to the normal to S at N_0 cut the part of S lying inside the sphere in not more than one point.

3. If ϑ is the acute angle formed by the normals to S at two points N_1 and N_2, and $r_{1,2}$ is the distance between these two points, there exist two positive numbers a and α, independent of the choice of N_1 and N_2, such that we have

$$\vartheta \leqslant a r_{1,2}^{\alpha} \qquad (\alpha \leqslant 1) \tag{3}$$

for any positions of N_1 and N_2 on S.

Closed surfaces that satisfy these conditions are generally called *Lyapunov surfaces.* We shall later introduce some further assumptions regarding the surface S, but for the moment certain corollaries of the above assumptions will be deduced.

It follows at once from (3) that the tangent plane varies continuously as the point of contact varies along the surface. Let N_0 be a point of S, which we take as origin; the Z axis is taken along the outward normal to S at N_0, whilst the X and Y axes are chosen arbitrarily in the tangent plane. The equation of the piece of S lying inside the sphere C_0 with centre N_0 and radius d can now be written explicitly as

$$\zeta = \zeta(\xi, \eta). \tag{4}$$

We shall always use (ξ, η, ζ) to denote the coordinates of a variable point N of S, and (x, y, z) to denote the coordinates of any point M of space. The coordinate axes described may be termed *local axes at the point* N_0.

The existence of the tangent plane and its continuous variation imply the existence and continuity of the first order derivatives $\zeta_\xi(\xi, \eta)$ and $\zeta_\eta(\xi, \eta)$. Let d be assumed sufficiently small. For instance, we can take the condition:

$$a d^{\alpha} \leqslant 1, \tag{5}$$

so that the angle ϑ_0 between the normal at N_0 and the normal at any point N of the piece of S lying inside the sphere C_0 does not exceed $\pi/2$. On writing r_0 for the distance N_0N ($r_0 < d$), we have:

$$\cos\vartheta_0 \geqslant 1 - \frac{1}{2}\vartheta_0^2 \geqslant 1 - \frac{1}{2}a^2 r_0^{2\alpha}, \tag{6}$$

whence

$$\frac{1}{\cos\vartheta_0} = \sqrt{1 + \zeta_\xi^2 + \zeta_\eta^2} \leqslant 1 + a^2 r_0^{2\alpha} \leqslant 2, \tag{7}$$

and consequently, by (5):

$$\zeta_\xi^2 + \zeta_\eta^2 \leqslant 2a^2 r_0^{2\alpha} + a^4 r_0^{4\alpha} \leqslant 3a^2 r_0^{2\alpha}. \tag{8}$$

We introduce polar coordinates:

$$\zeta = \varrho_0 \cos\theta; \quad \eta = \varrho_0 \sin\theta.$$

We have:

$$\zeta_{\varrho_0}^2 = (\zeta_\xi \cos\theta + \zeta_\eta \sin\theta)^2 \leqslant \zeta_\xi^2 + \zeta_\eta^2,$$

whence, by (8):

$$|\zeta_{\varrho_0}| \leqslant \sqrt{3}ar_0^a, \tag{9}$$

and

$$|\zeta| \leqslant \sqrt{3}\, ad^a\varrho_0 \leqslant \sqrt{3}\varrho_0, \tag{10}$$

so that

$$r_0 = \sqrt{\varrho_0^2 + \zeta^2} \leqslant 2\varrho_0. \tag{11}$$

Inequality (9) gives:

$$|\zeta_{\varrho_0}| \leqslant \sqrt{3}a2^a\varrho_0^a, \tag{12}$$

whence

$$|\zeta| \leqslant \frac{\sqrt{3}\; 2^a}{a+1}\, a\varrho_0^{a+1},$$

so that certainly:

$$|\zeta| \leqslant 2a\varrho_0^{a+1}. \tag{13}$$

since $2^a \leqslant a + 1$ for $a \leqslant 1$. Finally, it follows from (6) that

$$1 - \cos\vartheta_0 \leqslant 2^{2a-1}\, a^2\varrho_0^{2a}. \tag{14}$$

Inequalities can also be found for $\cos(\mathbf{n}, X)$ and $\cos(\mathbf{n}, Y)$, where $\mathbf{n}$ is the unit outward normal to S at the point N. We have, by (8):

$$|\cos(\mathbf{n}, X)| = \frac{|\zeta_\xi|}{\sqrt{1 + \zeta_\xi^2 + \zeta_\eta^2}} \leqslant |\zeta_\xi| \leqslant \sqrt{3}ar_0^a,$$

and similarly,

$$|\cos(\mathbf{n}, Y)| \leqslant \sqrt{3}ar_0^a.$$

Further, we have

$$\cos(\mathbf{n}, Z) = \cos\vartheta_0.$$

Let us collect all the above inequalities:

$$|\zeta| \leqslant c\varrho_0^{1+a}; \quad |\cos(\mathbf{n}, X)| \leqslant c\varrho_0^a; \quad |\cos(\mathbf{n}, Y)| \leqslant c\varrho_0^a$$

$$1 - \cos(\mathbf{n}, Z)| \leqslant c\varrho_0^{2a}; \quad |\cos(\mathbf{n}, Z) \geqslant \frac{1}{2}, \tag{15}$$

where, for the sake of simplifying later writing, we have put c for the maximum of the constants appearing in the respective inequalities. These inequalities obviously still hold if we replace ϱ_0 by r_0 on the right-hand sides. We have $r_0 = d$ at points of intersection of S with C_0,

and it follows from (11) that $\varrho_0 \geqslant d/2$. The part of S cut out by a cylinder whose axis coincides with the Z axis (the normal at N_0), and of radius $d/3$, is thus seen to lie inside C_0. We shall write τ_0 in future for this part of S. Its projection σ_0' on the XY plane (the tangent plane at N_0) is a circle:

$$\xi^2 + \eta^2 \leqslant \frac{d^2}{9}\,. \tag{16}$$

Formulae (15) hold at all points N lying on σ_0. We also bring into the discussion the part σ_1 of S which is cut from S by the circular cylinder whose axis is the Z axis and whose base radius is equal to a number d_1, where $d_1 < d/2$. We shall later make use of the fact that the choice of d_1 is arbitrary. Inequalities (15) will also hold on σ_1. The projection σ_1' of σ_1 on the tangent plane at N_0 is a circle:

$$\xi^2 + \eta^2 \leqslant d_1^2 \quad \left(d_1 < \frac{d}{2}\right). \tag{17}$$

Let us turn to an investigation of the properties of the potentials of a simple layer, as also of certain other potentials, viz. those of a double layer, which, like the potentials of a simple layer, can be written as integrals over the surface S.

193. The potential of a double layer. The singular solution $1/r$ of Laplace's equation plays a fundamental part in forming the functions (1) and (2). We now introduce another singular solution of this equation. Let N be a point of space and l a fixed direction drawn from N. We take a segment NN' of length ε in the direction l and locate a charge $(1/\varepsilon)$ at N, and $(-1/\varepsilon)$ at N'. On writing r and r' for the distances from the variable point M to the points N and N', we obtain the following potential for the two charges:

$$u_0(M) = \frac{1}{\varepsilon}\left(\frac{1}{r} - \frac{1}{r'}\right) = \frac{1}{\varepsilon}\cdot\frac{r'-r}{rr'} = \frac{1}{\varepsilon}\cdot\frac{r'^2 - r^2}{(r'+r)\,rr'}\,.$$

Let us introduce the angle $\varphi = (r, l)$, where the direction r is reckoned from the point M to N.

On taking into account the obvious equation $r'^2 = r^2 + \varepsilon^2 + 2r\varepsilon\cos\varphi$, we can write:

$$u_0(M) = \frac{\varepsilon + 2r\cos\varphi}{(r'+r)\,rr'}\,,$$

and in the limit as $\varepsilon \to 0$ we obtain the potential of a dipole of unit intensity with the direction l:

$$u_0(M) = \frac{\cos\varphi}{r^2}.$$

It may easily be shown that we can write this potential as the derivative of $1/r$ with respect to the direction l, the differentiation being performed with respect to the point M:

$$\frac{\cos\varphi}{r^2} = \frac{\partial}{\partial l}\left(\frac{1}{r}\right). \tag{18}$$

For, if (ξ, η, ζ) denote the coordinates of N, and (x, y, z) the coordinates of M, we have:

$$\frac{\partial}{\partial l}\left(\frac{1}{r}\right) = \frac{(\xi - x)\cos(l, x) + (\eta - y)\cos(l, y) + (\zeta - z)\cos(l, z)}{r^3},$$

whence, on taking into account the expression:

$$\cos\varphi = \frac{\xi - x}{r}\cos(l, x) + \frac{\eta - y}{r}\cos(l, y) + \frac{\zeta - z}{r}\cos(l, z),$$

we in fact arrive at (18). Function (18) obviously satisfies Laplace's equation and has a singularity at the point N. Let us cover the sur-

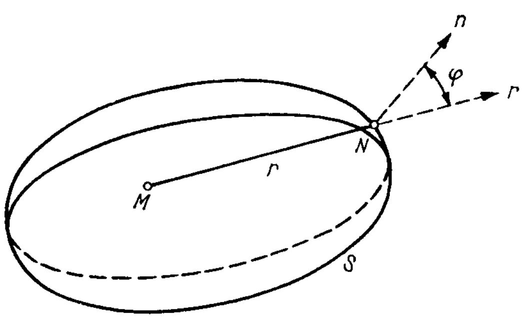

FIG. 9

face S with dipoles in such a way that the direction of the dipole at any point of the surface coincides with the direction of the outward normal n to the surface, and let $\mu(N)$ be the intensity of the dipole located at the point N. We thus arrive at the potential of a double layer, which is defined by the equation (Fig. 9):

$$w(M) = \iint_S \mu(N) \frac{\cos\varphi}{r^2}\, dS \qquad [\varphi = (r, n)]. \tag{19}$$

Function (19) has derivatives of all orders everywhere outside S and satisfies Laplace's equation. It can be differentiated under the integral sign with respect to the coordinates of the point N. If M coincides with a point N_0 lying on the surface, r vanishes when N coincides with N_0, and (19) becomes an improper integral.

Let us show that it has a meaning.

It is sufficient to investigate the integrand on the part σ_0 of the surface close to N_0. We can make use here of equation (4) of the surface referred to local axes at the point N_0.

We find an expression for $\cos\varphi_0 = \cos(\mathbf{r}_0, n)$, where $\mathbf{r}_0$ is the direction N_0N:

$$\cos\varphi_0 = \frac{\xi}{r_0}\cos\ (n, X) + \frac{\eta}{r_0}\cos\ (n, Y) + \frac{\zeta}{r_0}\cos(n, Z) \qquad (20)$$
$$\big((n, Z) = \vartheta_0\big),$$

where (ξ, η, ζ) are the coordinates of N and $r_0 = \sqrt{\xi^2 + \eta^2 + \zeta^2}$. On taking into account inequalities (15), together with the obvious inequalities: $|\xi| \leqslant \varrho_0$; $|\eta| \leqslant \varrho_0$; $\varrho_0 \leqslant r_0$, we obtain:

$$\left|\frac{\cos\varphi_0}{r^2}\right| \leqslant \frac{3c\varrho_0^\alpha}{\varrho_0^2} \qquad \left(\varrho_0 = \sqrt{\xi^2+\eta^2}\right),$$

i.e.

$$\left|\frac{\cos\varphi_0}{r_0^2}\right| \leqslant \frac{b}{\varrho_0^{2-\alpha}}, \qquad (21)$$

where b is a constant. In addition, we have for the continuous function $\mu(N)$:

$$|\mu(N)| \leqslant A \qquad (N \text{ on } S), \qquad (22)$$

where the constant $A = \max|\mu(N)|$ when N varies on S. On replacing the integral over σ_0 by the integral over the projection σ_0' of σ_0 on the XY plane (a circle with centre N_0 and radius $d/3$), we obtain

$$\iint\limits_{\sigma_0'} \mu(\xi,\eta)\frac{\cos\varphi_0}{r_0^2}\cdot\frac{d\xi\, d\eta}{\cos\vartheta_0},$$

where the following inequality holds for the potential function, by (21), (22) and (15):

$$\left|\mu(\xi,\eta)\ \frac{\cos\varphi}{r_0^2\cos\vartheta_0}\right| \leqslant \frac{2Ab}{\varrho_0^{2-\alpha}},$$

whence the convergence of integral (19) follows, when the point M lies on S. Thus *function* (19) *is defined throughout space.*

Let us consider integral (19) with $\mu(N) \equiv 1$. On taking (18) into account, we can write:

$$w_1(M) = \iint_S \frac{\cos\varphi}{r^2}\,dS = -\iint_S \frac{\partial \frac{1}{r}}{\partial n}\,dS, \tag{23}$$

where it is assumed that the differentiation with respect to the direction n is performed with reference to the point N, which is the variable of integration. In view of this, we have put a minus sign in front of the integral.

Suppose first that the point lies outside the closed surface S. In this case $1/r$ is a harmonic function inside S with continuous derivatives of all orders as far as S, and we have by virtue of a fundamental property of harmonic functions [II, 194]:

$$w_1(M) = -\iint_S \frac{\partial \frac{1}{r}}{\partial n}\,dS = 0 \qquad (M \text{ outside } S).$$

Let M lie inside S. We isolate it by a small sphere C with centre M and radius ϱ. In the part of space D' between C and S the function $1/r$ is harmonic, and we have:

$$\iint_S \frac{\partial \frac{1}{r}}{\partial n}\,dS + \iint_C \frac{\partial \frac{1}{r}}{\partial n}\,dS = 0.$$

The outward normal with respect to the domain D' is directed to the centre of the sphere, so that

$$\left.\frac{\partial \frac{1}{r}}{\partial n}\right|_C = \frac{1}{\varrho^2},$$

and the previous equation can be rewritten as:

$$\iint_S \frac{\partial \frac{1}{r}}{\partial n}\,dS + \frac{1}{\varrho^2}\iint_C dS = 0 \quad \text{or} \quad \iint_S \frac{\partial \frac{1}{r}}{\partial n}\,dS + 4\pi = 0,$$

whence

$$w_1(M) = -\iint_S \frac{\partial \frac{1}{r}}{\partial n}\,dS = 4\pi \qquad (M \text{ inside } S).$$

Finally, let M coincide with a point N_0 lying on the surface. We draw a sphere C with centre N_0 and radius $d_1 < d/2$ and replace the part σ_1 of S lying inside C by the part C' of the sphere C such that N_0 lies outside the surface obtained, which consists of $(S - \sigma_1)$ and the part C' of C. We have:

$$\iint\limits_{S-\sigma_1} \frac{\partial \frac{1}{r}}{\partial n} \, dS + \iint\limits_{C'} \frac{\partial \frac{1}{r}}{\partial n} \, dS = 0 . \tag{24}$$

The second term is calculated as above, and is equal to the solid angle subtended by the part C' of C at the centre N_0 of C:

$$\iint\limits_{C'} \frac{\partial \frac{1}{r}}{\partial n} \, dS = \frac{1}{d_1^2} \iint\limits_{C'} dS . \tag{25}$$

The curve l of intersection of the sphere C with S has the property that, by (15), $|\zeta| \leqslant c d_1^{1+\alpha}$ for the ζ coordinate of any point of l; as $d_1 \to 0$, points of l indefinitely approach the XY plane.

It follows from this that the solid angle (25) tends to 2π as d_1 tends to zero, and (24) gives in the limit:

$$w_1(M) = - \iint\limits_{S} \frac{\partial \frac{1}{r}}{\partial n} \, dS = 2\pi \qquad (M \text{ on } S).$$

We thus have

$$\iint\limits_{S} \frac{\cos\varphi}{r^2} \, dS = \begin{cases} 4\pi & (M \text{ inside } S) \\ 0 & (M \text{ outside } S) \\ 2\pi & (M \text{ on } S). \end{cases} \tag{26}$$

Let us also consider an unclosed surface S_1 and the integral

$$w_2(M) = \iint\limits_{S_1} \frac{\cos\varphi}{r^2} \, dS, \tag{27}$$

the point M being assumed to lie outside S_1. We draw the cone with vertex M and base S_1, and let σ_1 be the part of the sphere with centre M and sufficiently small radius ϱ lying inside this cone.

We consider the domain D of space bounded by S_1, σ and the lateral surface Γ of our cone (Fig. 10). (We are assuming that the surfaces mentioned bound some domain D).

The function $1/r$ is harmonic inside D, so that

$$\iint\limits_{S_1} \frac{\partial \frac{1}{r}}{\partial n} \mathrm{d}S + \iint\limits_{\sigma_1} \frac{\partial \frac{1}{r}}{\partial n} \mathrm{d}S + \iint\limits_{\Gamma} \frac{\partial \frac{1}{r}}{\partial n} \mathrm{d}S = 0 .$$

On the surface Γ,

$$\frac{\partial \frac{1}{r}}{\partial n} = - \frac{\cos \varphi}{r^2} = 0 .$$

The direction n is opposite to that of r on σ_1 and $\partial(1/r)/\partial n = 1/\varrho^2$. If ω is the solid angle subtended by S_1 at the point M, we obtain from the previous formula:

$$\omega = - \iint\limits_{S_1} \frac{\partial \frac{1}{r}}{\partial n} \mathrm{d}S = \iint\limits_{S_1} \frac{\cos \varphi}{r^2} \mathrm{d}S,$$

i.e. *integral* (27) *gives the solid angle subtended by* S_1 *at the point* M. The direction n on S_1 is here reckoned outwards from the domain D. The radius vector of M can cut S_1 at several points. If we have say three points of intersection, $\cos \varphi > 0$ at two of them, and $\cos \varphi < 0$ at the third (Fig. 10). The element of our integral, i.e. $(\cos \varphi/r^2)\,\mathrm{d}S$, represents the elementary solid angle $\mathrm{d}\omega$ subtended by an elementary area of the surface at M, this angle being positive if $\cos \varphi > 0$ and negative if $\cos \varphi < 0$. If M lies on S_1, (27) must be regarded as an improper integral, as was the case above for a closed surface. Formulae (26) can again be obtained from our above arguments.

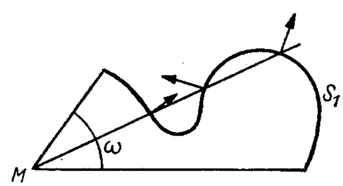

FIG. 10

We shall assume in future that the surface S is such that, for any position of the point M:

$$\iint\limits_{S} \frac{|\cos \varphi|}{r^2} \mathrm{d}s \leqslant c , \tag{28}$$

where c is a definite positive number. Suppose, say, that a positive integer k exists such that, for any position of M, S can be divided into separate parts, the number of which does not exceed k, so that a straight line passing through M cuts each piece in not more than one

point, whilst $\cos\varphi$ retains its sign on each piece. Condition (28) is now fulfilled if we take $c = 4\pi k$.

Formulae (26) show that, with $\mu(N) \equiv 1$, the potential of the double layer (19) has a discontinuity when M passes through S. Let us discuss this question for an arbitrary continuous density.

Let N_0 be a fixed point of S. We form the potential of the double layer:

$$w_0(M) = \iint\limits_S [\mu(N) - \mu(N_0)] \frac{\cos\varphi}{r^2} \, dS, \tag{29}$$

and show that it retains its continuity when M passes through the surface at a point N_0. Let ε be a given positive number. We isolate a piece σ of S containing N_0 as an interior point, on which the following inequality is satisfied:

$$|\mu(N) - \mu(N_0)| \leqslant \frac{\varepsilon}{4c} \qquad (N \text{ on } \sigma), \tag{30}$$

where c is the constant appearing in condition (28). On dividing S into two pieces, σ and $S - \sigma$, we can write:

$$w_0(M) = w_0^{(1)}(M) + w_0^{(2)}(M), \tag{31}$$

where

$$\begin{aligned} w_0^{(1)}(M) &= \iint\limits_\sigma [\mu(N) - \mu(N_0)] \frac{\cos\varphi}{r^2} \, dS; \\ w_0^{(2)}(M) &= \iint\limits_{S-\sigma} [(\mu(N) - \mu(N_0))] \frac{\cos\varphi}{r^2} \, dS. \end{aligned} \tag{32}$$

We have, for any position of the point M:

$$|w_0^{(1)}(M)| \leqslant \iint\limits_\sigma |\mu(N) - \mu(N_0)| \frac{|\cos\varphi|}{r^2} \, dS,$$

whence, by (28) and (30),

$$|w_0^{(1)}(M)| \leqslant \frac{\varepsilon}{4} \, . \tag{33}$$

It follows from (31) that

$$w_0(M) - w_0(N_0) = w_0^{(1)}(M) - w_0^{(1)}(N_0) + [w_0^{(2)}(M) - w_0^{(2)}(N_0)],$$

whence

$$|w_0(M) - w_0(N_0)| \leqslant |w_0^{(1)}(M)| + |w_0^{(1)}(N_0)| + |w_0^{(2)}(M) - w_0^{(2)}(N_0)|,$$

or, by (33):

$$|w_0(M) - w_0(N)| \leqslant \frac{\varepsilon}{2} + |w_0^{(2)}(M) - w_0^{(2)}(N_0)|. \tag{34}$$

In the double layer potential $w_0^{(2)}(M)$, the integration is performed over $(S - \sigma)$, whilst the point N_0 lies inside σ, so that $w_0^{(2)}(M)$ is a continuous function at and in the neighbourhood of N_0 (and has derivatives of all orders). We thus have $| w_0^{(2)}(M) - w_0^{(2)}(N_0) | \leqslant \varepsilon/2$ for all M sufficiently close to N_0, and by (34), $| w_0(M) - w_0(N_0) | \leqslant \varepsilon$, whence it follows, since ε is arbitrary, that the function $w_0(M)$ defined by (29) is continuous at N_0. We can write:

$$w_0(M) = w(M) - \mu(N_0) \iint\limits_S \frac{\cos\varphi}{r^2} \, dS, \tag{35}$$

where $w(M)$ is the potential of the double layer (19). Suppose first that M lies on S. We shall denote it by N. Now, by (26):

$$w_0(N) = w(N) - 2\pi\mu(N_0) \tag{36}$$

and

$$w_0(N_0) = w(N_0) - 2\pi\mu(N_0), \tag{37}$$

where $w(N_0)$ is the value of integral (19) at N_0. We shall now let the point N on S tend to N_0. Since we have proved that $w_0(M)$ is continuous, we have:

$$w_0(N) \to w_0(N_0) = w(N_0) - 2\pi\mu(N_0).$$

On returning to the right-hand side of (36), we see that $w(N)$ now has a limit $w(N_0)$, i.e. *the function $w(M)$ defined by* (19) *is continuous on the surface S.*

Now let M be inside S. We have here, by (26):

$$w_0(M) = w(M) - 4\pi\mu(N_0). \tag{38}$$

Now let M, lying inside S, tend to N_0. We have, by virtue of the proved continuity of $w_0(M)$:

$$w_0(M) \to w_0(N_0) = w(N_0) - 2\pi\mu(N_0). \tag{39}$$

On returning to the right-hand side of (38), we see that $w(M)$ has a limit. Let us write this limit as $w_i(N_0)$. It follows from (38) and (39) that

$$w_i(N_0) - 4\pi\mu(N_1) = w(N_0) - 2\pi\mu(N_0),$$

i.e.

$$w_i(N_0) = w(N_0) + 2\pi\mu(N_0). \tag{40}$$

Hence it is clear that the limit $w_i(N_0)$ and the value $w(N_0)$ of the function $w(M)$ at N_0 are different if $\mu(N_0) \neq 0$. If M is outside S, we have instead of (38):

$$w_0(M) = w(M)$$

and it may be seen, arguing as above, that $w(M)$ has a limit when M tends to N_0 outside S. On writing $w_e(N_0)$ for this limit, we have with the aid of (39):

$$w_e(N_0) = w(N_0) - 2\pi\mu(N_0). \tag{41}$$

On writing r_0 and φ_0 for the values of r and φ when M and N_0 coincide, (40) and (41) can be rewritten as

$$\left.\begin{aligned} w_i(N_0) &= w(N_0) + 2\pi\mu(N_0) = \iint_S \mu(N)\frac{\cos\varphi_0}{r_0^2}\,\mathrm{d}S + 2\pi\mu(N_0) \\ w_e(N_0) &= w(N_0) - 2\pi\mu(N_0) = \iint_S \mu(N)\frac{\cos\varphi_0}{r_0^2}\,\mathrm{d}S - 2\pi\mu(N_0). \end{aligned}\right\} \tag{42}$$

Here, φ_0 is the angle formed by the direction $\overline{N_0N}$ with the outward normal n at the variable point N, i.e. $\varphi_0 = (\mathbf{r}_0, n)$. On taking into account these formulae and the continuity of $w(N_0)$ when N_0 varies on S, we can say that the function $w(M)$ defined by (19) is continuous inside and up to S. Similarly, it is continuous outside and up to S. We recall that this function has derivatives of all orders inside and outside S. It may easily be seen that $w(M)$ tends to zero as M moves away indefinitely. For, if D denotes the shortest distance from M outside S to S [II, 89], we have

$$|w(M)| \leqslant \iint_S \left|\mu(N)\frac{\cos\varphi}{r^2}\right| \mathrm{d}S \leqslant \frac{A}{D^2}\cdot \text{area } S. \tag{43}$$

Hence it follows that $w(M) \to 0$ as M moves away indefinitely. More precisely, if O is any fixed point, then given any positive ε there exists a positive number B such that $|w(M)| \leqslant \varepsilon$ provided M lies outside the sphere with centre O and radius B.

194. Properties of the potential of a simple layer. The potential of a simple layer

$$u(M) = \iint_S \frac{\mu(N)}{r}\,\mathrm{d}S \tag{44}$$

is an improper integral if M lies on S. Let M coincide with a point N_0 on S. We show that the improper integral (44) now has a meaning. As in [193], it is sufficient to consider it on the part σ_0 of S containing N_0 as an interior point. We make use of equation (4) for σ_0 in

local coordinates. We have:

$$\iint\limits_{\sigma_0} \frac{\mu(N)}{r_0} \,\mathrm{d}S = \iint\limits_{\sigma_0'} \frac{\mu(\xi, \eta)}{r_0 \cos \vartheta_0} \,\mathrm{d}\xi \,\mathrm{d}\eta.$$

Using (15), (22) and the fact that $r_0 \leqslant \varrho_0$, we obtain the inequality for the integrand:

$$\left| \frac{\mu(\xi, \eta)}{r_0 \cos \vartheta_0} \right| \leqslant \frac{2A}{\varrho_0},$$

whence the convergence of integral (44), when M lies on S, follows at once. Formula (44) therefore defines $u(M)$ for any position of the point M. The function $u(M)$ is continuous at points M outside S. We show that $u(M)$ is also continuous at any point N_0 on S. Let ε be any given positive number, and σ_1 the part of S defined by inequality (17). We show that it is possible to choose d_1 so small that, for any position of M in some neighbourhood of N_0:

$$\left| \iint\limits_{\sigma_1} \frac{\mu(N)}{r} \,\mathrm{d}S \right| \leqslant \frac{\varepsilon}{4}. \tag{45}$$

We have:

$$\left| \iint\limits_{\sigma_1} \frac{\mu(N)}{r} \,\mathrm{d}S \right| \leqslant \iint\limits_{\sigma_1'} \frac{A}{\varrho_1} \,\mathrm{d}\xi \,\mathrm{d}\eta, \tag{46}$$

where σ_1' is the circle with centre N_0 and radius d_1, and ϱ_1 is the length of the projection M_1N_1 of the segment MN on the tangent plane. Suppose that M lies inside the sphere with centre N_0 and radius d_1. M_1 now lies in the circle σ_1', and if we take the circle σ_1'' with centre M_1 and radius $2d_1$, it will contain the whole of circle σ_1', so that, by (46):

$$\left| \iint\limits_{\sigma_1} \frac{\mu(N)}{r} \,\mathrm{d}S \right| \leqslant A \iint\limits_{\varrho_1 \leqslant 2d_1} \frac{\mathrm{d}\xi \,\mathrm{d}\eta}{\varrho_1} = A \int\limits_0^{2\pi} \int\limits_0^{2d_1} \frac{\varrho_1 \,\mathrm{d}\varrho_1 \,\mathrm{d}\theta}{\varrho_1} = 4\pi d_1 A.$$

It remains to fix d_1 such that $4\pi d_1 A \leqslant \varepsilon/4$, and we obtain inequality (45) for any position of M in the sphere with centre N_0 and radius d_1. We can further write function (44) as

$$u(M) = u_1(M) + u_2(M),$$

where

$$u_1(M) = \iint\limits_{\sigma_1} \frac{\mu(N)}{r} \,\mathrm{d}S; \quad u_2(M) = \iint\limits_{S-\sigma_1} \frac{\mu(N)}{r} \,\mathrm{d}S,$$

and $u_2(M)$ is continuous at N_0; the proof that $u(M)$ is continuous at N_0 follows precisely the same lines as in [193] for function (29). We thus

have the following result: *the potential of a simple layer* (44) *is defined throughout space and is continuous throughout space.* It can be shown, exactly as in [193], that $u(M) \to 0$ as the point M becomes infinitely remote.

195. The normal derivative of the potential of a simple layer. Let n_0 be the direction of the outward normal at a point N_0 of the surface S. Assuming that M does not lie on S, let us form the derivative of function (44) with respect to the direction n_0. Only the factor $1/r$ depends on M, and we can differentiate under the integral sign:

$$\frac{\partial u(M)}{\partial n_0} = \iint_S \mu(N) \frac{\partial \frac{1}{r}}{\partial n_0} \mathrm{d}S = \iint_S \mu(N) \frac{\cos \psi}{r^2} \mathrm{d}S. \tag{47}$$

Notice the difference between the last integral and integral (19), giving the potential of a double layer. In integral (19), $\varphi = (\mathbf{r}, \mathbf{n})$, where

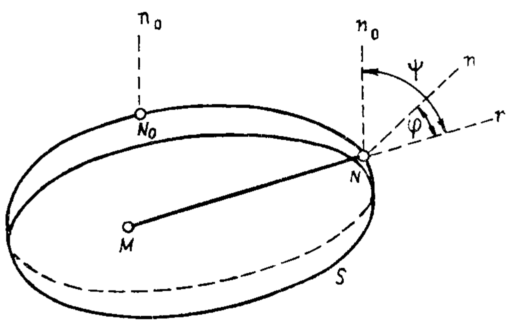

FIG. 11

$\mathbf{n}$ is the unit outward normal vector at N, the variable point of the integration, whilst in integral (47), $\psi = (\mathbf{r}, \mathbf{n}_0)$, where $\mathbf{n}_0$ is the unit outward normal vector at the fixed point N_0. In both cases r is the direction $\overline{MN}$ (Fig. 11). We show that integral (47) exists even when M coincides with the point N_0 mentioned above. We shall write integral (47) in this latter case in the form:

$$\iint_S \mu(N) \frac{\cos \psi_0}{r_0^2} \mathrm{d}S = \iint_S \mu(N) \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \mathrm{d}S, \tag{48}$$

where r_0 is the distance $|N_0N|$, and the angle $\psi_0 = (\mathbf{r}_0, \mathbf{n}_0)$ is the angle between the directions $\overline{N_0N}$ and n_0. We show further that when

M approaches N_0 from inside or from outside the surface along the normal, derivative (47) has definite limits, for which the following expressions hold:

$$\left.\begin{aligned}\left(\frac{\partial u(N_0)}{\partial n_0}\right)_i &= \iint_S \mu(N)\frac{\cos\psi_0}{r_0^2}\,dS + 2\pi\mu(N_0)\\ \left(\frac{\partial u(N_0)}{\partial n_0}\right)_e &= \iint_S \mu(N)\frac{\cos\psi_0}{r_0^2}\,dS - 2\pi\mu(N_0),\end{aligned}\right\} \tag{49}$$

where, as in [193], the subscripts i and e mean that the limits of $\partial u(M)/\partial n_0$ have to be taken as M tends to N_0 from inside and from outside the surface, and the left-hand sides merely represent a notation for these limits.

In the local systems of coordinates with origin at N_0 the direction n_0 coincides with the direction of the Z axis. We shall write (x, y, z) as above for the coordinates of M, and (ξ, η, ζ) for the coordinates of N in the local system. On isolating the piece σ_0 of the surface S as usual, we can write integral (47) as

$$\iint_{\sigma_0} \mu(N)\frac{\zeta - z}{r^3}\,dS. \tag{50}$$

If M coincides with N_0, $z = 0$, and the integral becomes

$$\iint_{\sigma_1} \mu(N)\,\frac{\zeta}{r_0^3}\,dS = \iint_{\sigma_1'} \mu(\xi,\eta)\frac{\zeta(\xi,\eta)}{r_0^3\cos(n, Z)}\,d\xi\,d\eta,$$

where ζ is replaced in accordance with (4). On taking (15) into account, and the fact that $r_0 \geqslant \varrho_0$, it can be seen at once that the integral written above has a meaning. We have thus proved the existence of integral (50). Let us turn to the proof of (49).

We form the difference between integral (47) and the potential of a double layer with the same density $\mu(N)$:

$$\frac{\partial u(M)}{\partial n_0} - w(M) = \iint_S \mu(N)\frac{\cos\psi - \cos\varphi}{r^2}\,dS. \tag{51}$$

This last integral has a meaning if M does not lie on S or if M coincides with N_0.

We show that this difference remains continuous when M passes through the surface S at N_0. As in previous sections, it is sufficient here to show that the last integral, taken over the small piece σ_1 of S given by condition (17), can be made as small as desired in absolute value. We shall assume in future inequalities that M is on the normal

to S at N_0, i.e. that $x = y = 0$ in the local coordinate system. We now have

$$\frac{\cos\psi - \cos\varphi}{r_2} = -\frac{\xi}{r^3}\cos(\mathbf{n}, X) - \frac{\eta}{r^3}\cos(\mathbf{n}, Y) - \frac{\zeta - z}{r^3}(\cos\vartheta_0 - 1). \quad (52)$$

On taking into account (15) and the inequalities

$$|\xi| \leqslant \varrho_0, \quad |\eta| \leqslant \varrho_0; \quad r \geqslant \varrho_0 \geqslant \frac{1}{2}r; \quad |\zeta - z| \leqslant r \leqslant 2\varrho_0,$$

where $\varrho_0 = \sqrt{\xi^2 + \eta^2}$ is the length of the projection of $\overline{MN}$ on the XY plane, we obtain:

$$\frac{|\cos\psi - \cos\varphi|}{r^2} \leqslant \frac{b_1}{\varrho_0^{2-\alpha}},$$

where b_1 is a constant. Thus, on taking (22) into account, we have

$$\left|\int_{\sigma_1}\!\!\int \mu(N)\frac{|\cos\psi - \cos\varphi|}{r^2}\,\mathrm{d}S\right| \leqslant \int\!\!\int_{\varrho_0 \leqslant d_1} \frac{2Ab_1}{\varrho_0^{2-\alpha}}\,\mathrm{d}\xi\,\mathrm{d}\eta =$$

$$= 2Ab_1\int_0^{2\pi}\!\!\int_0^{d_1}\frac{\mathrm{d}\varrho_0\,\mathrm{d}\varphi}{\varrho_0^{1-\alpha}} = b_2 d_1^{\alpha}, \quad (53)$$

where b_2 is a constant. This inequality holds for any position of M on the normal to S at N_0, including M coinciding with N_0. Hence it follows that, with d_1 sufficiently small, the integral on the right-hand side of (51), taken over σ_1, will have an absolute value less than any previously assigned positive number for a suitable choice of d_1. This shows that the difference (51) must be continuous at N_0. But $w(M)$ has a limit as M tends to N_0 from inside or from outside S. Hence it follows that (47) also has a limit in both cases. By using the continuity of the difference (51) we obtain

$$\left(\frac{\partial u(N_0)}{\partial n_0}\right)_i - w_i(N_0) = \int_S\!\!\int \mu(N)\frac{\cos\psi_0}{r_0^2}\,\mathrm{d}S - w(N_0),$$

and if we take into account the first of expressions (42), we get the first of expressions (49). The second of expressions (49) is obtained in a similar way. These formulae lead us directly to the size of the jump in the normal derivative of the potential of a single layer:

$$\left(\frac{\partial u(N_0)}{\partial n_0}\right)_i - \left(\frac{\partial u(N_0)}{\partial n_0}\right)_e = 4\pi\mu(N_0). \quad (54)$$

196. The normal derivative of the potential of a simple layer (continued). It is important for what follows to be able to show that the normal derivative tends to its limits

$$\left(\frac{\partial u\,(N_0)}{\partial n_0}\right)_i \text{ and } \left(\frac{\partial u\,(N_0)}{\partial n_0}\right)_e$$

uniformly for the whole of the surface S as M tends to N_0 along the normal. To do this, we must first prove the uniform convergence of the integral in (51). Let $\omega(M)$ denote this integral. As already mentioned, this function has a meaning if M coincides with N_0. We have to show that, given any positive ε, there exists a positive η, not depending on the position of N_0 on S, such that $|\,\omega(M) - \omega(N_0)\,| \leqslant \varepsilon$ if $|\,MN_0\,| \leqslant \eta$, M being on the normal to S at N_0.

We fix d_1 so that $b_2\, d_1^a \leqslant \varepsilon/4$ and write $\omega(M) = \omega_1(M) + \omega_2(M)$, where

$$\omega_1(M) = \iint\limits_{\sigma_1} \mu\,(N)\,\frac{\cos\psi - \cos\varphi}{r^2}\,dS;\quad \omega_2(M) = \iint\limits_{S-\sigma_1} \mu\,(N)\,\frac{\cos\psi - \cos\varphi}{r^2}\,dS.$$

Now, by (53), we have $|\,\omega_1(M)\,| \leqslant \varepsilon/4$ for any position of M on the normal to S at N_0. Further,

$$\omega\,(M) - \omega\,(N_0) = \omega_1\,(M) - \omega_1\,(N_0) + \left[\omega_2\,(M) - \omega_2\,(N_0)\right],$$

whence

$$\begin{aligned}\omega\,(M) - \omega\,(N_0)\,| &\leqslant |\,\omega_1\,(M)\,| + |\,\omega_1\,(N_0)\,| + |\,\omega_2\,(M) - \omega_2(N_0)\,| \leqslant \\ &\leqslant \frac{\varepsilon}{2} + |\,\omega_2\,(M) - \omega_2(N_0)\,|. \qquad (55)\end{aligned}$$

On taking (52) into account, we obtain

$$\begin{aligned}&\left[\frac{\cos\psi - \cos\varphi}{r^2}\right]_M - \left[\frac{\cos\psi - \cos\varphi}{r_2}\right]_{N_0} = \\ &= \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\left[\xi\cos(\mathbf{n}, X) + \eta\cos(\mathbf{n}, Y) + \zeta\,(\cos\vartheta_0 - 1)\right] + \\ &\quad + \frac{z}{r^3}\,(\cos\vartheta_0 - 1) \qquad \left(\vartheta_0 = (\mathbf{n}, Y)\right). \qquad (56)\end{aligned}$$

When integrating over $(S - \sigma_1)$ we have $r \geqslant d_1$ and $r_0 \geqslant d_1$. In addition, for any positions of N and N_0 on S, the absolute values of ξ, η, ζ do not exceed the diameter D of the surface S, i.e. the maximum distance between points of S. We have further: $|\,r - r_0\,| \leqslant |\,z\,|$ and

$$\left|\frac{1}{r_0^3} - \frac{1}{r^3}\right| = |\,r - r_0\,|\left(\frac{1}{r_0^3 r} + \frac{1}{r_0^2 r^2} + \frac{1}{r_0 r^3}\right) \leqslant \frac{3\,|\,z\,|}{d_1^4};\quad \frac{|\,z\,|}{r^3} \leqslant \frac{|\,z\,|}{d_1^3},$$

and we obtain, in accordance with (56):

$$\left| \left[\frac{\cos\psi - \cos\varphi}{r^2} \right]_M - \left[\frac{\cos\psi - \cos\varphi}{r^2} \right]_{N_0} \right| \leqslant c_1 |z|,$$

where c_1 is a definite constant, not depending on the position of N_0. Obviously it depends on the choice of d_1. We obtain on taking into account the expression for $\omega_2(M)$:

$$|w_2(M) - w_2(N_0)| \leqslant \iint\limits_{S-\sigma_1} |\mu(N)| c_1 |z| \, dS \leqslant Ac_1 |z| \cdot \text{area } S.$$

If we take:

$$|z| \leqslant \frac{\varepsilon}{2Ac_1 \cdot \text{area } S}, \tag{57}$$

we have $|\omega_2(M) - \omega_2(N_0)| \leqslant \varepsilon/2$, and, by (55): $|\omega(M) - \omega(N_0)| \leqslant \varepsilon$. We can therefore take the right-hand side of inequality (57) as the required number η.

We have shown that the difference

$$\frac{\partial u(M)}{\partial n_0} - w(M)$$

tends uniformly, with respect to the position of N_0 on S, to its limit as M tends to N_0 along the normal. On the other hand, the double layer potential $w(M)$ is a continuous function as far as S, so that $w(M)$ also tends uniformly to its limits on S. Hence it follows that the normal derivative $\partial u(M)/\partial n_0$ also tends to its limiting values (49) uniformly on S. We shall say, following A. M. Lyapunov, that a function $u(M)$, harmonic inside or outside S, has a *regular normal derivative* if, as M tends to N_0 along the normal to S, the normal derivative $\partial u(M)/\partial n_0$ tends to its limits uniformly with respect to the point N_0 on S. We can therefore assert that:

THEOREM. *The potential of a simple layer with continuous density has regular normal derivatives both inside and outside S.*

Having fixed the positive value $|z|$, with M lying either inside or outside S, we can assume that the value of the normal derivative $\partial u(M)/\partial n_0$ is a function of N_0 that also depends on the parameter $|z|$, this function being a continuous function of N_0, since $u(M)$ has continuous derivatives inside and outside S, and the direction n on S varies continuously.

Since the convergence is uniform as $|z| \to 0$, we can say that limits (49) are also continuous functions of N_0, whilst it follows from this

that the *integral on the right-hand sides of expressions* (49) *is a continuous function of N_0 on S.* This integral is called *the direct value of the normal derivative of the simple layer potential on S.*

197. The direct value of the normal derivative. Let $F(N)$ denote the direct value of the normal derivative on S:

$$F(N_0) = \iint_S \mu(N) \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \, dS. \tag{58}$$

We have seen that $F(N_0)$ is a continuous function of the position of N_0 on S. We shall now prove a theorem that makes this property of $F(N_0)$ more precise. It was first proved by Lyapunov.

THEOREM. *With a continuous density $\mu(N)$, the function $F(N_0)$ satifies the condition:*

$$| F(N_1) - F(N_0) | \leqslant B r_{0,1}^{\beta}, \tag{59}$$

where B and β are positive constants and $r_{0,1} = | N_0 N_1 |$.

We shall in future refer to (59) as a *Lipschitz condition.* If $r_{0,1}$ is greater than some positive quantity, given any positive β, we can satisfy this inequality by means of a suitable choice of constant B. For we know that $F(N)$ is continuous on S and therefore bounded, i.e. $| F(N) | \leqslant A_1$, and if $r_{0,1} \geqslant h > 0$, by taking $B = 2A_1/h^{\beta}$, we can obviously obtain (59) with $r_{0,1} \geqslant h$. If we obtain a different B in (59) when $r_{0,1} < h$, by taking the greater of these two values of B we can write (59) for all values of $r_{0,1}$. We can therefore assume that say $r_{0,1} < d/10$. We have:

$$F(N_1) - F(N_0) = \iint_S \mu(N) \left[\frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \right] dS,$$

where $\mathbf{r}_0$ and $\mathbf{r}_1$ are the vectors $\overline{N_0 N}$ and $\overline{N_1 N}$, whilst r_0 and r_1 are their lengths, so that we obtain, on taking (22) into account:

$$| F(N_1) - F(N_0) | \leqslant A \iint_S \left| \frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \right| dS. \tag{60}$$

We cut out a part σ_1 of S with the aid of a circular cylinder, the axis of which is the normal to S at N_0 and the base radius $2r_{0,1}$. We divide the integral over S into integrals over σ_1 and $S - \sigma_1$:

$$\left. \begin{aligned} J_1 &= \iint_{\sigma_1} \left| \frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \right| dS; \\ J_2 &= \iint_{S-\sigma_1} \left| \frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \right| dS. \end{aligned} \right\} \tag{61}$$

We introduce the scalar products of vectors and write

$$\frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} = \frac{\mathbf{r}_1 \cdot \mathbf{n}_1}{r_1^3} - \frac{\mathbf{r}_0 \cdot \mathbf{n}_0}{r_0^3} =$$

$$= \frac{\mathbf{r}_1 \cdot \mathbf{n}_0 - \mathbf{r}_0 \cdot \mathbf{n}_0}{r_1^3} + \frac{\mathbf{r}_1 \cdot \mathbf{n}_1 + \mathbf{r}_1 \cdot \mathbf{n}_0}{r_1^3} + \mathbf{r}_0 \cdot \mathbf{n}_0 \left(\frac{1}{r_1^3} - \frac{1}{r_0^3}\right),$$

where, as usual, $\mathbf{n}_0$ and $\mathbf{n}_1$ are the unit outward normal vectors at N_0 and N_1.

It follows from the above that

$$\left|\frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2}\right| \leqslant$$

$$\leqslant \frac{|\mathbf{n}_1 \cdot \mathbf{n}_0 - \mathbf{r}_0 \cdot \mathbf{n}_0|}{r_1^3} + \frac{|\mathbf{r}_1 \cdot \mathbf{n}_1 - \mathbf{r}_1 \cdot \mathbf{n}_0|}{r_1^3} + |\mathbf{r}_0 \cdot \mathbf{n}_0| \left|\frac{1}{r_1^3} - \frac{1}{r_0^3}\right|. \tag{62}$$

We write inequalities for the separate terms:

$$|\mathbf{r}_1 \cdot \mathbf{n}_1 - \mathbf{r}_1 \cdot \mathbf{n}_0| = |\mathbf{r}_1 \cdot (\mathbf{n}_1 - \mathbf{n}_0)| \leqslant r_1 |\mathbf{n}_1 - \mathbf{n}_0|.$$

On forming the triangle with sides $\mathbf{n}_0$ and $\mathbf{n}_1$, we find that $|\mathbf{n}_1 - \mathbf{n}_0| \leqslant \theta$, where θ is the angle between $\mathbf{n}_0$ and $\mathbf{n}_1$. On taking condition (3) into account, we can write:

$$|\mathbf{r}_1 \cdot \mathbf{n}_1 - \mathbf{r}_1 \cdot \mathbf{n}_0| \leqslant a r_1 r_{0,1}^{\alpha},$$

where a is a constant. Further:

$$\mathbf{r}_1 \cdot \mathbf{n}_0 - \mathbf{r}_0 \cdot \mathbf{n}_0| = |(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}_0| = |\mathbf{r}_{0,1} \cdot \mathbf{n}_0| = |\zeta_1|,$$

where ζ_1 is the coordinate of N_1 in the local system with origin at N_0. On taking (15) into account, we have:

$$|\mathbf{r}_1 \cdot \mathbf{n}_0 - \mathbf{r}_0 \cdot \mathbf{n}_0| \leqslant c r_{0,1}^{1+\alpha}.$$

Finally, if the point of integration N is sufficiently close to N_0, we have, by (15), $|\mathbf{r}_0 \cdot \mathbf{n}_0| = |\zeta| \leqslant c r_0^{1+\alpha}$. But, as in the case of (59), we can assume that this last inequality holds for all r_0. On substituting all these inequalities in (62), we have

$$\left|\frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2}\right| \leqslant$$

$$\leqslant \frac{r_1 r_{0,1}^{\alpha}}{r_1^3} + \frac{r_{0,1}^{1+\alpha}}{r_1^3} + c_1 r_0^{1+\alpha} |r_1 - r_0| \left(\frac{1}{r_0^3 r_1} + \frac{1}{r_0^2 r_1^2} + \frac{1}{r_0 r_1^3}\right), \tag{63}$$

where c_1 is the greater of the constants a and c. From the triangle N_0N_1N: $r_1 + r_{0,1} \geqslant r_0$. But we have $r_{0,1} \leqslant r_0/2$ when integrating over $(S - \sigma_1)$, so that $r_1 \geqslant r_0/2$. On using these inequalities, together with the inequality

$|r_1 - r_0| \leqslant r_{0,1}$, we can write instead of (63):

$$\left|\frac{\cos(\mathbf{r}_1, \mathbf{n}_1)}{r_1^2} - \frac{\cos(r_0, \mathbf{n}_0)}{r_0^2}\right| \leqslant c_1 r_{0,1}^{\alpha}\left(\frac{1}{r_1^2} + \frac{r_{0,1}}{r_1^3} + \frac{r_0^{1+\alpha} r_{0,1}^{1-\alpha}}{r_0^3 r_1} + \frac{r_0^{1+\alpha} r_1^{1-\alpha}}{r_0^2 r_1^2}\frac{r_0^{1+\alpha} r_1^{1-\alpha}}{r_0 r_1^3}\right)$$

$$\leqslant c_1 r_{0,1}^{\alpha}\left(\frac{4}{r_0^2} + \frac{4}{r_0^2} + \frac{2}{r_0^2} + \frac{4}{r_0^2} + \frac{8}{r_0^2}\right) = \frac{22 c_1 r_{0,1}^{\alpha}}{r_0^2}.$$

We obtain on returning to the second of expressions (61):

$$|J_2| \leqslant c_0 r_{0,1}^{\alpha} \iint_{S-\sigma_1} \frac{\mathrm{d}S}{r_0^2}, \tag{64}$$

where $c_0 = 22c_1$. The radius of the cylinder which cuts out σ_1 from S was taken as $2r_{0,1}$. Let us take the cylinder with the same axis and fixed radius $d/3$. It cuts out a piece σ_0 from S, where σ_0 contains σ_1 in its interior.

We have:

$$\iint_{S-\sigma_1} \frac{\mathrm{d}S}{r_0^2} = \iint_{\sigma_0-\sigma_1} + \frac{\mathrm{d}S}{r_0^2} \iint_{S-\sigma_0} + \frac{\mathrm{d}S}{r_0^2}.$$

Since $r_0 \geqslant d/3$ in the second integral,

$$\iint_{S-\sigma_0} \frac{\mathrm{d}S}{r_0^2} \leqslant \frac{9}{d^2} \cdot \text{area}\, S.$$

When integrating over $(\sigma_0 - \sigma_1)$, we can perform the integration on the tangent plane at N_0 and obtain, by means of the usual inequalities ($r_0 \geqslant \varrho_0$ and $\cos(\mathbf{n}, Z) \geqslant 1/2$):

$$\iint_{\sigma_0-\sigma_1} \frac{\mathrm{d}S}{r_0^2} \leqslant \int_0^{2\pi}\int_{2r_{0,1}}^{\frac{d}{3}} \frac{2\varrho_0 \mathrm{d}\varrho_0 \mathrm{d}\theta}{\varrho_0^2} = 4\pi\left(\log\frac{d}{3} - \log 2r_{0,1}\right).$$

On substituting in (64), we get an inequality of the form:

$$J_2 \leqslant A_1 r_{0,1}^{\alpha} \log r_{0,1} + B_1 r_{0,1}^{\alpha},$$

where A_1 and B_1 are constants. This can be replaced by an inequality of the form:

$$J_2 \leqslant A_2 r_{0,1}^{\beta},$$

if we take a positive β less than α.

As regards an inequality for J_1, we have

$$J_1 \leqslant \iint_{\sigma_1} \frac{|\cos(\mathbf{r}_1, \mathbf{n}_1)|}{r_1^2}\mathrm{d}S + \iint_{\sigma_1} \frac{|\cos(\mathbf{r}_0, \mathbf{n}_0)|}{r_0^2}\mathrm{d}S. \tag{65}$$

Application of the usual inequalities gives

$$\iint\limits_{\sigma_1} \frac{|\cos(\mathbf{r}_0, \mathbf{n}_0)|}{r_0^2} \, dS = \iint\limits_{\sigma_1} \frac{|\zeta|}{r_0^3} \, dS \leqslant c \int\limits_0^{2\pi} \int\limits_0^{2r_{0,1}} \frac{\varrho_0^{1+\alpha} \varrho_0 \, d\varrho_0 \, d\theta}{\varrho_0^3} = A_3 r_{0,1}^\alpha,$$

where A_3 is a constant. To obtain an inequality for the first of integrals (65), we draw a sphere with centre N_1 and radius $4r_{0,1}$, observing that $4r_{0,1} < 2d/5$. This cuts out from S a piece σ_2 containing the piece σ_1. This piece σ_2 has an explicit equation in the local coordinates with origin N_1, and we can apply the usual inequalities on this piece, the integration being performed on the tangent plane at N_1. The domain of integration will be part of a circle with centre N_0 and radius $4r_{0,1}$. On integrating over the whole of the circle, we get the inequality

$$\iint\limits_{\sigma_1} \frac{|\cos(\mathbf{r}_1, \mathbf{n}_1)|}{r_1^2} \, dS \leqslant \iint\limits_{\sigma_2} \frac{|\cos(\mathbf{r}_1 \, \mathbf{n}_1)|}{r_1^2} \, dS < A_4 r_{0,1}^\alpha.$$

On substituting all the inequalities obtained in (60), we have

$$|F(N_1) - F(N_0)| \leqslant A(A_2 r_{0,1}^\beta + A_5 r_{0,1}^\alpha),$$

and finally we can write (59), where β is any positive number less than α.

198. The derivative of the potential of a simple layer with respect to any direction. We investigated in [195] the limits of the normal derivative of the potential of a simple layer as M approaches N_0 along the normal. If we assume more than continuity as regards the density $\mu(N)$, it can be shown that limits exist for the derivatives with respect to any fixed direction, and that, moreover, these limits do not depend on the law by which M approaches N_0. Suppose that the density satisfies the Lipschitz condition:

$$|\mu(N_2) - \mu(N_1)| \leqslant B r_{1,2}^\delta, \tag{66}$$

where $r_{1,2} = |N_1 N_2|$, and B and δ are positive constants ($\delta \leqslant 1$). Let XYZ be the local coordinate system at N_0 on S. We take the derivative of $u(M)$ with respect to the direction x, lying in the tangent plane to S at N_0. We shall assume for the present that M lies on the normal to S at N_0. Let us suppose for the sake of definiteness that M lies inside S. We have:

$$\frac{\partial u(M)}{\partial x} = \iint\limits_S \mu(N) \frac{\xi}{r^3} \, dS \qquad (r = |MN|). \tag{67}$$

We introduce the quantity $r' = \sqrt{\xi^2 + \eta^2 + z^2}$ and consider

$$\iint\limits_{\sigma_0} \mu(N_0) \frac{\xi}{r'^3} \cos(\mathbf{n}, Z) \, dS = \mu(N_0) \iint\limits_{\sigma_0'} \frac{\xi}{(\xi^2 + \eta^2 + z^2)^{3/2}} \, d\xi \, d\eta \; (z \neq 0), \tag{68}$$

where σ_0' is the circle $\xi^2 + \eta^2 \leqslant d^2/9$. We obviously have:

$$\iint_{\sigma_0'} \frac{\xi}{(\xi^2 + \eta^2 + z^2)^{3/2}} d\xi\, d\eta = \int_0^{2\pi} \cos\theta\, d\theta \int_0^{\frac{d}{3}} \frac{\varrho_0^2}{\sqrt{\varrho_0^2 + z^2}} d\varrho_0 = 0.$$

We can write instead of (67):

$$\frac{\partial u\,(M)}{\partial x} = \iint_{\sigma_0} \mu\,(N) \frac{\xi}{r^3} dS + \iint_{S-\sigma_0} \mu\,(N) \frac{\xi}{r^3} dS = v_1\,(M) + v_2\,(M). \tag{69}$$

On using the fact that integral (68) vanishes, we have

$$v_1\,(M) = \iint_{\sigma_0} \xi \left[\frac{\mu\,(N)}{r^3} - \frac{\mu\,(N_0) \cos\,(\mathbf{n}, Z)}{r'^3}\right] dS. \tag{70}$$

The difference appearing under the integral sign can be written as

$$\frac{\mu\,(N)}{r^3} - \frac{\mu\,(N_0) \cos\,(\mathbf{n}, Z)}{r'^3} = \frac{\mu\,(N) - \mu\,(N_0)}{r^3} + \frac{\mu\,(N_0)\,[1 - \cos(\mathbf{n}, Z)]}{r^3} +$$
$$+ \mu\,(N_0) \cos(\mathbf{n}, Z) \left(\frac{1}{r^3} - \frac{1}{r'^3}\right). \tag{71}$$

We shall find inequalities for each of the terms on the right-hand side. From (66):

$$\frac{|\,\mu\,(N) - \mu\,(N_0)\,|}{r^3} \leqslant \frac{b r_0^\beta}{r^3},$$

whence, since $r_0 \leqslant 2\varrho_0$ and $r > \varrho_0$, we find that

$$\frac{|\,\mu\,(N) - \mu\,(N_0)\,|}{r^3} \leqslant \frac{2^\beta b}{\varrho_0^{3-\beta}}. \tag{72}$$

Further, it follows from (15) and (22) that

$$\frac{|\,\mu\,(N_0)\,|\,[1 - \cos(\mathbf{n}, Z)\,]}{r^3} \leqslant \frac{cA}{\varrho_0^{3-2\alpha}}. \tag{73}$$

We shall find an inequality for the third term on the right-hand side of (71): r' is the length of the vector from M to N', N' being the projection of N on the XY plane, so that, from triangle MNN':

$$|\,r - r'\,| \leqslant |\,\zeta\,| \leqslant 2a\varrho_0^{1+\alpha},$$

whence it follows that

$$\left|\frac{1}{r^3} - \frac{1}{r'^3}\right| \leqslant 2a\varrho_0^{1+\alpha} \left(\frac{1}{r^3 r'} + \frac{1}{r^2 r'^2} + \frac{1}{r r'^3}\right) \leqslant \frac{6a}{\varrho_0^{3-\alpha}},$$

since r and $r' \geqslant \varrho_0$, and

$$\left|\mu\,(N_0) \cos(\mathbf{n}, Z) \left(\frac{1}{r^3} - \frac{1}{r'^3}\right)\right| \leqslant \frac{6aA}{\varrho_0^{3-\alpha}}. \tag{74}$$

We could have assumed when deducing the inequalities that M coincides with N_0. In this case $\alpha = 0$ and $r = r_0$.

On substituting (71) in integral (70), $v_1(M)$ is split into three integrals:

$$v_1(M) = v_{1,1}(M) + v_{1,2}(M) + v_{1,3}(M) \tag{75}$$

over σ_0, each of which has a meaning for any position of M on the normal at N_0, and in particular when M coincides with N_0. We have inequalities (72), (73) and (74) for the integrands of these integrals, with

$$|\xi| \frac{C_1}{\varrho_0^{3-\beta}}, \quad |\xi| \frac{C_2}{\varrho_0^{3-2\alpha}}, \quad |\xi| \frac{C_3}{\varrho_0^{3-\alpha}}, \tag{76}$$

respectively on the right-hand sides, where the constants C_1, C_2 and C_3 do not depend on the position of N_0 on S or of M on the normal. Hence it follows that $v_{1,k}(M)$ $(k = 1, 2, 3)$ tend to limits which are equal to $v_{1,k}(N_0)$ as M tends to N_0 uniformly with respect to the position of N_0 on S. Let us prove this for $v_{1,1}(M)$. Let ε be any given positive number. We isolate the part σ_1 of σ_0 defined by $\xi^2 + \eta^2 \leqslant d_1$, and choose d_1 so small that the integral

$$\iint_{\sigma_1} |\xi| \frac{|\mu(N) - \mu(N_0)|}{r^3} \, dS$$

remains less than or equal to $\varepsilon/4$ in absolute value for any position of M on the normal. This can be done by virtue of the first of inequalities (76). Further, we can write $v_{1,1}(M)$ as

$$v_{1,1}(M) = \iint_{\sigma_1} \xi \frac{\mu(N) - \mu(N_0)}{r^3} \, dS + + \iint_{\sigma_0 - \sigma_1} \xi \frac{\mu(N) - \mu(N_0)}{r^3} \, dS =$$

$$= v_{1,1}^{(1)}(M) + v_{1,1}^{(2)}(M)$$

and obtain

$$v_{1,1}(M) - v_{1,1}(N_0) = v_{1,1}^{(1)}(M) - v_{1,1}^{(1)}(N_0) + [v_{1,1}^{(2)}(M) - v_{1,1}^{(2)}(N_0)],$$

whence

$$|v_{1,1}(M) - v_{1,1}(N_0)| \leqslant \frac{\varepsilon}{2} + |v_{1,1}^{(2)}(M) - v_{1,1}^{(2)}(N_0)|. \tag{77}$$

The integral $v_{1,1}^{(2)}(M)$ is taken over a surface, every point of which is not less than a distance d_1 from N_0 and M, so that we have, precisely as in [196]:

$$|v_{,1}^{(2)}(M) - v_{1,1}^{(2)}(N_0)| \leqslant C_4 |z|,$$

where C_4 does not depend on the position of N_0 on S. Now, (77) gives

$$|v_{1,1}(M) - v_{1,1}(N_0)| \leqslant \frac{\varepsilon}{2} + C_4 |z|,$$

and we obtain for $|z| \leqslant \varepsilon/2C_4$:

$$|v_{1,1}(M) - v_{1,1}(N_0)| \leqslant \varepsilon,$$

whence it follows that $v_{1,1}(M) \to v_{1,1}(N_0)$ uniformly with respect to the position of N_0 on S.

We see on returning to expression (75) that $v_1(M)$ tends uniformly to the limit $v_1(N_0)$ as $M \to N_0$. We notice that this limit is the same whether M tends to N_0 from inside or from outside S. Expressed more simply, the function $v_1(M)$ is continuous at N_0 when M moves along the normal.

The integral $v_2(M)$ is taken over the piece $(S - \sigma_0)$ of S, all the points of which are not closer than the distance $d/3$ from M and N_0. It follows from this, as above, that

$$| v_2 (M) - v_2 (N_0) | \leqslant C_5 | z | ,$$

where the constant C_5 does not depend on the position of N_0 on S, so that $v_2(M) \to v_2(N_0)$ uniformly with respect to N_0. Finally, we can say that the derivative $\partial u(M)/\partial x$ tends uniformly to a limit as M tends to N_0 along the normal, this limit being the same when $M \to N_0$ from outside and inside S. Similarly, the same can evidently be said for $\partial u(M)/\partial y$. We proved in [196] that the derivative $\partial u(M)/\partial z$ tends uniformly to a limit. There, however, we had different limits from inside and outside. If l is any direction forming angles a_1, a_2, a_3 with the X, Y, Z axes, it follows at once from the above that the derivative

$$\frac{\partial u (M)}{\partial l} = \frac{\partial u (M)}{\partial x} \cos a_1 + \frac{\partial u (M)}{\partial y} \cos a_2 + \frac{\partial u (M)}{\partial z} \cos a_3 \tag{78}$$

also tends uniformly to limiting values when M tends to N_0 from inside or outside S.

In view of the uniform convergence of derivative (78) to its limits from inside and outside, we can say that these limits are continuous functions of the point N_0 on S.

Let us finally show that derivative (78) tends to the above-mentioned limit however M tends to N_0 (i.e. not necessarily along the normal). Suppose for definiteness that M tends to N_0 from inside, and let $\omega(N_0)$ denote the limiting values on S of derivative (78). Given any positive ε, we have to show that there exists a positive η such that

$$\left| \frac{\partial u (M)}{\partial l} - \omega (N_0) \right| \leqslant \varepsilon, \tag{79}$$

provided $| MN_0 | \leqslant \eta$, where M lies inside S. We draw the sphere with centre N_0 and radius δ so small that $| \omega(N) - \omega(N_0) | \leqslant \varepsilon/2$ on the part σ' of S lying inside this sphere. We assume further that M lies inside the sphere with centre N_0 and radius η, this latter number being chosen so that

$$\left| \frac{\partial u (M)}{\partial n} - \omega (N) \right| \leqslant \frac{\varepsilon}{2} ,$$

provided M lies on the normal to S at N and $| MN | \leqslant \eta$. This is possible by the proved uniform convergence of $\partial u(M)/\partial l$ to $\omega(N)$ on S. In addition, we further assume that $\eta \leqslant \delta/3$. If the distance of M from N_0 is not greater than η, the distance from N will certainly be not greater than η, where N is a point of σ' on the normal on which M lies. We have:

$$\frac{\partial u (M)}{\partial l} - \omega (N_0) = \frac{\partial u (M)}{\partial l} - \omega (N) + \omega (N) - \omega (N_0)$$

and

$$\left|\frac{\partial u(M)}{\partial l} - \omega(N_0)\right| \leqslant \left|\frac{\partial u(M)}{\partial n} - \omega(N)\right| + |\omega(N) - \omega N_0)|.$$

By what has been said above, both terms on the right are $\leqslant \varepsilon/2$, so that

$$\left|\frac{\partial u(M)}{\partial l} - \omega(N_0)\right| \leqslant \varepsilon \quad \text{for} \quad |MN_0| \leqslant \eta. \tag{80}$$

We have used above the following elementary proposition: the shortest distance from M to the surface S is the length of the normal MN to S through M.

We remark further that integrals (67) and (68) do not have a meaning when $z = 0$, i.e. when M and N_0 coincide; but their difference has a meaning, as we have seen.

The above discussion leads us to the following theorem, first proved by Lyapunov:

THEOREM. *If the density $\mu(N)$ satisfies the Lipschitz condition* (66), *the derivative of the potential of a simple layer with respect to any fixed direction is continuous up to S both from inside and outside. The derivative with respect to any direction tangential to S at N_0 varies continuously on passage of the point M of the surface to N_0.*

Greater difficulties are involved in investigating the behaviour of the derivatives of the potential of a double layer on approaching the surface S. The fundamental results in this direction were also obtained by Lyapunov in the work already quoted.

199. Logarithmic potential. In the case of a plane, we have the basic singular solution $\log(1/r)$ instead of $1/r$ [II, 193]. Let l be a closed contour on the XY plane and l_0 its length. The potential of a simple layer is given by

$$u(M) = \int_l \mu(N)\log\frac{1}{r}\,ds = \int_l \mu(s)\log\frac{1}{r}\,ds. \tag{81}$$

The second singular solution, analogous to a dipole in three-dimensional space, is

$$\frac{\cos\varphi}{r} = \frac{\partial}{\partial l}\left(\log\frac{1}{r}\right),$$

and the potential of a double layer is given by

$$w(M) = \int_l \mu(N)\frac{\cos\varphi}{r}\,ds, \tag{82}$$

where $\varphi = (\mathbf{r}, \mathbf{n})$. The expression $\cos\varphi\, ds/r$ gives the angle subtended by the element ds of the contour at the point M, the angle being positive if $\cos\varphi > 0$, and negative if $\cos\varphi < 0$. The

analogue of (26) is as follows:

$$\int_l \frac{\cos\varphi}{r}\,\mathrm{d}s = \begin{cases} 2\pi & (M \text{ inside } l) \\ 0 & (M \text{ outside } l) \\ \pi & (M \text{ on } l). \end{cases} \tag{83}$$

Similar assumptions can be made regarding the contour l to those made above regarding the surface S.

Suppose now that the functions $x(s)$, $y(s)$, giving the equation of l parametrically and being periodic with period l_0, have continuous derivatives up to the second order. The function $\mu(N) = \mu(s)$ will be assumed continuous. Let us investigate the kernel of the potential of the double layer, on the assumption that M lies on l and coincides with a point N_0 of l. Since the direction-cosines of the direction n are given by the derivatives $y'(s)$ and $-x'(s)$, we can write:

$$\frac{\cos\varphi}{r} = \frac{\cos(r, n)}{r} = \frac{[x(s) - x(s_0)]\,y'(s) - [y(s) - y(s_0)]\,x'(s)}{[x(s) - x(s_0)^2 + [y(s) - y(s_0)]^2}. \tag{84}$$

If s and s_0 are different, this expression is a continuous function of s and s_0. Now let s and s_0 tend to the common limit s_1. On applying Taylor's formula, we can write:

$$x(s) - x(s_0) = x'(s_0)(s - s_0) + \frac{1}{2}x''(s_0')(s - s_0)^2$$

$$x'(s) = x'(s_0) + x''(s_0'')(s - s_0)$$

$$y(s) - y(s_0) = y'(s_0)(s - s_0) + \frac{1}{2}y''(s_0''')(s - s_0)$$

$$y'(s) = y'(s_0) + y''(s_0'''')(s - s_0),$$

where s_0', s_0'', s_0''', s_0'''' lie between s and s_0. On substituting in (84) and cancelling $(s - s_0)^2$, we obtain in the limit:

$$\frac{x'(s_1)\,y''(s_1) - y'(s_1)\,x''(s_1)}{2\,[x'^2(s_1) + y'^2(s_1)]} = \frac{x'(s_1)\,y''(s_1) - y'(s_1)\,x''(s_1)}{2},$$

which is equal to half the curvature of the curve at $s = s_1$. Function (84) is therefore continuous in s and s_0 along l. On writing $L(s_0, s)$ for this function, we can say that the potential of a double layer:

$$w(N_0) = w(s_0) = \int_0^l \mu(s)\,L(s_0, s)\,\mathrm{d}s$$

is a continuous function of N_0 if N_0 lies on l.

Hence, given our assumptions regarding $x(s)$ and $y(s)$, (84) is a continuous function of s and s_0 on l. In the three-dimensional case the function $(\cos\varphi)/r^2$ in general had a polarity when N and N_0 coincided. We can prove formulae analogous to (42) for the double layer potential (82):

$$
\begin{aligned}
w_i(N_0) &= w(N_0) + \pi\mu(N_0) = \int_l \mu(N)\frac{\cos(\mathbf{r}_0,\mathbf{n})}{r_0}\,\mathrm{d}s + \pi\mu(N_0)\\
w_e(N_0) &= w(N_0) - \pi\mu(N_0) = \int_l \mu(N)\frac{\cos(\mathbf{r}_0,\mathbf{n})}{r_0}\,\mathrm{d}s - \pi\mu(N_0)
\end{aligned}
\tag{85}
$$

where $r_0 = |N_0N|$ and $(\mathbf{r}_0, \mathbf{n})$ is the angle formed by the direction $\overline{N_0N}$ with the direction n of the outward normal to l at N. It follows from (85) that

$$w_i(N_0) - w_e(N_0) = 2\pi\mu(N_0). \tag{86}$$

The simple layer potential (81) is defined and continuous throughout the plane.

Let N_0 be a point on S and $\mathbf{n}_0$ the direction of the normal at this point. We have, if M is not on S:

$$\frac{\partial u(M)}{\partial n_0} = \int_l \mu(N)\frac{\partial\log\frac{1}{r}}{\partial n_0}\,\mathrm{d}s = \int_l \mu(N)\frac{\cos(\mathbf{r},\mathbf{n}_0)}{r}\,\mathrm{d}s. \tag{87}$$

As M approaches N_0 along the normal from inside and from outside S, derivative (87) has limits which are given by

$$
\begin{aligned}
\left(\frac{\partial u(N_0)}{\partial n_0}\right)_i &= \int_l \mu(N)\frac{\cos(\mathbf{r}_0,\mathbf{n}_0)}{r_0}\,\mathrm{d}s + \pi\mu(N_0)\\
\left(\frac{\partial u(N_0)}{\partial n_0}\right)_e &= \int_l \mu(N)\frac{\cos(\mathbf{r}_0,\mathbf{n}_0)}{r_0}\,\mathrm{d}s - \pi\mu(N_0),
\end{aligned}
\tag{88}
$$

from which it follows that

$$\left(\frac{\partial u(N_0)}{\partial n_0}\right)_i - \left(\frac{\partial u(N_0)}{\partial n_0}\right)_e = 2\pi\mu(N_0). \tag{89}$$

We have instead of (84):

$$\frac{\cos(\mathbf{r}_0,\mathbf{n}_0)}{r_0} = \frac{[x(s)-x(s_0)]\,y'(s_0) - [y(s)-y(s_0)]\,x'(s_0)}{[x(s)-x(s_0)]^2 + [y(s)-(s_0)]^2},$$

and it can be shown, as above, that this expression remains continuous when s and s_0 coincide. We remark that the simple layer potential (81) does not in general vanish at infinity.

200. Integral formulae and parallel surfaces. We shall require later the following integral formulae, enabling integrals over three-dimensional volumes to be transformed to surface integrals [II, 193]:

$$\iiint\limits_{D_i} \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\right)\mathrm{d}\tau =$$

$$= \iint\limits_{S} u\frac{\partial v}{\partial n}\,\mathrm{d}S - \iiint\limits_{D_i} u\,\Delta v\,\mathrm{d}\tau, \tag{90}$$

$$\iiint\limits_{D_i} (u\,\Delta v - v\,\Delta u)\,\mathrm{d}\tau = \iint\limits_{S}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S, \tag{91}$$

where D_i is the part of space bounded by the surface S, and n is the direction of the outward normal to S. These formulae are deduced on the following assumptions: u, v and their first order partial derivatives are continuous in D_i as far as S, the second order partial derivatives are continuous inside D_i, and the integrals over D_i containing Δu and Δv exist. If Δu and Δv do not possess continuity as far as S, the integrals are improper, and are obtained as the limits for any sequence of domains $D_i^{(n)}$ which are contained inside D_i when the $D_i^{(n)}$ tend to D_i in such a way that every point inside D_i lies inside the $D_i^{(n)}$ from a certain n onward. We shall be concerned in future with harmonic functions, so that $\Delta u = \Delta v = 0$, and we shall take $u = v$ in (91). The above formulae become in this case:

$$\iiint\limits_{D_i}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}\tau = \iint\limits_{S} u\left(\frac{\partial u}{\partial n}\right)_i\mathrm{d}S, \tag{92}$$

$$\iint\limits_{S}\left[u\left(\frac{\partial v}{\partial n}\right)_i - v\left(\frac{\partial u}{\partial n}\right)_i\right]\mathrm{d}S = 0. \tag{93}$$

These formulae also hold for and infinite domain D_e lying outside S:

$$\iiint\limits_{D_e}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}\tau = \iint\limits_{S} u\left(\frac{\partial u}{\partial n}\right)_e\mathrm{d}S, \tag{94}$$

$$\iint\limits_{S}\left[u\left(\frac{\partial v}{\partial n}\right)_e - v\left(\frac{\partial u}{\partial n}\right)_e\right]\mathrm{d}S = 0, \tag{95}$$

provided that the functions u and v, harmonic outside S, are continuous together with their first order derivatives as far as S and tend to zero as the point M moves away indefinitely, so that the inequalities hold:

$$\begin{aligned} R\,|u(M)| \leqslant A; \quad R^2\left|\frac{\partial u(M)}{\partial l}\right| \leqslant A; \\ R\,|v(M)| \leqslant A; \quad R^2\left|\frac{\partial v(M)}{\partial l}\right| \leqslant A, \end{aligned} \tag{96}$$

where R is the distance from M to any definite point O of space, A is a numerical constant and l is any fixed direction. In (94) and (95) n is the direction of the normal to S, outward with respect to D_e, i.e. directed into S.

To prove (94) and (95), we have to apply them to the finite domain bounded by S and the sphere with centre O and a sufficiently large radius. As the radius tends to infinity the integral over the surface of the sphere tends to zero, since the products $u\partial v/\partial n$ and $v\partial u/\partial n$ will be of order $1/R^3$, whilst the surface area is $\sim 4\pi R^2$. Hence we obtain (94) and (95) [cf. II, 194].

As we shall see in a later section, conditions (96) are fulfilled with the single assumption that the harmonic functions $u(M)$ and $v(M)$ tend to zero when M becomes infinitely remote. The following formula is a corollary of (93) and (95) [II, 194]:

$$u(M) = \frac{1}{4\pi}\iint\limits_S \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial \frac{1}{r}}{\partial n}\right] dS, \tag{97}$$

where n is the direction of the outward normal with respect to D_i or D_e, depending on the case to which (97) is applied.

We shall now prove more general conditions for the above formulae to be applicable. We mark off a segment of constant length δ, directed inwards into S, along the normals at every point of S. We assume that, for all sufficiently small δ, the locus of the ends P of these segments forms a closed surface, which does not cut itself, and which lies inside S and has a continuously varying tangent plane. Let S_δ denote this surface. For every point N on S there is a corresponding definite point P on S_δ, which lies on the normal to S at N; and conversely, for every point P on S_δ there is a corresponding definite point N on S. We shall show that the normal to S at a point N_0 is also normal to S_δ at P. Let (x, y, z) be the coordinates of points of S and (x', y', z') the coordinates of the corresponding points P in some system of coordinates.

We have:

$$\begin{aligned} x' &= x - \delta \cos (\mathbf{n}, X) \\ y' &= y - \delta \cos (\mathbf{n}, Y) \\ z' &= z - \delta \cos (\mathbf{n}, Z), \end{aligned} \tag{98}$$

where $\mathbf{n}$ is the outward normal to S. We shall assume that some piece of the surface S has the explicit equation $z = z(x, y)$, where $z(x, y)$ has continuous derivatives up to the second order. The direction-cosines of the normal will now be continuously differentiable functions of the coordinates.

Suppose that N describes some curve l on the above-mentioned piece of S, so that (x, y, z) are continuously differentiable functions of some parameter t. Now, (x', y', z') will also be continuously differentiable functions of t. On differentiating with respect to t the obvious equation

$$(x' - x)^2 + (y' - y)^2 + (z' - z)^2 = \delta^2,$$

we obtain:

$$\left[(x' - x)\, x'_t + (y' - y)\, y'_t + (z' - z)\, z'_t\right] + \left[(x' - x)\, x_t + \right. \\ \left. + (y' - y)\, y_t + (z' - z)\, z_t\right] = 0.$$

But the second square bracket vanishes, since PN is the normal to S. Hence it follows that the first bracket also vanishes, which is equivalent to the fact that the tangent to l' is perpendicular to PN. It follows immediately from this that PN is also normal to S_δ. We shall assume that every point of S can be located inside a piece of surface with the above properties. The surface S_δ is said to be *parallel to the surface* S.

Now let the functions u and v, harmonic inside S, have regular normal derivatives when M tends to N along the normal, u and v being themselves continuous in the closed domain $\overline{D}_i$. We can now apply all the above formulae for the domain bounded by the surface S_δ. On recalling that u, v and their normal derivatives tend uniformly to their limits, and that the normals to S_δ and S coincide, we obtain all these formulae for D_i also as $\delta \to 0$. The triple integral over D_i must be regarded here as improper, i.e. as the limit of integrals over interior domains as these tend to D_i. Since the integrand is positive, the precise way in which these interior domains tend to D_i is of no significance. In particular, domains bounded by S_δ can be used. When passing to the limit we must also bear in mind the variation of the surface area.

An element of this area is expressed in terms of the coefficients of the first Gaussian form as [II, 130]:

$$dS = \sqrt{EG - F^2}\, dx\, dy,$$

if we take say x and y as parameters, and it follows from (98) that E, G and F are second degree polynomials in δ. The above arguments can also be used for D_e. In this case, the minus must be replaced by a plus in formulae (98). If the harmonic functions $u(M)$ and $v(M)$ are expressible as the potentials of simple layers with continuous densities, they are continuous as far as S and have regular normal derivatives.

We thus have:

THEOREM. *If it is possible to construct parallel surfaces inside and outside S with the above-mentioned properties, the above formulae can be applied for the simple layer potentials $u(M)$ and $v(M)$ with continuous densities.*

We shall now indicate some sufficient conditions for the existence of surfaces S_δ parallel to S. Let S be a Lyapunov surface, with $\alpha = 1$ in condition (3). We show that the surface S_δ is now, for sufficiently small δ, a closed surface with no multiple points, i.e. the points P corresponding to different points N on S are distinct. Suppose for the moment that $\delta < d/3$, and that we obtain the same point P for different points $N_1(x_1, y_1, z_1)$ and $N_2(x_2, y_2, z_2)$, i.e.

$$\begin{aligned} x_1 - \delta \cos(n_1, X) &= x_2 - \delta \cos(n_2, X) \\ y_1 - \delta \cos(n_1, Y) &= y_2 - \delta \cos(n_2, Y) \\ z_1 - \delta \cos(n_1, Z) &= z_2 - \delta \cos(n_2, Z) \end{aligned} \tag{99}$$

where n_1 and n_2 are the directions of the outward normals to S at N_1 and N_2. We observe that N_2 lies inside the sphere with centre N_1 and radius d. If $r_{1,2}$ denotes the distance $|N_1N_2|$, we have by (99):

$$r_{1.2} = \delta \sqrt{2(1 - \cos\theta)},$$

where θ is the angle between n_1 and n_2. From (6) with $\alpha = 1$ we have $1 - \cos\theta \leqslant a^2 r_{1,2}^2/2$, so that, by (99_1): $r_{1,2} \leqslant a\delta r_{1,2}$.

If we take $\delta < 1/a$, we arrive at a contradiction. Thus, for a Lyapunov surface with $\alpha = 1$, the surface S_δ has no multiple points if $\delta < d/3$ and $\delta < 1/a$. Moreover, it follows at once from the conditions imposed on S [192] that all the points P with $\delta < d$ lie inside (or outside) S. If we further assume that the equation of a part of S in local coordinates: $z = z(x, y)$, is such that $z(x, y)$ has continuous derivatives up to

the second order, the surface S_δ will have a continuously varying tangent plane. The fact that S_δ is closed follows at once from the fact that, when an interior point M of D_i moves continuously, the shortest distance from M to S will be equal to δ for some position of M.

Note. Suppose that $u(M)$ is continuous inside S, has continuous first order derivatives and has a regular normal derivative. The limiting values of this latter, $(\partial u(N)/\partial n)_i$, will now form a function continuous on S [196], whence it follows that a number B exists such that

$$\left|\left(\frac{\partial u(N)}{\partial n}\right)_i\right| \leqslant B \qquad (N \text{ on } S).$$

On the other hand, by virtue of the uniform convergence of the normal derivative to a limit, given any positive ε there exists a number η such that

$$\left|\frac{\partial u(M)}{\partial n} - \left(\frac{\partial u(M)}{\partial n}\right)_t\right| \leqslant \varepsilon \text{ for } |MN| \leqslant \eta,$$

where M lies inside S and on the normal to S at N. Having fixed ε, we obtain $|\partial u(M)/\partial n| \leqslant (B+\varepsilon)$ for $|MN| \leqslant \eta$, whence $|u(M_2) - u(M_1)| \leqslant (B+\varepsilon)\,\delta_{1,2}$, where $\delta_{1,2} = |M_1M_2|$. Hence it follows that $u(M)$ has a definite limit $u(N)$ as $M \to N$ along the normal. We can further write:

$$u(M) - u(N) = \int\limits_0^\delta \frac{\partial u(M_1)}{\partial n}\, d\delta_1,$$

where M_1 is a variable point on the normal, $\delta_1 = |NM_1|$ and $\delta = |NM|$, and $\delta_1 \leqslant \delta \leqslant \eta$. It follows from the previous inequality for the normal derivative that $|u(M) - u(N)| \leqslant (B+\varepsilon)\delta$, whence it is clear that $u(M) \to u(N)$ uniformly with respect to the position of N on S. On taking this into account, it is easily shown [cf. 198] that $u(M)$ tends to $u(N)$ whatever the law by which M tends to N, and that $u(M)$ is continuous up to S. Similar arguments can be applied for D_e. Thus, *$u(M)$ is continuous up to S when it has a regular normal derivative.*

Therefore, we *can only be sure that the above-mentioned integral formulae are applicable when $u(M)$ and $v(M)$ have regular normal derivatives.*

Everything said above for D_i can be carried over to the case of a plane. The case of an infinite domain on a plane is somewhat different, and will be discussed below.

201. Sequences of harmonic functions. Before turning to the solution of boundary value problems for Laplace's equation with the aid of the potentials of a simple and a double layer, we must establish some properties of harmonic functions, supplementary to those described earlier. We consider a sequence of harmonic functions, or what amounts to the same thing, a series whose terms are harmonic functions. We shall give all our proofs for the case of a plane. They are precisely similar for three-dimensional space. We merely have to use the formula giving the solution of the Dirichlet problem for a sphere instead of Poisson's formula.

The fundamental theorem on uniformly convergent series of harmonic functions is strikingly similar to the analogous theorem from the theory of regular functions of a complex variable [III_2, 12]:

If the terms of the series

$$\sum_{k=1}^{\infty} u_k(x, y) \tag{100}$$

are harmonic functions inside a bounded domain B and are continuous functions in the closed domain $\overline{B}$, and the series is uniformly convergent on the contour l of this domain, it must be uniformly convergent throughout the closed domain and the sum of the series must be a harmonic function inside B.

Let ε be a previously assigned positive number. In view of the uniform convergence on the contour l, there exists an N such that, for any $n \geqslant N$ and any positive ε, we have

$$\left|\sum_{k=1}^{n+p} u_k(x, y)\right| \leqslant \varepsilon \qquad \big[(x, y) \text{ on } l\big].$$

The above finite sum of harmonic functions will be a harmonic function inside B and will be continuous in the closed domain $\overline{B}$, and in view of the fundamental property of harmonic functions that they attain their extrema on the contour [II, 194], we can say that, since the above inequality is satisfied on the contour, it will certainly be satisfied at all interior points, or in other words, it will be satisfied throughout the closed domain; and this gives us the uniform convergence of series (100) throughout the closed domain. Thus the sum $S(x, y)$ of series (100) is a continuous function in the closed domain. We show that it is a harmonic function inside the domain. Let M_0 be any point inside B. We describe the circle Σ_0 with centre M_0 and radius R such

that the whole of the circle lies inside B. Let $S_n(x, y)$ denote the sum of the first n terms of series (100). This finite sum will be a harmonic function, and its values inside the circle Σ_0 will be expressible in terms of its values on the circumference in accordance with Poisson's formula:

$$S_n(\varrho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} S_n(R, \psi) \frac{R^2 - \varrho^2}{R^2 - 2R\varrho \cos(\psi - \varphi) + \varrho^2} \, d\psi,$$

where (ϱ, φ) are the polar coordinates of $M(x, y)$ if M_0 is taken as origin. On the circumference of our circle $S_n(R, \psi) \to S(R, \psi)$ uniformly with respect to ψ, and we have on passing to the limit:

$$S(\varrho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} S(R, \psi) \frac{R^2 - \varrho^2}{R^2 - 2R\varrho \cos(\psi - \varphi) + \varrho^2} \, d\psi,$$

i.e. the sum of series (100) is expressible by Poisson's integral inside our circle, and is therefore a harmonic function. We recall that M_0 was any point inside B, and notice that it could have been proved in precisely the same way that series (100) can be differentiated inside B with respect to the variables (ϱ, φ) as many times as desired. For it follows at once from Poisson's formula that

$$\frac{\partial u_k(\varrho, \varphi)}{\partial \varrho} = \frac{1}{2\pi} \int_0^{2\pi} u_k(R, \psi) \frac{\partial}{\partial \varrho} \frac{R^2 - \varrho^2}{R^2 - 2R\varrho \cos(\psi - \varphi) + \varrho^2} \, d\psi.$$

On multiplying both sides of series (100) by

$$\frac{\partial}{\partial \varrho} \frac{R^2 - \varrho_2}{R^2 - 2R\varrho \cos(\psi - \varphi) + \varrho_2}$$

and integrating over the circumference of Σ_0, we have

$$\frac{\partial S(\varrho, \varphi)}{\partial \varrho} = \sum_{k=1}^{\infty} \frac{\partial u_k(\varrho, \varphi)}{\partial \varrho}.$$

Our theorem can obviously also be stated in terms of a sequence of harmonic functions, viz: *if a sequence $S_n(x, y)$ of functions harmonic inside B and continuous in the closed domain $\overline{B}$ tends uniformly to the limit function $S(x, y)$ on the contour l, it must be uniformly convergent to the limit function throughout the closed domain $\overline{B}$. The limit function is harmonic inside B, and the sequence can be differentiated any number of times inside B.*

We shall prove a further theorem, relating to the particular case when the terms of series (100) are positive functions. To begin with,

a corollary of Poisson's formula must be mentioned. A function $u(\varrho, \varphi)$, harmonic inside the circle $\varrho < R$ with centre M_0 and continuous in the closed circular domain, is expressible in this domain by Poisson's formula:

$$u(\varrho, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} u(R, \psi) \frac{R^2 - \varrho^2}{R^2 - 2R\varrho \cos(\psi - \varphi) + \varrho^2} \, d\psi .$$

Suppose, in the addition, that this function is positive. Since $|\cos(\psi - \varphi)| \leqslant 1$, we can write the inequality:

$$(R - \varrho)^2 \leqslant R^2 - 2R\varrho \cos(\psi - \varphi) + \varrho^2 \leqslant (R + \varrho)^2,$$

and it follows at once from Poisson's formula that:

$$\frac{R-\varrho}{R+\varrho}\cdot\frac{1}{2\pi}\int_0^{2\pi} u(R, \psi)\, d\psi \leqslant u(\varrho, \varphi) \leqslant \frac{R+\varrho}{R-\varrho}\cdot\frac{1}{2\pi}\int_0^{2\pi} u(R, \psi)\, d\psi,$$

or, using the mean value theorem [II, 194]:

$$\frac{R-\varrho}{R+\varrho} u(M_0) \leqslant u(\varrho, \varphi) \leqslant \frac{R+\varrho}{R-\varrho} u(M_0). \tag{101}$$

This inequality for the values of a positive harmonic function at any point inside a circle in terms of its value at the centre of the circle is usually known as Carnac's inequality. We can use this inequality to prove the following theorem:

If $S_n(M)$ is an increasing sequence of functions harmonic inside B, and if it has a finite limit at any one interior point M_0 of B, it is convergent everywhere inside B and is uniformly convergent in any closed domain B_1 which is contained, together with its boundary, inside B.

By hypothesis, $S_{n+1}(M) \geqslant S_n(M)$ inside B. In view of the convergence of the sequence at M_0, given any positive ε, there exists an N such that

$$[S_{n+p}(M_0) - S_n(M_0)] \leqslant \varepsilon$$

for $n \geqslant N$ and any positive p. Let Σ_0 be a circle with centre M_0 and radius R lying inside B. Since the difference written above is a positive harmonic function, we can write

$$0 \leqslant S_{n+p}(M) - S_n(M) \leqslant \frac{R+\varrho}{R-\varrho}\varepsilon,$$

where M is any interior point of the circle and ϱ is the distance from M to M_0. On taking the circle Σ_0' with centre M_0 and radius $(R - a)$,

where a is any given small positive number, we have

$$0 \leqslant S_{n+p}(M) - S_n(M) \leqslant \frac{2R}{a}\varepsilon$$

in Σ_0', whence follows the uniform convergence of $S_n(M)$ in Σ_0'. Since the sequence is seen to be convergent at an interior point M_1 of the circle Σ_0, we can now use the above arguments to prove its uniform convergence inside a circle with centre at M_1 and lying inside B. On proceeding in this way, we can show, precisely as in the case of analytic continuation, that the sequence is uniformly convergent in any closed circle inside B. Any closed domain B_1 which, together with its boundary, lies inside B, can be covered by a finite number of circles lying inside B, whence follows the uniform convergence of the sequence in B_1. We remark further that, *by the above theorem, the uniform convergence of the sequence implies that its limit function is harmonic inside B.*

This last theorem can be stated in terms of series, viz: *let the terms of series* (100) *be harmonic functions inside B and be positive from a certain n onward. If the series is convergent at some interior point of B, it is convergent at all interior points of B, and is uniformly convergent in any closed domain B_1 which, together with its boundary, is contained in B.* Of course we could have taken a decreasing instead of increasing series in the last theorem, with negative instead of positive functions.

202. Formulation of interior boundary value problems for Laplace's equation. Let D_i be a finite domain of three-dimensional space bounded by a surface S. As we know, the interior Dirichlet problem consists in seeking the function $u(M)$ which is harmonic inside D_i, is continuous in the closed domain $\overline{D}_i$ and takes on S given values representing a function continuous on S. The solution must be unique [II, 194]. We shall later prove the existence of a solution, given certain assumptions regarding the boundary S. The problem is essentially the same in the case of a plane.

In Neumann's problem, instead of specifying the function itself on the boundary, we specify the limiting values $f(N)$ of the normal derivative $\partial u(M)/\partial n$, on the assumption that $M \to N$ along a normal. If we further assume that $u(M)$ has a regular normal derivative, we can apply (93) to $u(M)$ and $v(M) \equiv 1$, and obtain:

$$\iint_S f(N)\,\mathrm{d}S = 0, \tag{102}$$

which is therefore the necessary condition for the interior Neumann problem to be soluble when a regular normal derivative exists. We remark that, if a function $u(M)$ gives a solution of the Neumann interior problem, the function $u(M) + C$, where C is an arbitrary constant, also gives a solution of the problem with the same boundary condition $f(N)$. The uniqueness theorem for the solution of the interior Neumann problem consists in asserting that these in fact exhaust all the solutions of the problem, i.e. *if* $u_1(M)$ *and* $u_2(M)$ *are two solutions of the Neumann problem with the same boundary condition* $f(N)$, *the difference* $u_2(M) - u_1(M)$ *must be constant in D.*

This proposition is easily proved if $u_1(M)$ and $u_2(M)$ are assumed to have regular normal derivatives. In this case the difference $v(M) = u_2(M) - u_1(M)$ also has a regular normal derivative, the boundary values of which are zero; hence $v(M)$ is continuous as far as S. and we obtain on applying (92) to $v(M)$:

$$\iiint\limits_{D_i} \left[\left(\frac{\partial v(M)}{\partial x}\right)^2 + \left(\frac{\partial v(M)}{\partial y}\right)^2 + \left(\frac{\partial v(M)}{\partial z}\right)^2\right] d\tau = 0,$$

whence it follows that $v(M)$ is in fact constant inside D_i. A more general proof of the uniqueness of the solution of Neumann's problem will be given in a later section.

It could have been assumed, when formulating the Dirichlet and Neumann interior problems, that the boundary S consists of several closed surfaces.

The third fundamental boundary value problem connected with Laplace's equation consists in finding a harmonic function inside S *when a linear combination of the normal derivative and the function is given on the boundary*, i.e. the boundary condition has the form:

$$\left(\frac{\partial u(N)}{\partial n}\right)_i + p(N)\, u = f(N) \quad (N \text{ on } S), \tag{103}$$

where $p(N)$ and $f(N)$ are continuous functions given on S, it being assumed that $p(N) > 0$. Let us prove the uniqueness theorem on the assumption that $u(M)$ has a regular normal derivative. If two solutions were to exist, their difference $v(N)$ would satisfy the single boundary condition:

$$\left(\frac{\partial v(N)}{\partial n}\right)_i + p(N)\, v(N) = 0. \tag{104}$$

On applying (92) to $v(M)$ and using (104), we obtain

$$\iiint\limits_{D_i} \left[\left(\frac{\partial v(M)}{\partial x}\right)^2 + \left(\frac{\partial v(M)}{\partial y}\right)^2 + \left(\frac{\partial v(M)}{\partial z}\right)^2\right] d\tau = -\iint\limits_{S} p(N)\, v^2(N)\, dS.$$

The integral on the right-hand side cannot be positive, whilst the integral on the left cannot be negative, i.e. they must both vanish, whence it follows at once that $v(M) = 0$.

All the above results also hold in the case of a plane.

We have so far considered the so-called interior problems, in which it is required to find a harmonic function in a bounded domain with some given boundary condition. We now turn to the exterior problem, where a harmonic function is required in the infinite part of space lying outside some closed surface (or outside several closed surfaces). A similar problem can be formulated for a plane. Here, the essential role is played by the condition imposed on the required function in the neighbourhood of an infinitely remote point. This problem receives different treatments for a plane and for space. We shall start by considering the condition at infinity in the plane case.

203. The exterior problem in the case of a plane. A function $u(M)$, harmonic in the neighbourhood of an infinitely remote point, is said to be *regular at the infinitely remote point* if $u(M)$ has a finite limit as M tends to infinity. Let us explain the meaning of this definition. We construct in the neighbourhood of an infinitely remote point the harmonic function $v(M)$ conjugate to $u(M)$ [III, 2]. The function $v(M)$ can acquire an added constant, call it γ, on a circuit counterclockwise round the infinitely remote point. The function

$$f(z) = u(z) + i\,v(z) - \frac{\gamma}{2\pi}\log z$$

of the complex variable z will be single-valued and regular in the neighbourhood of the infinitely remote point, and it must therefore be possible to expand it at this point in a Laurent series in integral powers of z. Let us show that this expansion contains no terms at all with positive powers of z. In fact, if there were an infinite set of such terms, as $|z| \to \infty$ the function $f(z)$ would have to take values as close as desired to any previously assigned number [III_2, 17], whereas in fact the real part of the function, i.e. $u(z) - (\gamma/2\pi)\log|z|$ either tends to infinity if $\gamma \neq 0$ because $u(z)$ has a finite limit by hypothesis, or else, if $\gamma = 0$, it has a finite limit.

If there were a finite number of terms with positive powers, i.e. if

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_0 + \frac{a_{-1}}{z} + \ldots \quad (a_m \neq 0),$$

we should have

$$u(z) - \frac{\gamma}{2\pi}\log\varrho = r\varrho^m \cos(m\varphi + \psi) + \\ + \mathcal{R}\left[a_{m-1}z^{m-1} + \ldots + a_0 + \frac{a_{-1}}{z} + \ldots\right]$$

$$(z = \varrho e^{i\varphi}; \quad a_m = re^{i\psi}; \quad \mathcal{R} \text{ indicates the real part}).$$

If both sides are divided by ϱ^m and ϱ tends to infinity with fixed φ, the left-hand side will obviously tend to zero, whilst the right-hand side will have a limit $r\cos(m\varphi + \psi)$, depending on φ, which is not always zero. We thus arrive at a contradiction, i.e. the expansion of $f(z)$ contains only a constant term and terms in negative powers:

$$f(z) = a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots \tag{105}$$

As $|z| \to \infty$ the function $f(z)$ has a finite limit a_0, and it immediately follows that the constant γ must be zero, i.e. *if* $u(M)$ *is regular at an infintely remote point and* $v(M)$ *is the conjugate function,* $f(z) = u(z) + iv(z)$ *has the expansion* (105) *in the neighbourhood of the infinitely remote point.* It follows at once from the previous arguments that, to obtain this result, we need only suppose that $u(M)$ is simply bounded in absolute value in the neighbourhood of the infinitely remote point. Expansion (105) will follow from this alone, and hence the existence of a finite limit of $u(M)$ when the point M tends to infinity.

The exterior Dirichlet problem amounts to finding a function $u(M)$ which is harmonic outside a closed contour l, is regular at infinity and takes on l given values $f(N)$. Let z_0 be a point inside l. We carry out the conformal representation $w = 1/(z - z_0)$. The part of the plane outside l becomes a bounded domain B, a harmonic function becomes a harmonic function [III_2, 29], the point $z = \infty$ becomes $w = 0$, and $f(z)$ becomes a function of w regular at $w = 0$. Our exterior Dirichlet problem becomes an interior problem for the transformed domain, and there can obviously be only one solution of the problem.

On using (105), differentiating it with respect to z and observing that

$$z^2 f'(z) \to -a_{-1} \text{ as } |z| \to \infty,$$

we can say that, if a harmonic function $u(M)$ is regular at an infinitely remote point, the products $\varrho^2\,\partial u/\partial x$ and $\varrho^2\,\partial u/\partial y$, where $\varrho = |z|$, remain bounded when M moves away indefinitely. Hence it follows at once that the product $\varrho^2\,\partial u/\partial m$, where m is any direction, which can

in fact vary as M varies, also remains bounded as M becomes infinitely remote. If B is the part of the plane lying outside the closed contour l, and $u(M)$ and $v(M)$ are functions harmonic in B, continuous at an infinitely remote point and continuous together with their first order derivatives up to the contour, we have:

$$\int\limits_B\int \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \mathrm{d}S = \int\limits_l u \left(\frac{\partial u}{\partial n}\right)_{\mathrm{e}} \mathrm{d}s, \tag{106}$$

$$\int\limits_B\int \left[u\left(\frac{\partial v}{\partial n}\right)_{\mathrm{e}} - v\left(\frac{\partial u}{\partial n}\right)_{\mathrm{e}}\right] \mathrm{d}S = 0, \tag{107}$$

where n is the direction of the normal to l, outward with respect to the domain B. These formulae are proved precisely as in [200] for the case of three-dimensional space. We need only bear in mind that the derivatives $v(\partial u/\partial n)$, and $u(\partial v/\partial n)$ are of order $1/R^2$ on the circle C with centre at a fixed point O and radius R, whilst the length of the circumference is $2\pi R$. As in [200], formulae (106) and (107) still hold if we require the existence of regular normal derivatives of $u(M)$ and $v(M)$ instead of the continuity of the first order derivatives as far as l.

We turn to the exterior Neumann problem, where we have on l the boundary condition:

$$\left(\frac{\partial u}{\partial n}\right)_{\mathrm{e}} = f(N) \qquad (N \text{ on } l) \tag{108}$$

as M tends to N along the normal, and the requirement that $u(M)$ be regular at infinity is retained. Let a solution $u(M)$ of the problem exist, and let $u(M)$ be assumed to have a regular normal derivative on l. On drawing a circle C of sufficiently large radius R and applying (107) to $u(M)$ and $v(M) \equiv 1$, for the domain bounded by l and C, we obtain

$$\int\limits_{l\circlearrowright} \left(\frac{\partial u}{\partial n}\right)_{\mathrm{e}} \mathrm{d}s + \int\limits_{C\circlearrowright} \left(\frac{\partial u}{\partial n}\right)_{\mathrm{e}} \mathrm{d}s = 0. \tag{109}$$

But the derivative $(\partial u/\partial n)_{\mathrm{e}}$ is of order $1/R^2$ on C, whence it follows that the integral over C tends to zero as $R \to \infty$, and we obtain in the limit, from (108),

$$\int\limits_l f(N)\, \mathrm{d}s = 0. \tag{110}$$

This necessary condition was also obtained for the interior Neumann problem. On using (106), it can be shown that the solution of the

exterior Neumann problem is unique if we assume that the normal derivative is regular [cf. 202]. There is no condition analogous to (110) for solubility of the exterior Neumann problem in three-dimensional space.

It is worth observing that the basic singular solution $\log(1/r)$ is not regular at an infinitely remote point. As $r \to \infty$ it tends to infinity. The second singular solution $(\cos\varphi)/r$, corresponding to a dipole, is regular at infinity and vanishes there. In three-dimensional space, the basic singular solution $1/r$, as well as the dipole potential, vanishes at an infinitely remote point. The potential of a simple layer (81), which is a harmonic function outside l, is not in general regular at an infinitely remote point. If the total charge is zero, i.e. if

$$\int_l \mu(N)\, ds = 0, \tag{111}$$

potential (81) is regular in this particular case. For, let R be the distance of the point M from the origin. On introducing into integral (111) the factor $\log R$, which does not depend on the variable point of integration N, we can write potential (81) as

$$u(M) = \int_l \mu(N) \log\frac{R}{r}\, ds,$$

and $\log(R/r)$ tends to zero uniformly with respect to points N on l as M moves away to infinity. The potential is thus seen to be regular at an infinitely remote point, where it vanishes.

Let us prove a further property of harmonic functions. Suppose that *$u(M)$ is a harmonic function in a circle whose centre N_0 is taken as origin, with the possible exception of the origin itself, and is bounded in absolute value in this circle. We show that the limit of $u(M)$ exists as $M \to N_0$, and if this limit is taken as the value $u(N_0)$, $u(M)$ is harmonic throughout the circle including the origin.* To see this, we only need to repeat the arguments that led us to expansion (105), with the infinitely remote point replaced by the origin. We have instead of (105):

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

whence our assertion follows.

204. Kelvin's transformation. When considering harmonic functions in three-dimensional space we no longer have most of the valuable auxiliary equipment such as is provided, in the case of a plane, by the theory of functions of a complex variable, and in particular, by the

conformal transformation mapping every harmonic function into another harmonic function. Nevertheless, in the case of three-dimensional space there is a point transformation of a special type which has the same property: if $u(x, y, z)$ is a harmonic function in a domain D, the function

$$v(x', y', z') = \frac{1}{r'} u\left(\frac{x'}{r'^2}, \frac{y'}{r'^2}, \frac{z'}{r'^2}\right) \quad (r'^2 = x'^2 + y'^2 + z'^2) \qquad (112)$$

is harmonic in the domain D' obtained from D with the aid of the transformation:

$$x' = \frac{x}{r^2}; \quad y' = \frac{y}{r^2}; \quad z' = \frac{z}{r^2} \qquad (r^2 = x^2 + y^2 + z^2). \qquad (113)$$

We first of all remark that $r' = 1/r$ and that the inverse transformation to (113) has the same form:

$$x = \frac{x'}{r'^2}; \quad y = \frac{y'}{r'^2}; \quad z = \frac{z'}{r'^2}. \qquad (113_1)$$

If we introduce spherical coordinates, (112) transforms to

$$v(r', \theta, \varphi) = \frac{1}{r'} u\left(\frac{1}{r'}, \theta, \varphi\right).$$

Since $u(r, \theta, \varphi)$ satisfies Laplace's equation:

$$r^2 u_{rr} + 2ru_r + \frac{1}{\sin\theta}(\sin\theta\, u_\theta)_\theta + \frac{1}{\sin^2\theta} u_{\varphi\varphi} = 0$$

and we have the obvious identity

$$r'^5\left(v_{r'r'} + \frac{2}{r'} v_{r'}\right) = u_{rr} + \frac{2}{r} u_r,$$

it can easily be seen that the function $v(r', \theta, \varphi)$ also satisfies Laplace's equation. Transformation (113) is a symmetry transformation with respect to the unit sphere with centre at the origin [cf. II, 197]. We could obviously also take the centre of the sphere at any point (a, b, c) and its radius R as arbitrary. Formulae (112) and (113) can now be written as

$$x' - a = \frac{R^2}{r^2}(x - a); \quad y' - b = \frac{R^2}{r^2}(y - b); \quad z' - c = \frac{R^2}{r^2}(z - c)$$

$$v(x', y', z') = \frac{R}{r'} u\left[a + \frac{R^2}{r'^2}(x' - a), \ldots\right] \qquad (114)$$

$$r^2 = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2};$$

$$r'^2 = \sqrt{(x-a)^2 + (y-b)^2 + (z'-c)^2}.$$

Transformation (114) is known as *Kelvin's transformation*. Before explaining the concept of regularity of a harmonic function at an infi-

nitely remote point in three-dimensional space, let us prove the property of harmonic functions which we proved for the plane case at the end of the last section. *Let $u(M)$ be a harmonic function in some sphere S_0 with centre at the origin, with the possible exception of the origin itself, and let it be bounded in absolute value in this sphere. We show that the limit of $u(M)$ exists as M tends to the origin, and if this limit is taken as the value of $u(M)$ at the origin, $u(M)$ is also harmonic at the origin.*

By using the integral mentioned in [II, 197], we can construct a function $u_1(M)$, harmonic in the sphere S_0 including the origin, and taking the same boundary values as $u(M)$ on the surface of the sphere.

We apply Kelvin's transformation to the difference $u_1(M) - u(M)$, with respect to the sphere S_0. The transformed function is harmonic outside S_0, vanishes on the surface of S_0 and tends to zero as the point M' tends to infinity. This last fact follows immediately from the form of (114) and the fact that $u(M)$ is bounded by hypothesis in the neighbourhood of the origin. Since the extrema of a harmonic function must be attained on the boundary of the domain, we can assert that the function $u_1(M) - u(M)$ must be identically zero, i.e. $u_1(M)$ coincides with $u(M)$, so that this latter is also harmonic at the origin.

Let $u(M)$ be a function harmonic in the neighbourhood of a point O which we take as origin, and at O itself. On carrying out the Kelvin transformation with centre at the origin and radius at least equal to unity, we obtain a transformed function $v(M')$ which is harmonic in the neighbourhood of an infinitely remote point. This function tends to zero as $r' \to \infty$ and furthermore, it follows directly from (112) that the product $r'v(M)$ remains bounded as $r' \to \infty$, and the same can be said of the products $r'^2(\partial v/\partial x'), r'^2(\partial v/\partial y'), r'^2(\partial v/\partial z')$. This last result is a direct consequence of the fact that the derivatives of $u(M)$ are bounded in the neighbourhood of the origin. Conversely, if $u(M)$ is a function which is harmonic in the neighbourhood of a point at infinity and is such that the product $ru(M)$ remains bounded as $r \to \infty$, it can be seen by carrying out Kelvin's transformation that the transformed function $v(M') = 1/r' \cdot u(M)$ is harmonic and bounded in the neighbourhood of the origin, and is therefore harmonic at the origin as well. But it now follows at once from the above arguments that the products $r^2(\partial u/\partial x)$, $r^2(\partial u/\partial y)$, $r^2(\partial u/\partial z)$ are bounded as $r \to \infty$. Suppose, finally, that we only know, regarding a function $u(M)$ harmonic in the neighbourhood of an infinitely remote point, that $u(M) \to 0$ as $r \to \infty$, i.e. given any positive ε, there exists a positive number A such that $| u(M) | \leqslant \varepsilon$ provided $r \geqslant A$. We draw a sphere S_0 with centre

at the origin and so large a radius that $u(M)$ is harmonic outside S_0 and on the surface of S_0. We can construct a function $v_1(M')$, harmonic inside S_0 and having the same boundary values as $u(M)$ on the surface of S_0.

Let $u_1(M)$ be the result of applying Kelvin's transformation to the function $v_1(M')$ with respect to the sphere S_0. The difference $u(M) - u_1(M)$ is a harmonic function outside S_0, vanishes on S_0 and tends to zero as $r \to \infty$. We have seen that such a function must vanish identically. Hence our original function $u(M)$ must coincide with the function $u_1(M)$ obtained from $v_1(M')$ by Kelvin's transformation, $v_1(M')$ being harmonic inside S_0. We have seen that, for such a function, the products

$$ru_1(M), \quad r^2 \frac{\partial u_1(M)}{\partial x}, \quad r^2 \frac{\partial u_1(M)}{\partial y}, \quad r^2 \frac{\partial u_1(M)}{\partial z} \tag{115}$$

must be bounded as $r \to \infty$. Hence we see that *the fact that* $u(M) \to 0$ *as* $r \to \infty$ *implies that products* (115) *for* $u(M)$ *must remain bounded as* $r \to \infty$.

We describe the function $u(M)$, harmonic in the neighbourhood of a point at infinity, as *regular at infinity* if $u(M) \to 0$ as $r \to \infty$. If we only know that $u(M)$ tends to a finite limit b, we can say that such a function is equal to the sum of a constant b and a harmonic function regular at infinity. If the products (115) for functions $u(M)$ and $v(M)$ harmonic outside S remain bounded, and if these functions have regular normal derivatives from the outside on S, then as we saw in [200], (94) and (95) hold for these functions, the integration being carried out over the part of space outside S.

The exterior Dirichlet problem consists in finding a function $u(M)$, harmonic outside S, regular at an infintely remote point, continuous up to S and having previously assigned values $f(N)$ on the surface S. On taking an interior point M_0 of S as origin and carrying out Kelvin's transformation, the exterior Dirichlet problem is reduced to an interior problem for the transformed domain. We can prove with the aid of the usual arguments that the solution of the exterior Dirichlet problem is unique. The existence of the solution amounts to the existence of the solution of the interior problem, and this latter can be proved with general assumptions regarding the surface provided that the boundary data are continuous.

We may note the difference in the statements of the exterior Dirichlet problem for a plane and for space. In the plane case we speci-

fied the boundary values and merely required that the function tend to a finite limit as $r \to \infty$. In the case of three-dimensional space we specify the limit itself, i.e. we assume that it is zero. We might have assumed that our function tends to a given limit b as $r \to \infty$. In this case we should arrive at our previous statement of the problem by considering the difference $u(M) - b$. It may easily be seen that, in the case of three-dimensional space, it is not sufficient to require that $u(M)$ have a finite limit as $r \to \infty$ in order to obtain a definite statement of the exterior Dirichlet problem. For suppose that a certain amount of electricity is in equilibrium on a conducting surface S. The electrical potential of the simple layer will have a certain constant value c on S, whilst it is easily seen that $u(M)$ is a harmonic function outside S and tends to zero as $r \to \infty$. The constant c is itself a harmonic function outside S and has the same boundary values on S, but it is no longer regular, according to our definition, at an infinitely remote point. This argument does not apply in the plane case, since the electrical potential of a simple layer on a curve l becomes infinite at an infinitely remote point. We further remark that a function $u(M)$ is sometimes said to be harmonic outside a surface S only when it is regular at an infinitely remote point, i.e. some authors include regularity at an infinitely remote point in their definition of a function harmonic outside S.

The exterior Neumann problem consists in finding a function, harmonic outside S, regular at infinity, and with a normal derivative having specified boundary values on S. In this case the boundary values of the normal derivative no longer need to satisfy condition (110). The proof of this condition that we gave in the plane case is not suitable in space, since the surface area of a sphere of radius R is of order R^2, and the integral of $\partial u/\partial n$ over a sphere of sufficiently large radius does not necessarily tend to zero as $R \to \infty$. If we assume that the solution of the exterior Neumann problem has a regular normal derivative, the uniqueness of this solution follows at once from (94). Similar arguments have been used for the interior Neumann problem.

We conclude the present section by remarking that the above-mentioned properties of a harmonic function in the neighbourhood of an infinitely remote point could be obtained directly from the expansion of the function in spherical functions in this neighbourhood.

205. Uniqueness of the solution of Neumann's problem. The present section gives a proof of the uniqueness of the solution of Neumann's interior problem, without the assumption that the normal derivative

is regular. As a preliminary we take a solid of a special form and construct in the solid a harmonic function having certain properties which will be indicated below. Let $T(a, k, h)$ be a solid bounded by the surface

$$Z = k\,(x^2 + y^2)^{\frac{1+a}{2}} \tag{116}$$

and the plane $Z = h$, where k, a and h are positive constants. We write N_0 for the point $(0, 0, 0)$ and call it the vertex of the solid. Let σ' denote the part of the boundary of the solid that lies in the plane $Z = h$, and σ'' the remaining part of the boundary. Further, let u_0 and u_1 be two real constants with $u_0 < u_1$. We construct a function $w(M)$ harmonic inside the solid $T(a, k, h)$, continuous up to the boundary, and such that $w(M) < u_1$ on σ', $w(M) \leqslant u_0$ on σ'' and $w(N_0) = u_0$. If $r = \sqrt{x^2 + y^2 + z^2}$, θ is the angle between the radius vector and the z axis and $P_n(x)$ is Legendre's function [III_2, 141], we know from [III_2, 135] that, given any n, the function $r^n P_n(\cos\theta)$ is harmonic inside $T(a, k, h)$. We shall construct $w(M)$ in the form

$$w(M) = \gamma\left[r\cos\theta + r^{1+\beta} P_{1+\beta}(\cos\theta)\right] + u_0, \tag{117}$$

where γ and β are positive constants which will be defined later. We obviously have $w(N_0) = u_0$. We show first of all that, for all β sufficiently close to zero:

$$P_{1+\beta}(0) < 0. \tag{118}$$

The function $P_n(x)$ is the sum of the hypergeometric series:

$$P_n(x) = F\left(n+1, -n, 1; \frac{1-x}{2}\right) \qquad (|x| < 1),$$

in view of which we can write

$$P_n(0) = 1 - \frac{(n+1)\,n}{2} + \sum_{k=2}^{\infty} (-1)^k \frac{(n+k)(n+k-1)\ldots(n-k+1)}{(k!)^2\, 2^k},$$

or

$$P_n(0) = -\frac{(n-1)(n+2)}{2} + \sum_{k=2}^{\infty} (-1)^k \left[\prod_{s=1}^{k} \frac{(n+s)(n-s+1)}{2s^2}\right].$$

We take $1 < n < 2$, so that the factors corresponding to $s = 1$ and $s = 2$ are positive, whilst the remaining $(k - 2)$ are negative. On taking $(-1)^{k-2}$ out of these last factors, we can write them as

$$0 < \frac{(n+s)(s-n-1)}{2s^2} = \frac{s^2 - n^2 - s - n}{2s^2} < \frac{1}{2} \quad (s = 3, 4, \ldots).$$

Hence we obtain, on separating out the first two factors:

$$P_n(0) < -\frac{(n-1)(n+2)}{2} + \sum_{k=2}^{\infty} \frac{(n+2)(n+1)n(n-1)}{16} \cdot \frac{1}{2^{k-2}},$$

i.e.

$$P_n(0) < \frac{(n-1)(n+2)}{2}\left[\frac{(n+1)n}{8} - 1\right],$$

whence it follows that $P_n(0) < 0$ if $n^2 + n < 8$. Thus (118) is proved for all positive β sufficiently close to zero. We fix β so that it satisfies this condition, and also the condition $\beta < \alpha$, (see equation (116)). Since $z = r \cos\theta$, we obtain on surface (116):

$$r\cos\theta + r^{1+\beta} P_{1+\beta}(\cos\theta) = k\varrho^{1+\alpha} + r^{1+\beta} P_{1+\beta}(\cos\theta),$$

where $\varrho = \sqrt{x^2 + y^2}$, and finally, on surface (116):

$$r\cos\theta + r^{1+\beta} P_{1+\beta}(\cos\theta) = r^{1+\beta}\left[kr^{\alpha-\beta}\sin^{1+\alpha}\theta + P_{1+\beta}(\cos\theta)\right].$$

If $r \to 0$, then $\theta \to \pi/2$, and by (118) and the fact that $\beta < \alpha$, the square bracket now has a negative limit. We can therefore fix a positive h so small that we have, throughout the part σ'' of the surface $T(\alpha\ k\ h)$,

$$r\cos\theta + r^{1+\beta} P_{1+\beta}(\cos\theta) \leqslant 0. \tag{119}$$

We finally choose a positive γ so small that (117) gives $w(M) < u_1$ on σ'. The constructed function $w(M)$ satisfies all the above-mentioned conditions. We have on the axis of $T(\alpha, k, h)$, i.e. on the z axis:

$$w(M) = \gamma(z + z^{1+\beta}) + u_0 \qquad (z > 0).$$

If M is a variable point on this axis, then

$$\frac{w(M) - w(N_0)}{z} = \gamma + \gamma z^{\beta} > \gamma. \tag{120}$$

We now prove the theorem:

THEOREM. *If $u(M)$ is a function harmonic inside D_i and not equal to a constant, and if u_0 is the finite strict lower bound of $u(M)$ inside D_i, and there exists a point N_0 on S such that $u(M) \to u_0$ as M approaches N_0 from inside, then the ratio*

$$\frac{u(M) - u(N_0)}{|N_0 M|} \tag{121}$$

will remain greater than some positive number as M approaches N_0 along a normal.

We assume that there exists a solid $T(\alpha, k, h)$ which touches S at N_0 and all the points of which, except N_0, lie inside S. The number h is

chosen small enough for (119) to hold throughout the part σ'' of the surface of the solid. Let u_1 be the minimum of $u(M)$ on the part σ' of the surface of $T(a, k, h)$. Since $u(M)$ is not a constant, $u_0 < u_1$, and by choosing γ sufficiently small, we can construct $w(M)$ with the above-mentioned properties. Here, $w(N_0) = u(N_0)$, and $w(N) < u(N)$ on the remaining part of the surface of $T(a, k, h)$. On directing the z axis along the inward normal to S at N_0, we now obtain:

$$\frac{u(M) - u(N_0)}{|N_0 M|} = \frac{u(M) - u(N_0)}{z} > \frac{w(M) - w(N_0)}{z} > \gamma, \tag{122}$$

which proves the theorem.

A direct consequence of this theorem is the uniqueness of the solution of Neumann's problem in the following sense:

If a function $u(M)$, harmonic inside D_i, is continuous up to S, and $(\partial u(N)/\partial n)_i = 0$ throughout the surface S, $u(M)$ is constant. Let N_0 be the point of S where $u(M)$ has its minimum value. It follows at once from (122) that the derivative with respect to the normal at N_0 cannot tend to zero when $M \to N_0$ whilst remaining on the normal. If this were so, we should obtain from the formula of finite increments:

$$\frac{u(M) - u(N_0)}{z} \to 0,$$

which contradicts (122).

The proof of uniqueness is similar for Neumann's exterior problem.

The above proof was given in a joint paper of M. V. Keldysh and M. A. Lavrent'ev (*Dokl. Akad. Nauk SSSR*, t. XVI, no. 3, 1937).

If it is possible to touch the surface S from inside by a sphere, the proof of the uniqueness theorem is elementary. (S. Zaremba, *Uspekhi matematicheskikh nauk*, t. I, vyp. 3—4).

206. The solution of boundary value problems in the three-dimensional case. Let us consider the interior Dirichlet and Neumann problems for a domain D_i bounded by a surface S. We shall seek the solution of the interior Dirichlet problem in the form of the potential of a double layer:

$$u(M) = \iint_S \mu(N) \frac{\cos(\mathbf{r}, \mathbf{n})}{r^2} \, dS \qquad (v = |MN|), \tag{123}$$

where $\mathbf{r}$ is the direction MN and $\mathbf{n}$ is the direction of the outward normal at N to the surface. The unknown is the density $\mu(N)$. In

accordance with the first of formulae (42), the interior Dirichlet problem with boundary values

$$u|_S = f(N) \tag{124}$$

is equivalent to the following integral equation for the density $\mu(N)$.

$$f(N_0) = \iint_S \mu(N) \frac{\cos(\mathbf{r}_0, \mathbf{n})}{r_0^2} \, dS + 2\pi\mu(N_0) \qquad (r_0 = |N_0 N|).$$

On introducing the kernel

$$K(N_0; N) = -\frac{1}{2\pi} \frac{\cos(\mathbf{r}_0, \mathbf{n})}{r_0^2},$$

we can rewrite the last equation as

$$\mu(N_0) = \frac{1}{2\pi} f(N_0) + \iint_S \mu(N) K(N_0; N) \, dS. \tag{125}$$

The kernel $K(N_0; N)$ is not symmetrical, since the normal is taken at N and $\mathbf{r}_0$ denotes the direction $\overline{N_0 N}$. We obtain the transposed kernel of the adjoint equation [9] if we take the normal at N_0 and reckon $\mathbf{r}_0$ from N to N_0. This transposed kernel $K_1(N_0; N)$ is therefore given by

$$K_1(N_0; N) = K(N; N_0) = \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2}, \tag{126}$$

where $\mathbf{n}_0$ is the direction of the outward normal at N_0. We have changed the sign of the kernel, since the direction of $\mathbf{r}_0$ must be reversed in $K_1(N_0; N)$, whilst $\mathbf{r}_0$ in (126) denotes the direction $\overline{N_0 N}$ as before. The solution of the interior Neumann problem with the boundary condition:

$$\left(\frac{\partial u(N)}{\partial n}\right)_i = f(N) \tag{127}$$

is sought in the form of the potential of a simple layer:

$$u(M) = \iint_S \frac{\mu(N)}{r} \, dS. \tag{128}$$

On using the first of formulae (49), we arrive at an integral equation equivalent to the problem:

$$f(N_0) = \iint_S \mu(N) \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \, dS + 2\pi\mu(N_0).$$

This equation can also be written as

$$\mu(N_0) = \frac{1}{2\pi} f(N_0) - \iint\limits_S \mu(N) K_1(N_0; N) \, dS. \tag{129}$$

Similarly, by using the second of formulae (42) and (49), we obtain for the exterior Dirichlet and Neumann problems with the boundary conditions

$$u|_S = f(N); \tag{130_1}$$

$$\left(\frac{\partial u(N)}{\partial n}\right)_e = f(N), \tag{130_2}$$

the integral equations

$$\mu(N_0) = -\frac{1}{2\pi} f(N_0) - \iint\limits_S \mu(N) \frac{\cos(\mathbf{r}_0, \mathbf{n})}{r_0^2} \, dS, \tag{131_1}$$

$$\mu(N_0) = -\frac{1}{2\pi} f(N_0) + \iint\limits_S \mu(N) \frac{\cos(\mathbf{r}_0, \mathbf{n}_0)}{r_0^2} \, dS, \tag{131_2}$$

where, as above, the solution of the Dirichlet problem is sought in form (123) and the solution of the Neumann problem in form (128). Let us write the equations containing the parameter λ:

$$\mu(N_0) = \varphi(N_0) + \lambda \iint\limits_S \mu(N) K(N_0; N) \, dS, \tag{132}$$

$$\mu(N_0) = \varphi(N_0 + \lambda \iint\limits_S \mu(N) K_1(N_0; N) \, dS. \tag{133}$$

Equation (132) with $\lambda = 1$ and $\varphi(N_0) = f(N_0)/2\pi$ corresponds to the interior Dirichlet problem, and with $\lambda = -1$ and $\varphi(N_0) = f(N_0)/2\pi$ to the exterior Dirichlet problem. Equation (133) with $\lambda = 1$ and $\varphi(N_0) = - f(N_0)/2\pi$ corresponds to the exterior Neumann problem, and with $\lambda = -1$ and $\varphi(N_0) = f(N_0)/2\pi$ to the interior Neumann problem. If S is a Lyapunov surface, and $a = 1$ in condition (3), we obtain the inequality for the kernel of the integral equation, on the basis of the results of [192]:

$$|K(N_0; N)| \leqslant \frac{C}{r}, \tag{134}$$

and we can assume that the fundamental theorems of the theory of integral equations [19] hold for equations (132) and (133).

207. Investigation of the integral equations. We consider the homogeneous equation:

$$\mu(N_0) = \lambda \iint_S \mu(N) K(N_0; N) \, \mathrm{d}S. \tag{135}$$

The investigation of this is extremely simple in the case of a convex surface. In this case $\cos(\mathbf{r}_0, \mathbf{n}_0) \geqslant 0$ and $K(N_0; N) \leqslant 0$. Let $\mu(N)$ be a non-zero solution of equation (135), and let N_0 be the point at which $|\mu(N)|$ has its maximum value. If $\mu(N)$ is not constant, we get

$$|\mu(N_0)| < |\lambda| |\mu(N_0)| \iint_S \frac{\cos(\mathbf{r}_0, \mathbf{n})}{2\pi r_0^2} \, \mathrm{d}S,$$

or; by (26),

$$|\mu(N_0)| < |\lambda| |\mu(N_0)| \qquad \big(\mu(N_0) \neq 0\big),$$

whence $|\lambda| > 1$. If we suppose that $\mu(N)$ is a non-zero constant in (135), on again using (26) we get $\lambda = -1$.

We thus find that:

If S is a convex surface, $\lambda = 1$ is not an eigenvalue of equation (135), whilst $\lambda = -1$ is an eigenvalue with unit rank, corresponding to the eigenfunction $\mu(N) = \text{const.}$ We can therefore say that, as regards equation (133), $\lambda = 1$ is again not an eigenvalue, whilst $\lambda = -1$ is an eigenvalue of unit rank.

We now show that the same results hold for any Lyapunov surface with $a = 1$; here we shall use the results of [200], based on the possibility of constructing parallel surfaces. We start from equation (133) and consider the corresponding homogeneous equation with $\lambda = 1$:

$$\mu(N_0) = \iint_S \mu(N) K_1(N_0; N) \, \mathrm{d}S. \tag{136}$$

Let $\mu_0(N)$ be a continuous solution of this equation. We have to show that $\mu_0(N) \equiv 0$. The potential of a simple layer with density $\mu_0(N)$ gives us a function $u_0(M)$, harmonic in D_i and D_e, continuous throughout space, and such that the values of the normal derivative $(\partial u_0(N)/\partial n)_e$ vanish on S. This latter result follows from the fact that $\mu_0(N)$ satisfies (136) by hypothesis. We can apply (94) to the simple layer potential $\mu_0(N)$, whence it follows that $u_0(N)$ is constant in D_e. The simple layer potential vanishes at infinity, so that $\mu_0 = 0$ in D_e, and in particular, on S. Now the harmonic function $u_0(M) = 0$

inside S as well, i.e. $u_0(M) \equiv 0$ throughout space. On taking (54) into account, we obtain

$$\mu_0'(N) = \frac{1}{4\pi}\left[\left(\frac{\partial u_0(N)}{\partial n}\right)_i - \left(\frac{\partial u_0(N)}{\partial n}\right)_e\right] \equiv 0 \qquad (\text{on } S),$$

which is what we wanted to prove. We can therefore say that $\lambda = 1$ is not an eigenvalue of equations (132) and (133). The homogeneous equation (135) with $\lambda = -1$ has, by (26), a solution equal to an arbitrary constant, i.e. $\lambda = -1$ is an eigenvalue of equations (132) and (133). Let us show that its rank is unity. We need only show that, with $\lambda = -1$, equation (133):

$$\mu(N_0) = -\iint_S \mu(N) K_1(N_0; N)\, dS \tag{137}$$

has only one eigenfunction (discounting an arbitrary constant factor).

Let $\mu_1(N)$ be an eigenfunction of (137). The potential of a simple layer with density $\mu_1(N)$ gives a function $u_1(M)$ harmonic in D_i, for which the limit of $(\partial u_1(N)/\partial n)_i$ on S is zero. As above, (92) shows that $u_1(M)$ is constant in D_i and on S, i.e. the density $\mu_1(N)$ gives a simple layer potential which remains constant on and inside S. In other words, $\mu_1(N)$ is an electrostatic density.

Let $\mu_2(N)$ be another solution of (137), differing from zero. We have to show that $\mu_2(N)$ only differs from $\mu_1(N)$ by a constant factor. We form the simple layer potential u_3 with density $\mu_3(N) = \mu_1(N) + c\mu_2(N)$, where c is a constant. As above, it will be constant in D_i and on S. We can choose c so that the constant value of u_3 in D_i and on S is zero. The simple layer potential u_3 will also vanish at infinity. Hence it follows that u_3 vanishes in D_e also, i.e. $u_3 \equiv 0$ throughout space. We conclude from this, on the basis of (54), that $\mu_3(N) = \mu_1(N) + c\mu_2(N) = 0$ on S, i.e. $\mu_2(N)$ in fact differs only by a constant factor from $\mu_1(N)$. It follows at once from the proof of $u_3 \equiv 0$, that the potentials of a simple layer with densities $\mu_1(N)$ and $\mu_2(N)$ differ from zero in $\overline{D}_i$. This makes it possible for us to determine the constant c.

Let us show that the integral

$$\iint_S \mu_1(N)\, dS, \tag{138}$$

giving the total amount of electricity in equilibrium on the conducting surface S, differs from zero. As already mentioned, u_1 has a con-

stant value in D_i, and (54) gives:

$$\mu_1(N) = -\frac{1}{4\pi}\left(\frac{\partial u_1(N)}{\partial n}\right)_e. \tag{139}$$

If integral (138) were to vanish, we should have:

$$\iint_S \left(\frac{\partial u_1(N)}{\partial n}\right)_e \mathrm{d}S = 0. \tag{140}$$

We apply (94) to u_1. The function u_1 has a constant value on S, and integral (140) vanishes. Hence it follows that both sides of (94) vanish for $u = u_1$, i.e. u_1 is constant in D_e, and (139) shows us that $\mu_1(N)$ is zero on S, which contradicts the fundamental assumption that $\mu_1(N)$ is a solution of homogeneous equation (137) which is not identically zero. Hence integral (138) is in fact non-zero. On multiplying $\mu_1(N)$ by a constant factor, we can give integral (138) any previously assigned value.

It follows from these arguments that equations (132) and (133) for $\lambda = 1$ have a definite solution with any function φ, and hence we get solutions of the interior Dirichlet and exterior Neumann problems. We now return to equation (133) with $\lambda = -1$. It gives the density of the potential (128) that solves the interior Neumann problem. The necessary and sufficient condition for the existence of a solution is that the function φ of the integral equation be orthogonal to the eigenfunction of the homogeneous adjoint equation, i.e. to a constant. This leads to the usual condition:

$$\iint_S f(N)\,\mathrm{d}S = 0, \tag{141}$$

the necessity of which has already been seen above. Given our assumptions, it is also a sufficient condition. If it is fulfilled, the solution of the non-homogeneous equation is determined apart from an added term, which is a solution of homogeneous equation (137), i.e. it is defined apart from a term which represents an electrostatic density. On substituting this term in the simple layer potential we obtain a constant potential; and a constant term does not play any essential role in the solution of the interior Neumann problem.

We now consider equation (132) with $\lambda = -1$, which corresponds to the exterior Dirichlet problem. The necessary and sufficient condition for the equation to be soluble is that the absolute term be orthogonal to the solution of homogeneous equation (137), i.e. to the electro-

static density $\mu_1(N)$:

$$\iint_S \mu_1(N) f(N)\, dS = 0. \tag{142}$$

This supplementary condition is not tied up with the core of the problem, and is only due to the fact that we seek the solution of the exterior Dirichlet problem as a double layer potential. It follows at once from the form of such a potential that it vanishes as $r \to \infty$ with order $1/r^2$. This rapid tending to zero at infinity is not necessary for the solution of the exterior problem, and is in fact the result produced by the presence of the supplementary condition (142). Let us show that the exterior Dirichlet problem can be solved without any supplementary condition imposed on $f(N)$, with the aid of the sum of simple and double layer potentials. In fact, let $\mu_1(N)$ be the electrostatic density for which integral (138) is unity, and $u_1(M)$ the corresponding simple layer potential. The potential $u_1(M)$ has limiting values on S equal to some non-zero constant k. We choose the constant c such that

$$\iint_S \mu_1(N)\,[f(N) - c]\, dS = 0,$$

i.e. we put

$$c = \iint_S \mu_1(N) f(N)\, dS.$$

In accordance with the foregoing, we can form a double layer potential $w(M)$, which satisfies the exterior Dirichlet problem with boundary values $f(N) - c$. The sum

$$w(M) + \frac{c}{k} u_1(M)$$

will now satisfy the exterior Dirichlet problem with given boundary values $f(N)$.

Note. We have assumed in the present section that the domain for which the boundary problem is solved is bounded by a single surface S. The results are different if the finite domain D is bounded from the outside by a surface S_0 and from the inside by surfaces S_k $(k = 1, 2, \ldots, m)$. We shall seek the solution of the Dirichlet problem for this domain as a double layer potential. We obtain as before equation (132) with $\lambda = 1$ for the density, where S will be the total boundary of D, i.e. S will consist of surfaces S_0 and S_k $(k = 1, 2, \ldots, m)$, and the normal has to be directed outwards from D, i.e. inwards to the surface, on the S_k. We therefore obtain the interior Dirichlet problem. Equation (133) with $\lambda = 1$ corresponds to the exterior

Neumann problem. It amounts in the present case to finding a function harmonic inside each of the surfaces S_k and outside S_0, the values of its normal derivative being given on these surfaces. As usual, the function must be regular at infinity.

If the given values of the normal derivative are zero, we have homogeneous equation (133) with $\lambda = 1$. As we have just seen, this equation has only the zero solution if D is bounded by a single surface. In the present case the situation is different. In fact, imagine that all the surfaces are conducting. We locate a unit of positive electricity on the surface S_1, whilst we join S_0 to earth and suppose that the electrostatic state has been established on all the surfaces. On the surfaces S_l $(l = 2, \ldots, m)$ we have an induced distribution of electricity with zero total charge.

Let $\mu_1(N)$ be the density of the electrostatic distribution on S. The potential of a simple layer with such a density will obviously be constant inside each surface S_k and zero outside S_0, i.e. this potential will be a solution of the exterior Neumann problem with homogeneous boundary conditions. In other words, $\mu_1(N)$ will satisfy homogeneous equation (133) with $\lambda = 1$. On locating a unit of positive electricity successively on each of the surfaces S_k, we get m linearly independent solutions $\mu_k(N)$ of homogeneous equation (133) with $\lambda = 1$. It can be shown that these functions $\mu_k(N)$ form a complete system of linearly independent solutions of the homogeneous equation. Hence it is seen that $\lambda = 1$ is in the present case an eigenvalue of equations (132) and (133). The necessary and sufficient condition for (132) to be soluble is that $f(N)$ satisfy the m conditions:

$$\iint_S f(N)\, \mu_k(N)\, dS = 0.$$

If at least one of these conditions is not fulfilled, the interior Dirichlet problem has no solution in the form of a double layer potential. It can be shown that it is soluble as the sum of double and simple layer potentials, the proof being similar to the above for the exterior Dirichlet problem. A detailed account of problems of potential theory in the case of boundaries consisting of several surfaces may be found in N. M. Günther's *La théorie du potential* ... (Paris, 1934).

We have already proved the uniqueness of the solution of Neumann's problem under the condition that the required harmonic function be continuous up to S. It follows from the above discussion that this unique solution of Neumann's problem is expressible as a simple layer potential.

208. Summary of the results relating to the solution of boundary value problems. Let us formulate the results obtained above relating to the solution of the Dirichlet and Neumann problems, and obtain some new results in this field. For any function $f(N)$, continuous on S, equation (125) defines a continuous density $\mu(N)$ such that the double layer potential (123) gives a solution of the interior Dirichlet problem with boundary condition (124). Similarly, equation (131_2) gives a continuous density $\mu(N)$ such that the simple layer potential (128) satisfies the exterior Neumann problem with boundary condition (130_2). If $f(N)$ satisfies condition (141), equation (129) defines a density $\mu(N)$ such that the simple layer potential satisfies the interior Neumann problem.

Some important points regarding the solution of Neumann problems must be noticed. The integral term on the right-hand sides of equations (129) and (131_2) satisfies the Lipschitz condition [197]. If the function $f(N)$ also satisfies this condition, it follows from these equations that $\mu(N)$ satisfies the same condition. But now it follows from [198] that the corresponding simple layer potential, i.e. *the solution of Neumann's problem, is not only itself continuous up to S, but also has first order partial derivatives continuous up to S.*

We now turn to condition (141) for the interior Neumann problem to be soluble. We obtained (141) on the assumption that the solution $u(M)$ has a regular normal derivative. Therefore $u(M)$ must be continuous up to S [200]. Let us show that the necessity of condition (141) follows merely from the continuity of $u(M)$ up to S. Suppose that $f(N)$ does not satisfy this condition, but that there nevertheless exists a solution of the Neumann problem $u(M)$, continuous up to S. We show that this leads to a contradiction.

We have by hypothesis:

$$\int\int_S f(N)\, dS \neq 0,$$

and a non-zero constant C can be chosen such that

$$\int\int_S [f(N) - C]\, dS = 0.$$

By what has been proved above, we can construct a solution $u_1(M)$ of the interior Neumann problem with the boundary condition

$$\left(\frac{\partial u_1(N)}{\partial n}\right)_i = f(N) - C$$

in the form of a potential of a simple layer with continuous density, $u_1(M)$ being continuous up to S. The difference $u_2(M) = u(M) - u_1(M)$ is also continuous up to S and satisfies the boundary condition

$$\left(\frac{\partial u_2(N)}{\partial n}\right)_i = C.$$

By changing the sign of $u_2(M)$ if necessary, we can assume that $C > 0$. The function $u_2(M)$ attains its minimum on S at some point N_0, but this is in contradiction with the fact that, as M tends to N_0 along the normal, $\partial u(M)/\partial n$ tends to a positive value C. Therefore, the necessity of condition (141) for Neumann's interior problem to be soluble follows from the continuity of the required solution up to S.

We have indicated above the behaviour of the derivatives of the solution of Neumann's problem on approaching S. A similar study of the solution of Dirichlet's problem is more difficult, inasmuch as the solution is expressed as the potential of a double layer. A study of the potential of a double layer was carried out by Lyapunov in the work already quoted, and also in his work *"On Neumann's Fundamental Principle in Dirichlet's Problem"* (O fundamental'nom printsipe Neimana v zadache Dirikhle) (1902).

Let us enumerate the results obtained by Lyapunov in regard to the potential of a double layer and the solution of the Dirichlet problem.

1. The value at points of S of the potential $w(N)$ of a double layer with continuous density is expressible by a function satisfying the Lipschitz condition with any index less than unity.

2. If the potential of a double layer with continuous density has a regular normal derivative on S from one side of S, it has a regular normal derivative from the other side of S, and these normal derivatives are equal at all points of S.

3. The necessary and sufficient condition for the solution of the interior or exterior Dirichlet problem with continuous boundary values $f(N)$ on S to have a regular normal derivative on S is that the potential of the double layer with density $f(N)$ has a regular normal derivative on S. This theorem is proved on the assumption that the number a appearing in condition (3) is equal to unity.

4. Let $F(x, y, z)$ be a single-valued function which is continuous along with its derivatives of the first and second orders and which is defined in a neighbourhood of the surface S, and let $f(N)$ be the value of $F(x, y, z)$ on S. The derivative with respect to any fixed direction of the function $u(M)$, harmonic in D_i or D_e and taking the value $f(N)$ on S, is then continuous as far as S. This theorem is proved without the assumption that $a = 1$.

Lyapunov also established a sufficient condition for the double layer potential to have a regular normal derivative. Let us quote this. Let N_0 be any point of S which we take as the origin of polar coordinates (ϱ, θ) in the tangent plane to S at N_0. The density $\mu(N)$ at points N close to N_0 can be regarded as a function

of ϱ and θ by projecting N onto the tangent plane. We write:

$$\bar{\mu}(\varrho) = \frac{1}{2\pi} \int_0^{2\pi} \mu(\varrho, \theta)\, d\theta.$$

The condition mentioned amounts to the following. There exist two positive numbers b and β such that, for any choice of N_0,

$$|\bar{\mu}(\varrho) - \mu(N_0)| \leqslant b\varrho^{\beta+1}.$$

This theorem was proved by Lyapunov on the assumption that $\alpha = 1$.

Further results concerning the potential of a three-dimensional distribution, a simple and a double layer, and concerning the solution of the Dirichlet problem, can be found in the book by Günther mentioned above and in Kh. L. Smolitskii's article "Inequalities for the derivatives of fundamental functions" (Otsenki proizvoldnykh fundamental'nykh funktsii) (*Dokl. Akad. Nauk SSSR*, XXIV, 2. 1950).

We shall quote the results of this latter work relating to the solution of Dirichlet's problem.

Let $z = z(x, y)$ be the explicit equation of the surface S in the neighbourhood of the point N_0, in fact for $x^2 + y^2 \leqslant d^2/4$ [192]. If $z(x, y)$ has continuous derivatives up to order l, and all these derivatives have absolute values not exceeding some constant B, the same for all points N_0 of S, we shall say that S belongs to the class S_c.

Every function $f(N)$ defined on Sl is expressible as a function $f(x, y)$ of local coordinates in the neighbourhood of N_0. We shall write $D^k f(x, y)$ for any kth order derivative of $f(x, y)$. If the following inequalities hold:

$$|D^k f(x, y)| \leqslant A;$$

$$[D^k f(x, y)]_{(x_2 y_2)} - [D^k f(x, y)]_{(x_1, y_1)}| \leqslant B\left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)^{\beta} \quad (143)$$

$$(k = 0, 1, \ldots, l),$$

where $x_i^2 + y_i^2 \leqslant d^2/4$ $(i = 1, 2)$ and A, B and β are constants not depending on the choice of N_0, we shall say that $f(N)$ belongs to the class Lip $\beta(l, B)$. Let $f_1(M) = f_1(x, y, z)$ be defined in the closed domain $\bar{D}_i$, and have derivatives inside D_i up to order l, continuous as far as S, and let these derivatives satisfy relationships analogous to (143), in which D^k denotes any kth order derivative with respect to (x, y, z) and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is replaced by the distance between the points (x_2, y_2, z_2) and (x_1, y_1, z_1). We now say that the function $f_1(M)$ belongs to the class Lip $\beta(l, B)$. Smolitskii proves the following theorem in his work: If $f(N)$, defined on S, belongs to the class Lip $\beta(l, B)$ and the surface S belongs to the class Sl_{+5}, the solution $u(M)$ of the interior Dirichlet problem with boundary values $f(N)$ belongs to Lip $\beta(l, CB)$ where C is a constant not depending on the choice of $f(N)$.

209. Boundary value problems on a plane. The Dirichlet and Neumann problems on a plane can be treated in essentially the same way as in [206]. The solution of Dirichlet's problem is sought as the poten-

tial of a double layer:

$$u(M) = \int_l \mu(N) \frac{\cos(r, n)}{r} \, ds \tag{144}$$

and that of Neumann's problem as the potential of a simple layer:

$$u(M) = \int_l \mu(N) \log \frac{1}{r} \, ds. \tag{145}$$

We obtain for the density the integral equations

$$\mu(N_0) = \varphi(N_0) + \lambda \int_l \mu(N) K(N_0; N) \, ds, \tag{146}$$

$$\mu(N_0) = \varphi(N_0) + \lambda \int_l \mu(N) K_1(N_0; N) \, ds, \tag{147}$$

where

$$K(N_0; N) = -\frac{\cos(r_0, n)}{\pi r_0}; \; K_1(N_0; N) = \frac{\cos(r_0, n_0)}{r_0} \; (r_0 = |N_0 N|).$$

Equation (146) with $\lambda = 1$ and $\varphi(N_0) = 1/\pi \cdot f(N_0)$ corresponds to the interior Dirichlet problem, and with $\lambda = -1$ and $\varphi(N_0) = -1/\pi \cdot f(N_0)$ to the exterior Dirichlet problem. Equation (147) with $\lambda = 1$ and $\varphi(N_0) = -1/\pi \cdot f(N_0)$ corresponds to the exterior Neumann problem, and with $\lambda = -1$ and $\varphi(N_0) = 1/\pi \cdot f(N_0)$ to the interior Neumann problem. In all these cases $f(N_0)$ is the function appearing in the boundary condition.

Equation (146) can be written as

$$\mu(s_0) = \varphi(s_0) + \lambda \int_0^{l_0} \mu(s) K(s_0, s) \, ds,$$

where s and s_0 are the lengths of the arcs LN and LN_0 of the contour l, measured from some fixed point L in a definite direction, whilst l_0 is the length of l. Equation (147) may be written similarly. With the assumptions made in [199] regarding the contour l, $K(s_0; s)$ and $K_1(s_0; s)$ are continuous kernels.

As in [207], $\lambda = 1$ is not an eigenvalue, whilst $\lambda = -1$ is an eigenvalue of the first rank. The eigenfunction for equation (146) is now an arbitrary constant, and for equation (147) it is the electrostatic density $\mu_0(N)$, for which the potential of the simple layer (145) is equal to a constant on l and inside l.

On substituting the solution of (146) with $\lambda = 1$ and $\varphi(N_0) = 1/\pi \cdot f(N_0)$ in (144), we obtain the solution of the interior Dirichlet problem.

Substitution of the solution of (147) with $\lambda = 1$ in (145) does not in general give the solution of the exterior Neumann problem, since $\log(1/r)$ tends to infinity as $r \to \infty$. If $f(N)$ satisfies the condition:

$$\int_0^l f(N)\, ds = 0,$$

substitution of the solution of (147) with $\lambda = -1$ in (145) gives the solution of the interior Neumann problem.

We turn to the exterior Dirichlet problem. If $f(N)$ satisfies the condition:

$$\int_l \mu_0(N) f(N)\, ds = 0,$$

substitution of the solution of (146) with $\lambda = -1$ in (144) gives the solution of the problem.

If this condition if not fulfilled, we take a constant a such that [cf. 207]

$$\int_l \mu_0(N) [f(N) - a]\, ds = 0,$$

and obtain, as above, the solution $w(M)$ in accordance with (144) of the problem with boundary values $f(N) - a$, and the sum $w(M) + a$ is the required solution of the problem with boundary values $f(N)$. The addition of the constant is connected with the fact that (144) gives a harmonic function vanishing at infinity, whilst in the plane case the solution of the exterior problem does not require the solutions vanishing at infinity.

210. An integral equation for spherical functions. We take the homogeneous equation (133) for the case of a sphere Σ with centre the origin and unit radius. Here the direction n_0 is the direction of the radius $\overline{ON_0}$ and $\cos(r_0, n_0) = -r_0/2$, so that homogeneous equation (133) becomes

$$\mu(N_0) = -\frac{\lambda}{4\pi} \iint_\Sigma \frac{\mu(N)}{r_0}\, dS. \tag{148}$$

We would arrive at the same equation if we started from equation (132). The integral on the right is the value at the point $N_0(\theta_0, \varphi_0)$ of the potential of the spherical layer with density $-\lambda\mu(N) : 4\pi = -\lambda\mu(\theta, \varphi) : 4\pi$.

We first consider this potential at the point $M(\varrho, \theta', \varphi')$ situated inside the sphere. On writing r for the distance $|MN|$ and ϱ for

$|OM|$, we have the expansion [III$_2$, 132]:

$$\frac{1}{r} = \sum_{k=0}^{\infty} P_k(\cos\gamma)\,\varrho^k \qquad (\varrho < 1), \tag{149}$$

where the $P_k(x)$ are Legendre polynomials and γ is the angle formed of the radius vectors $\overline{OM}$ and $\overline{ON}$. We take a spherical function by order n for $\mu(\theta, \varphi)$:

$$\mu(\theta, \varphi) = Y_n(\theta, \varphi).$$

On using the expansion written above, which is uniformly convergent for $\varrho < 1$, we obtain:

$$\iint_{\Sigma} Y_n(\theta, \varphi)\frac{1}{r}\,\mathrm{d}\sigma = \frac{4\pi}{2n+1}\, Y_n(\theta', \varphi')\,\varrho^n,$$

which follows at once from the following expressions [III$_2$, 133]:

$$\iint_{\Sigma} Y_n(\theta, \varphi)\, P_m(\cos\gamma)\,\mathrm{d}S = 0 \qquad \text{for } m \neq n$$

$$\iint_{\Sigma} Y_n(\theta, \varphi)\, P_n(\cos\gamma)\,\mathrm{d}S = \frac{4\pi}{2n+1}\, Y_n(\theta'\,\varphi').$$

When M coincides with N_0, lying on the sphere, we have:

$$\iint_{\Sigma} Y_n(\theta, \varphi)\frac{1}{r_0}\,\mathrm{d}\sigma = \frac{4\pi}{2n+1}\, Y_n(\theta_0, \varphi_0).$$

It is clear from this that $\lambda_n = -(2n+1)$ are the eigenvalues of equation (148), and for every eigenvalue there are $(2n+1)$ corresponding eigenfunctions, these being spherical functions of order n. The eigenfunction corresponding to the first eigenvalue $\lambda_0 = -1$ is a constant (the electrostatic density for the case of a sphere).

We now show that equation (148) has no other eigenvalues and that there are no other eigenfunctions corresponding to the λ_n apart from those mentioned above (spherical functions). Let λ' be an eigenvalue of (148) different from those indicated, and let $\mu'(N)$ be a corresponding eigenfunction. The kernel of (148) is a symmetric function of N and N_0, so that $\mu'(N)$ must be orthogonal to all the spherical functions and, in particular, to $P_k(\cos\gamma)$:

$$\iint_{\Sigma} \mu'(\theta, \varphi)\, P_k(\cos\gamma)\,\mathrm{d}\sigma = 0.$$

It now follows from expansion (149) that the potential of the spherical layer with density $\mu'(N)$ is equal to zero everywhere inside the sphere,

and hence everywhere on the sphere. But integral equation (148) shows us now that $\mu'(N)$ is identically zero throughout the sphere, which cannot be true for an eigenfunction. We now consider the eigenvalue λ_n. If any eigenfunction which is not a spherical function of order n were to correspond to it, we would be able to assume that the eigenfunction is orthogonal to all the spherical functions and, by repeating the previous arguments, we would see that this function must vanish identically on the whole of the surface of the sphere.

Therefore, *the spherical functions represent the complete set of all eigenfunctions of integral equation* (148).

211. The heat equilibrium of a radiating body. Let us consider a third boundary problem for Laplace's value equation.

In the case of a steady heat flow the temperature $u(M)$ inside a body must satisfy Laplace's equation, whilst the condition

$$\frac{\partial u}{\partial n} + h\,(u - u_0) = 0$$

must be fulfilled on the boundary S, where h is the coefficient of external heat conductivity and u_0 is the temperature of the external medium in contact with the body. Both these quantities can be taken as functions of a point on the surface S, and we therefore arrive at the problem of finding a harmonic function inside the surface S which satisfies on S a boundary condition of the form

$$\frac{\partial u\,(N)}{\partial n} + p\,(N)\,u\,(N) = f\,(N), \tag{150}$$

where $p(N)$ and $f(N)$ are functions given on S and $p(N) > 0$. We shall seek the solution of this boundary value problem as the potential of a simple layer. The boundary condition (150) leads to the following integral equation for the density:

$$\iint\limits_S \mu\,(N)\frac{\cos\,(\boldsymbol{r}_0, \boldsymbol{n}_0)}{r_0^2}\,\mathrm{d}S + 2\pi\mu\,(N_0) + p\,(N_0)\iint\limits_S \mu\,(N)\frac{1}{r_0}\,\mathrm{d}S = f\,(N_0),$$

or

$$\mu\,(N_0) = \frac{1}{2\pi}f\,(N_0) - \iint\limits_S \mu\,(N)\left[\frac{p\,(N_0)}{2\pi\,r_0} + \frac{\cos\,(\boldsymbol{r}_0, \boldsymbol{n}_0)}{2\pi\,r_0^2}\right]\mathrm{d}S\,.$$

We show that, given the above assumptions, the homogeneous equation cannot have a non-zero solution. For we have seen above [202] that, with $p(N) > 0$, a harmonic function expressed by the potential

of a simple layer and hence having a regular normal derivative and satisfying the homogeneous boundary condition

$$\frac{\partial u(N)}{\partial n} + p(N)\, u(N) = 0, \tag{151}$$

is identically zero inside S. Suppose that the homogeneous equation has a solution $\mu(N)$. The potential of a simple layer with density $\mu(N)$ satisfies the homogeneous boundary condition (151), and therefore vanishes inside S. Since it also vanishes at infinity, we can conclude from this, as above, that it vanishes throughout space and that $\mu(N) \equiv 0$, i.e. the homogeneous equation has in fact no solutions, so that the non-homogeneous equation is soluble for any choice of the function $f(N_0)$. Suppose that S is a sphere Σ of unit radius and that the function $p(N)$ is a positive constant h. In this case, since $r_0 = -2\cos(r_0, n_0)$, we obtain the integral equation

$$\mu(N_0) = \frac{1-2h}{4\pi} \iint\limits_{\Sigma} \mu(N) \frac{1}{r_0}\, d\sigma + \frac{1}{2\pi} f(N_0),$$

which has been discussed in the previous section. If we take h as the parameter, the eigenvalues of this equation are given by the equation $1 - 2h = 2n + 1$, i.e. the eigenvalues are $h = 0, -1, -2, \ldots$ whilst the corresponding eigenfunctions are the spherical functions. This third boundary value problem can be considered for the plane case in essentially the same way.

212. Schwartz's method. A further method of solving the Dirichlet problem must be mentioned. Suppose that we know how to solve the Dirichlet problem with any continuous boundary values for the domains B_1 and B_2, these domains having a common part D as shown in Fig. 12. Schwartz's method enables us to solve the Dirichlet problem for the domain $B = B_1 + B_2$, obtained by combining B_1 and B_2. We shall give the arguments for the plane case, though they remain essentially the same in the case of three-dimensional space. The contours of B_1 and B_2 are divided by the points of intersection of the domains into the sections α_1 and β_1 for B_1, and α_2 and β_2 for B_2. Let $\omega(N)$ be a continuous function defined on the contour $l = \alpha_1 + \alpha_2$ of the domain B. The working

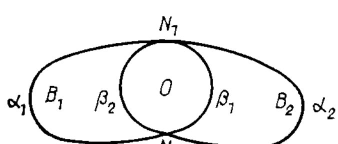

FIG. 12

in Schwartz's method is carried out as follows. The function $\omega(N)$, defined, in particular, on α_1, is continued on to β_1 in any manner whilst preserving its continuity. Let $\omega_1(N)$ be the function thus obtained on β_1. On solving the Dirichlet problem for B_1, we construct a function $u_1(M)$ harmonic in B_1 with the boundary values:

$$u_1(N) = \begin{cases} \omega\ (N) \text{ on } \alpha_1 \\ \omega_1(N) \text{ on } \beta_1 \end{cases}.$$

The values of this function on β_2 together with the values of $\omega(N)$ on α_2 are taken as the boundary values of a new function $v_1(M)$ harmonic in B_2:

$$v_1(M) = \begin{cases} \omega\ (N) \text{ on } \alpha_2 \\ u_1(N) \text{ on } \beta_2 \end{cases}.$$

We now construct a function $u_2(M)$ harmonic in B_1 with the boundary values:

$$u_2(N) = \begin{cases} \omega\ (N) \text{ on } \alpha_1 \\ v_1(N) \text{ on } \beta_1 \end{cases}.$$

Further, we construct a function $v_2(M)$ harmonic in B_2 with boundary values:

$$v_2(N) = \begin{cases} \omega\ (N) \text{ on } \alpha_2 \\ u_2(N) \text{ on } \beta_2 \end{cases}$$

and so on. In general:

$$\begin{aligned} &u_n(M) \text{ is harmonic in } B_1 \text{ and } u_n(N) = \begin{cases} \omega(N) \text{ on } \alpha_1, \\ v_{n-1}(N) \text{ on } \beta_1 \end{cases} \\ &v_n\ (M) \text{ is harmonic in } B_2 \text{ and } v_n\ (N) = \begin{cases} \omega\ (N) \text{ on } \alpha_2 \\ u_n(N) \text{ on } \beta_2 \end{cases}. \end{aligned} \tag{152}$$

We now show that $\lim u_n(M)$ in B_1 and $\lim v_n(M)$ in B_2 exist, and that these limits coincide in the common part of B_1 and B_2. For this, we make use of the following lemma. Let us first of all recall the assumptions made regarding the contours of the domains. We assume that the contours of B_1 and B_2 consist of a finite number of pieces, each having a continuously varying tangent. It is therefore possible to have a finite number of vertices on the contour. In addition, we assume that both contours have tangents at the points of intersection N_1 and N_2 (Fig. 12), and that these tangents at N_1 and N_2 form an angle different from zero. We shall now state the lemma.

If the contours of the domains B_1 and B_2 satisfy the above-mentioned conditions and if $w(M)$ is a function harmonic inside B_1, continuous in the closed domain, taking on α_1 the value zero and satisfying on β_1 the condition $|w(N)| \geqslant A$, then there exists a positive constant $q < 1$, depending only on the domains B_1 and B_2 and not on the choice of $w(M)$, such that $|w(M)| \leqslant qA$ on β_2. A similar proposition holds if we start from B_2 and estimate $w(M)$ on β_1. We can assume here that q is the same in both cases. We shall postpone the proof of the lemma to the next section, and apply it now to prove the convergence of Schwartz's process.

By construction,

$$
\begin{gathered}
u_{n+1}(N) - u_n(N) = \begin{cases} 0 & \text{on } \alpha_1 \\ v_n(N) - v_{n-1}(N) & \text{on } \beta_1 \end{cases} \\
v_n(N) - v_{n-1}(N) = \begin{cases} 0 & \text{on } \alpha_2 \\ u_n(N) - u_{n-1}(N) & \text{on } \beta_2 \end{cases}.
\end{gathered} \tag{153}
$$

We introduce the following notation:

$$
\begin{aligned}
M_n &= \max |u_{n+1}(N) - u_n(N)| = \max |v_n(N) - v_{n-1}(N)| \text{ on } \beta_1 \\
M'_n &= \max |v_{n+1}(N) - v_n(N)| = \max |u_{n+1}(N) - u_n(N)| \text{ on } \beta_2 .
\end{aligned}
$$

On taking into account boundary conditions (153) and the lemma, we get $M'_n \leqslant qM_n$ and $M_n \leqslant qM'_{n-1}$. Hence it follows that $M_n \leqslant q^2 M_{n-1}$ $(n = 2, 3, \ldots)$, i.e. $M_n \leqslant q^{2(n-1)} M_1$.

We form the series

$$
u_1(M) + \sum_{n=1}^{\infty} [u_{n+1}(M) - u_n(M)]. \tag{154}
$$

Its terms are zero on α_1 from the second onward, and satisfy $|u_{n+1}(N) - u_n(N)| \leqslant q^{2(n-1)} M_1$ on β_1. The series is therefore absolutely and uniformly convergent on the contour of B_1, and hence throughout the closed domain. Its sum $u(M)$ is continuous in the closed domain $\overline{B}_1$ and harmonic inside B_1. The sum of the first n terms of (154) is $u_n(M)$, and we can therefore say that $u_n(M) \to u(M)$ uniformly in the closed domain. Similarly, it can be shown that $v_n(M) \to v(M)$ uniformly in the closed domain $\overline{B}_2$, where $v(M)$ is continuous in the closed domain $\overline{B}_2$ and harmonic inside B_2. On the basis of (152):

$$
u_n(N) = v_{n-1}(N) \text{ on } \beta_1 \text{ and } v_n(N) = u_n(N) \text{ on } \beta_2 .
$$

On passing to the limit, we see that $u(N)$ and $v(N)$ coincide on β_1 and β_2. Hence it follows that they coincide everywhere in the common part D of domains B_1 and B_2. The functions $u(M)$ and $v(M)$ thus yield a single harmonic function inside $B = B_1 + B_2$. By (152), this harmonic function has the given boundary values $\omega(N)$ on the contour $l = a_1 + a_2$, so that Schwartz's method in fact solves our problem.

213. Proof of the lemma. We form the potential of a double layer distributed along the arc β_1 with unit density:

$$F(M) = -\int_{\beta_1} \frac{\partial \log \frac{1}{r}}{\partial n} dS. \tag{155}$$

This is the angle subtended by the arc β_1 at the point M, assuming that M belongs to B_1. The function (155), harmonic inside B_1, takes continuous boundary values at interior points of arcs a_1 and β_1 (Fig. 13).

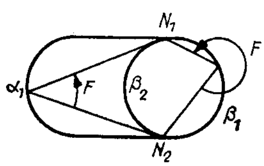

FIG. 13

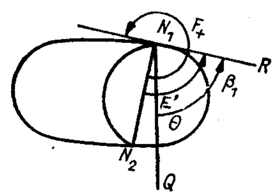

FIG. 14

Different limits will be obtained for the boundary values $F(N)$ of the function (155) according to whether a point N of the contour approaches N_1 along β_2 or along β_1; let $F_-(N_1)$ and $F_+(N_1)$ denote these limits.

The limits are the angles formed by the secant $\overline{N_1N_2}$ with the different directions of the tangent to the contour of B_1 at N_1 (Fig. 14), and we have:

$$F_+(N_1) - F_-(N_1) = \pi. \tag{156}$$

If we let the point M approach N_1 along any chord N_1Q, which forms an angle θ with the direction of the tangent at the point as indicated in Fig. 14, function (155) is easily seen from Fig. 14 to have the limit $F_+(N_1) - \theta$, which, in view of (156), can be written as

$$F_+(N_1) - \theta = \frac{\theta}{\pi} F_-(N_1) + \left(1 - \frac{\theta}{\pi}\right) F_+(N_1). \tag{157}$$

When M approaches N_1 in any manner, function (155) can have different limiting values, but they must lie between $F_-(N_1)$ and $F_+(N_1)$, and so the function (155) will be bounded in the neighbourhood of N_1. Similar results are obtained at the point N_2.

Let us define on the contour $l = \alpha_1 + \beta_1$ a function $f(N)$ equal to zero inside α_1 and to π inside β_1. On writing $F(N)$ as above for the limiting values of function (155), we can form the function $f_1(N) = F(N) - f(N)$ inside α_1 and β_1. It may easily be seen that this is continuous throughout the contour $l = \alpha_1 + \beta_1$, including the points N_1 and N_2, since $F(N)$ and $f(N)$ both have the same jump at these points. The value of $f_1(N)$, say at N_1, is equal to $F_-(N_1)$. Let $F_1(M)$ be harmonic in B_1 and have continuous boundary values $f_1(N)$ on the contour. We construct the harmonic function

$$G(M) = \frac{1}{\pi}\left[F(M) - F_1(M)\right]. \tag{158}$$

Its boundary values are zero inside α_1 and unity inside β_1. Moreover, from what has been said regarding $F(M)$, as M approaches N_1 or N_2 the limiting values of $G(M)$ must belong to the interval $[0, 1]$. By the maximum and minimum principle, all the interior values of function (158) will also lie inside this interval, i.e. $0 < G(M) < 1$, if M is inside B_1. Let θ_1 and θ_2 be the angles formed by the tangents to the curve β_2 at N_1 and N_2 with the tangents to the contour of the domain B_1 at these points. On approaching the point N_1 along β_2, $F(M)$ has the limit $[F_+(N_1) - \theta_1]$ by (157), and $F_1(M)$, with the continuous boundary values $f_1(N)$, will have the limit $f_1(N_1) = F_-(N_1)$, whilst function (158) will have the limit $1 - (\theta_1/\pi)$, by (156). Similarly, function (158) will have the limit $1 - (\theta_2/\pi)$ at N_2. Both these limits are less than unity, whilst we have $0 < G(M) < 1$ inside the domain. Hence it follows at once that a positive $q < 1$ exists such that $G(M) \leqslant q$ on β_2.

After these auxiliary constructions, we return to the function $w(M)$ mentioned in the lemma. On replacing this function by $w(M)/A$, we can assume that the number A, fixed in the lemma, is unity, i.e. the that harmonic function $w(M)$, continuous in the closed domain $\overline{B}_1$, has zero boundary values on α_1, and $| w(N) | \leqslant 1$ on β_1. The boundary values of $w(M)$ at N_1 and N_2 are obviously zero. We form the harmonic function $H(M) = G(M) - w(M)$. Its boundary values inside the arc α_1 are zero and are non-negative inside β_1, since we have $G(N) = 1$ and $| w(N) | \leqslant 1$ inside β_1. As M approaches N_1 and N_2

the limiting values of $H(M)$ must belong to the interval [0, 1]. Hence it follows at once that $H(M) \geqslant 0$ in the closed domain $\overline{B}_1$, i.e. $w(M) \leqslant G(M)$, so that we have $w(M) \leqslant G(M) \leqslant q$ on β_2. Similarly, $G(M) + w(M) \geqslant 0$ in $\overline{B}_1$, and it follows from this that $-w(M) \leqslant G(M) \leqslant q$ on β_2. The two inequalities obtained yield $|w(M)| \leqslant q$, which proves the lemma. This proof can be repeated for the three-dimensional case†.

214. Schwartz's method (continued). We have discussed the application of Schwartz's method in the case of the simplest mutual disposition of domains B_1 and B_2. The contours of these domains may intersect in more than two points (Fig. 15), or may have a common part

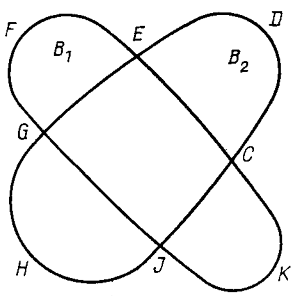

FIG. 15

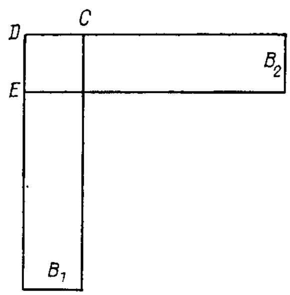

FIG. 16

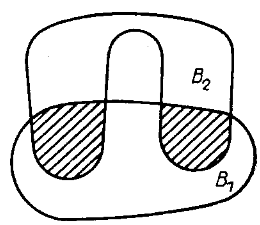

FIG. 17

(Fig. 16). It may happen that B_1 and B_2 are singly-connected, whilst their sum is multiply-connected (Fig. 17). In Fig. 15, the contour of the domain $B = B_1 + B_2$ is the curve $CDEFGHJKC$. In Fig. 16, the step-line CDE is the common part of the contours, whilst the shaded domains in Fig. 17 form the common part of B_1 and B_2. Whatever the case, the construction of the successive approximations in Schwartz's method is precisely the same as above. By slightly modifying the method of calculation, if we know how to solve the Dirichlet problem for B_1 and B_2, we can obtain the solution for the common part of these domains instead of for their sum. In the case of Fig. 16, this is the domain bounded by the contour $\beta_1 + \beta_2$. We are given the boundary values of $\omega(N)$ on this contour.
Given these boundary values, we seek the harmonic function as a sum:

$$w(M) = u(M) + v(M), \tag{159}$$

* See Courant and Hilbert: *Methoden der Mathematischen Physik*, vol. II.

where $u(M)$ is harmonic in B_1 and $v(M)$ harmonic in B_2. This decomposition of the harmonic function into two terms is obviously not unique, but this is of no importance for the construction. We continue in any manner the values of $\omega(N)$ given on β_1 on to the arc α_1 so as to obtain a continuous function, and denote this continuation by $\varphi_1(N)$.

We construct the successive approximations for $u(M)$ and $v(M)$ as solutions of Dirichlet's problem with the following boundary conditions:

$$u_1(N) = \begin{cases} \varphi_1(N) \text{ on } \alpha_1 \\ \omega(N) \text{ on } \beta_1 \end{cases}; \quad v_1(N) = \begin{cases} 0 & \text{on } \alpha_2 \\ \omega(N) - u_1(N) & \text{on } \beta_2 \end{cases}$$

Notice here that the difference $\omega(N) - u_1(N)$ vanishes at N_1 and N_2. To evaluate the subsequent approximations, we put

$$u_{n+1}(N) = \begin{cases} \varphi_1(N) & \text{on } \alpha_1 \\ \omega(N) - v_n(N) & \text{on } \beta_1 \end{cases}; \quad v_{n+1}(N) = \begin{cases} 0 & \text{on } \alpha_2 \\ \omega(N) - u_{n+1}(N) & \text{on } \beta_2. \end{cases}$$

The process is convergent and the sum (159) will give the solution of the problem.

A detailed account of this method may be found in *Approximation Methods of Higher Analysis* (Priblizhennye metody vysshego analiza) by L. V. Kantorovich and V. I. Krylov, where it is used for other equations of the elliptic type apart from Laplace's. The method can also be used for the three-dimensional case.

A further possible application of Schwartz's method may be mentioned. We shall shortly be concerned with the solution of Dirichlet's exterior problem, and we shall consider in this connection the three-dimensional instead of the plane case. Suppose we have n closed surfaces S_k $(k = 1, 2, \ldots, n)$ in space, each pair of which has no common points. Let D denote the part of space lying outside all the S_k, and D_k the part outside S_k. Suppose we know the solution of Dirichlet's problem for all the D_k with any continuous values on the S_k. It will now be shown how the Dirichlet problem can be solved for D. All the D_k and D contain the point at infinity as an interior point, and we assume as usual, for the solution of the Dirichlet problem, that the harmonic function vanishes at infinity.

We thus require to find a function, harmonic inside D and taking on the S_k the given continuous values

$$u|_{S_k} = f_k(N) \qquad (k = 1, 2, \ldots, n). \tag{160}$$

As a first step we find for each k the functions $u_{0,k}(M)$ ($k = 1, 2, \ldots, n$), harmonic inside D_k and taking the values $f_k(N)$ on S_k. Further, we find the functions $u_{1,k}(M)$ ($k = 1, 2, \ldots, n$), harmonic inside D_k and having the boundary values

$$u_{1,k}(M) = -\sum_{i \neq k} u_{0,i}(N) \text{ on } S_k \qquad (k = 1, 2, \ldots, n), \tag{161}$$

where the summation is carried out over all i from $i = 1$ to $i = n$, except for $i = k$.

In general, for each positive integer m we find functions $u_{m,k}(M)$ ($k = 1, 2, \ldots, n$), harmonic inside D_k and having the boundary values

$$u_{m,k}(N)|_{S_k} = -\sum_{i \neq k} u_{m-1,i}(N) \text{ on } S_k \qquad (k = 1, 2, \ldots, n). \tag{162}$$

The functions

$$\sum_{m=0}^{p} u_{m,k}(M) \qquad (k = 1, 2, \ldots, n),$$

are harmonic inside D_k and have the boundary values

$$\sum_{m=0}^{p} u_{m,k}(N) = f_k(N) - \sum_{m=0}^{p-1} \sum_{i \neq k} u_{m,i}(N) \text{ on } S_k \quad (k = 1, 2, \ldots, n).$$

On subtracting from both sides the sum

$$\sum_{m=0}^{p-1} u_{m,k}(N),$$

we can rewrite the previous equations as

$$\sum_{m=0}^{p-1} \sum_{i=1}^{n} u_{m,i}(N) = f_k(N) - u_{p,k}(N) \text{ on } S_k \quad (k = 1, 2, \ldots, n). \tag{163}$$

If we show that all the functions $u_{p,k}(M)$ ($k = 1, 2, \ldots, n$) tend uniformly to zero in the closed domain D as p increases indefinitely, it will follow from (163) that as p tends to infinity the functions

$$\sum_{m=0}^{p-1} \sum_{i=1}^{n} u_{m,i}(M),$$

harmonic inside D and continuous up to the boundary, in fact gives the solution of the Dirichlet problem for the domain D with the

boundary values $f_k(N)$ on S_k as:

$$u(M) = \sum_{m=0}^{\infty} \sum_{i=1}^{n} u_{m,i}(M). \tag{164}$$

Let us consider the conditions in which the functions $u_{p,k}(M)$ tend uniformly to zero in the closed domain D. Let $v_k(M)$ $(k = 1, 2, \ldots, n)$ denote a function harmonic inside D_k and equal to 1 on S_k. Now $v_k(M) \geqslant 0$ inside D_k, and since $v_k(M) \to 0$ as the point M moves to infinity, there exists a constant q_k satisfying the condition $0 < q_k < 1$ such that

$$v_k(M) \leqslant q_k \text{ on } S_i \text{ for } i \neq k \qquad (k = 1, 2, \ldots, n). \tag{165}$$

If $w_k(M)$ $(k = 1, 2, \ldots, n)$ are any functions harmonic inside D_k, continuous up to S_k and satisfying the conditions:

$$|w_k(M)| \leqslant a_k \text{ on } S_k \qquad (k = 1, 2, \ldots, n), \tag{166}$$

where a_k are constants, then $a_k v_k(M) - w_k(M)$ will be harmonic inside D_k and non-negative on S_k, whence it follows that $a_k v_k(M) - w_k(M) \geqslant 0$ in the closed domain D_k, i.e. $w_k(M) \leqslant a_k v_k(M)$ in $\overline{D}_k$. We can change the sign of the harmonic function $w_k(M)$ without affecting (166), and can therefore assume that $w_k(M) \geqslant 0$ at the point M. Hence it follows from the above discussion that

$$|w_k(M)| \leqslant a_k v_k(M), \tag{167}$$

so that, by (165):

$$|w_k(N)| \leqslant a_k q_k \text{ on } S_i \text{ for } i \neq k \qquad (k = 1, 2, \ldots, n). \tag{168}$$

This inequality is therefore a consequence of (166). Let a be a positive number such that $|f_k(N)| \leqslant a$ for $k = 1, 2, \ldots, n$, and let q be the maximum of $q_1, q_2, \ldots, q_n$, where obviously $0 < q < 1$. By (160), we have $|u_{0,k}(N)| < a$ on S_k. By (161) and (168), we also have: $|u_{1,k}(N)| \leqslant (n-1)aq$ on S_k $(k = 1, 2, \ldots, n)$. On further applying (162) with $m = 2$ and again using (168), we get $|u_{2,k}(N)| \leqslant (n-1)^2 aq^2$ on S_k, and in general: $|u_{p,k}(N)| \leqslant (n-1)^p aq^p$ on S_k, so that

$$|u_{p,k}(M)| \leqslant [(n-1)q]^p \; a \,(M \text{ in } D_k) \qquad (k = 1, 2, \ldots, n). \tag{169}$$

If the number of surfaces $n = 2$, it follows from this that $u_{p,k}(M) \to 0$ as $p \to \infty$ uniformly in $\overline{D}_k$ and certainly uniformly in the closed

domain D. If $n > 2$, we obtain the following sufficient condition for $u_{p,k}(M) \to 0$:

$$(n-1)q < 1. \tag{170}$$

The number q, by construction, does not depend on the boundary conditions $f_k(N)$ and is defined only by the domain D. We could have considered in precisely the same way the case when the domain D is finite, with an exterior boundary S_1 and interior boundaries S_2, $S_3, \ldots, S_n$. In this case we should have had the interior Dirichlet problem for the domain D_1 bounded by S_1, and the exterior problem as above for the D_k $(k = 2, \ldots, n)$. The construction given above is due to G. M. Goluzin (*Matematicheski sbornik*, **41,** 2, 1934). It is not applicable in the plane case, as may easily be seen by specifying constant values on separate closed contours l_k.

215. Sub- and superharmonic functions. When solving Dirichlet's problem by the method of integral equations it proved necessary to impose fairly severe restrictions on the boundary of the domain. We shall give another method of solution of Dirichlet's problem, which is suitable with very general assumptions regarding the boundary and the boundary values. This method is more of a theoretical nature and does not yield a construction for the approximate solution of the problem. It was proposed by Poincarè and made precise by Perron (*Math. Zeitschr.*, Bd. 18, 1923; see also an article by I. G. Petrovskii in *Uspekhi matematicheskikh nauk*, t. VIII, and his book on partial differential equations).

The present section will be concerned with certain new concepts, which will be useful when explaining the method. These new concepts are of general interest for mathematical physics. Our treatment will be confined to the case of a plane, though the three-dimensional case is essentially the same. The difference in the investigation of the boundary values of the harmonic function constructed by the method will be indicated at the end of our discussion.

An equation analogous to Laplace's for a function of a single independent variable $y(x)$ is $y''(x) = 0$, and its general solution is the first degree polynomial $y = ax + b$, its graph being a straight line. The Dirichlet problem, i.e. the problem of finding the solution of the equation $y''(x) = 0$ inside the interval $[a, b]$ with given values at the ends of the interval, amounts simply to drawing a straight line through two given points. A characteristic feature of a first

degree polynomial is that its value at any point $x = x_0$ is the arithmetic mean of its values at $x = x_0 + h$ and $x = x_0 - h$, equidistant from x_0. We now take a continuous curve that is concave upwards. Let $y = y(x)$ be its equation. We have at points of this curve:

$$y(x_0) < \frac{1}{2}\left[y(x_0 + h) + y(x_0 - h)\right]. \qquad (171_1)$$

Similarly, if the curve is convex upwards, we have

$$y(x_0) > \frac{1}{2}\left[y(x_0 + h) + y(x_0 - h)\right]. \qquad (171_2)$$

Inequality (171_1) follows at once from the fact that each piece of the curve here lies below its chord. Let us introduce a similar class of functions in the case of several variables. Let $f(M)$ be a function continuous inside a plane domain B. We describe it as *subharmonic inside B if there exists for every interior point P of B a positive δ such that $f(P)$ does not exceed the mean value of $f(M)$ on the circle with centre P and any radius $\varrho < \delta$.* If P has coordinates (x, y), this condition can be written as

$$f(x, y) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(x + \varrho\cos\varphi, y + \varrho\sin\varphi)\,d\varphi \qquad (\varrho < \delta). \qquad (172_1)$$

If $f(M)$ is a harmonic function inside B, the sign of equality holds in (172_1) for every interior point of B [II, 194], i.e. *a subharmonic function is a particular case of a harmonic function.* The definition can be extended directly to the three-dimensional case, though the circle has to be replaced by a sphere. A *superharmonic function* is similarly defined. Here we have, instead of (172_1), at every point inside B:

$$f(x, y) \geqslant \frac{1}{2\pi}\int_0^{2\pi} f(x + \varrho\cos\varphi,\; y + \varrho\sin\varphi)\,d\varphi. \qquad (172_2)$$

A harmonic function is also a particular case of a superharmonic function. It follows at once from the definition that, if $f(M)$ is subharmonic and C is a constant, $Cf(M)$ is subharmonic for $C > 0$ and superharmonic for $C < 0$. If $f(M)$ is superharmonic, $Cf(M)$ is superharmonic for $C > 0$ and subharmonic for $C < 0$. In addition, it follows directly from the definition that a finite sum of subharmonic functions is subharmonic, and a finite sum of superharmonic functions is superharmonic.

Let $f(M) = f(x, y)$ have continuous secon order derivatives inside B, and suppose

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \geqslant 0 \text{ (inside } B). \tag{173_1}$$

On applying Green's formula to the circle K_ϱ with centre $P(x, y)$ lying inside B, and putting $u = f$ and $v = 1$, we find:

$$\int\limits_{C_\varrho} \frac{\partial f}{\partial n} \, \mathrm{d}s = \iint\limits_{K_\varrho} \Delta f \, \mathrm{d}\sigma, \tag{174}$$

where C_ϱ is the circumference of K_ϱ. We also apply to the function f the formula [II, 193]:

$$f(x, y) = \frac{1}{2\pi} \int\limits_{C_\varrho} \left(f \frac{\partial \log r}{\partial n} - \log r \frac{\partial f}{\partial n} \right) \mathrm{d}s + \frac{1}{2\pi} \iint\limits_{K_\varrho} \Delta f \log r \, \mathrm{d}\sigma,$$

where r is the distance from (x, y) to the variable point of integration. The direction n coincides on C_ϱ with the direction r, and $\mathrm{d}s = \varrho \, \mathrm{d}\varphi$; we can use (174) to rewrite the formula as

$$f(x, y) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(x + \varrho \cos \varphi, y + \varrho \sin \varphi) \, \mathrm{d}\varphi + \frac{1}{2\pi} \iint\limits_{K_\sigma} \Delta f \log \frac{r}{\varrho} \, \mathrm{d}\sigma.$$

We have $r/\varrho \leqslant 1$ in K, and by (173_1), the last formula gives inequality (172_1), i.e. *given condition* (173_1), *$f(M)$ is subharmonic inside B*. Similarly, if

$$\Delta f \leqslant 0 \ \text{ (in } B), \tag{173_2}$$

$f(M)$ is superharmonic inside B. We clearly do not assume the existence of derivatives in the fundamental definition of sub- and superharmonic functions. Conditions (173_1) and (173_2) are analogous to the familiar conditions for convexity and concavity of a curve [I, 71].

Some simple properties of sub- and superharmonic functions may be mentioned. Let $f(M)$ be continuous in a closed domain and subharmonic inside the domain. It now follows at once from (172_1) that *a subharmonic function takes its greatest value on the contour*. Moreover, it cannot have a maximum, in the neighbourhood of which it is non-constant, at an interior point. Similarly, *a superharmonic function takes its minimum value on the contour.*

216. Auxiliary propositions. Let us prove some propositions concerning sub- and superharmonic functions which will be needed for solving Dirichlet's problem. We shall write $\overline{B}$ as usual for the bounded domain B together with its contour, i.e. for the closed domain.

THEOREM I. *Let $f_k(M)$ $(k = 1, \ldots, m)$ be functions continuous in $\overline{B}$ and subharmonic inside B. The function $\varphi(M)$, equal to the maximum of the values of $f_k(M)$ $(k = 1, \ldots, m)$ at every point of $\overline{B}$:*

$$\varphi(M) = \max\left[f_1(M), \ldots, f_m(M)\right], \tag{175$_1$}$$

is continuous in $\overline{B}$ and subharmonic inside B.

THEOREM I′. *Similarly, if the $f_k(M)$ are superharmonic and*

$$\psi(M) = \min\left[f_1(M), \ldots, f_m(M)\right], \tag{175$_2$}$$

$\psi(M)$ is also superharmonic.

The continuity of $\varphi(M)$ in $\overline{B}$ follows at once from the continuity of the $f_k(M)$. Let (x_0, y_0) be a point inside B and let $\varphi(x_0, y_0)$ be equal to $f_1(x_0, y_0)$ say. We have, since $f_1(x, y)$ is subharmonic:

$$\varphi(x_0, y_0) = f_1(x_0, y_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f_1(x_0 + \varrho\cos\varphi, y_0 + \varrho\sin\varphi)\,\mathrm{d}\varphi.$$

But, by (175_1), $\varphi(M) \geqslant f_1(M)$ along the circle over which the integration is performed, so that certainly

$$\varphi(x_0, y_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \varphi(x_0 + \varrho\cos\varphi, y_0 + \varrho\sin\varphi)\,\mathrm{d}\varphi,$$

which shows that $\varphi(M)$ is subharmonic.

THEOREM II. *Let $f(M)$ be subharmonic inside B and continuous in $\overline{B}$; let K be a circle contained in B, and $u_K(M)$ the function harmonic inside K whose values on the circumference of K coincide with the values of $f(M)$. Then*

$$f(M) \leqslant u_K(M) \quad (\textit{in } K). \tag{176$_1$}$$

THEOREM II′. *Similarly, if $f(M)$ is superharmonic, then*

$$f(M) \geqslant u_K(M) \quad (\textit{in } K). \tag{176$_2$}$$

The expression $f - u_K = f + (-u_K)$ is the sum of a subharmonic function $f(M)$ and a harmonic (i.e. also subharmonic) function $(-u_K)$. Thus $f - u_K$ is subharmonic inside K and vanishes on the contour.

Hence, by what was said in the previous section, $f - u_K \leqslant 0$ inside K, which leads to (176_1).

THEOREM III. *If in the conditions of Theorem* II, *we replace the values of $f(M)$ in the circle K by the values of $u_K(M)$ and denote the new function by $f_K(M)$, this function is continuous in $\overline{B}$ and subharmonic inside B.*

THEOREM III′. *The same construction for a superharmonic function yields a superharmonic function $f_K(M)$.*

Outside K the function f_K coincides with f, and condition (172_1) is obviously satisfied at every point outside K for sufficiently small δ. Inside K the function f_K is harmonic, and (172_1) is fulfilled with the sign of equality. It remains to verify (172_1) at points of the circumference of K. Let (x_0, y_0) be a point which is assumed to lie inside B if the circumference has points in common with the contour of B. We have:

$$f_K(x_0, y_0) = f(x_0, y_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(x_0 + \varrho\cos\varphi, y_0 + \varrho\sin\varphi)\,d\varphi \quad (\varrho < \delta).$$

Inside K, by Theorem II, $f_K \geqslant f$, whilst outside K, $f_K = f$, so that certainly

$$f_K(x_0, y_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f_K(x_0 + \varrho\cos\varphi, y_0 + \varrho\sin\varphi)\,d\varphi,$$

which is what we had to prove.

217. The method of lower and upper functions. We now turn to a new method of solving Dirichlet's problem. Let B be a bounded domain on the plane and l its contour, about which we make no assumptions at present. Let $\omega(N) = \omega(x, y)$ be a function defined on l; we assume for the present in regard to this function merely that it is bounded, i.e. that there exist two numbers a and b such that

$$a \leqslant \omega(N) \leqslant b. \tag{177}$$

We describe as a *lower function* any function $\varphi(M)$ which is continuous in the closed domain, subharmonic inside the domain, and satisfies $\varphi(N) \leqslant \omega(N)$ on the contour. Similarly, an *upper function* $\psi(M)$ is one which is superharmonic inside the domain and which satisfies $\psi(N) \geqslant \omega(N)$ on the contour.

Obviously an infinite set exists of each type of function. For instance, every constant which does not exceed a is a lower function. Let φ be a lower and ψ an upper function. The expression $\chi = \varphi - \psi = \varphi + (-\psi)$ is the sum of two subharmonic functions and is therefore subharmonic, and $\chi \leqslant 0$ on l. Hence it follows that $\chi \leqslant 0$ in $\overline{B}$, i.e. $\varphi \leqslant \psi$ in $\overline{B}$. In other words, every lower function is not greater than every upper function in B. It follows at once from Theorems I and I′ that, *if* $f_1(M), f_2(M), \ldots, f_m(M)$ *are lower functions,* $\varphi(M)$, *defined by* (175_1), *is also a lower function,* and similarly for upper functions and (175_2). Similarly, it follows from Theorems III and III′ that, *if* $f(M)$ *is a lower function,* $f_K(M)$ *is a lower function,* and likewise, *if* $f(M)$ *is an upper function,* $f_K(M)$ *is an upper function.*

It is quite obvious that lower functions are bounded from above by some number, and upper functions bounded from below. For instance, the number b, appearing in (177), is an upper function, and we have for any lower function: $\varphi(M) \leqslant b$. Similarly for any upper function $\psi(M) \geqslant a$. Hence, *the set of values of all possible upper functions* $\psi(M)$ *at any fixed point* M *inside* B *has a strict lower bound* [I, 42], which we denote by $u(M)$. This will be a function defined inside B. It follows from $\psi(M) \geqslant a$ that $u(M) \geqslant a$, and since the constant b is an upper function, we have $u(M) \leqslant b$, i.e. *the constructed function* $u(M)$ *satisfies the condition* $a \leqslant u(M) \leqslant b$. By the definition of strict lower bound, there exists for every interior point M_0 of B a sequence $\psi_n(M)$ of upper functions such that $\psi_n(M_0) \to u(M_0)$ as $n \to \infty$. If an upper function $\psi(M)$ exists such that $\psi(M_0) = u(M_0)$, we can take for instance $\psi_n(M) = \psi(M)$ for any subscript n. The sequence $\psi_n(M)$ may be different for different points M_0. We now prove a theorem.

THEOREM. *$u(M)$ is a harmonic function inside B.*

In future, we shall write superscripts in brackets instead of subscripts for the functions.

Let us prove a preliminary lemma.

LEMMA. *If P is an arbitrary fixed point inside B, there exists a monotonic sequence of upper functions:*

$$\varphi^{(1)}(M) \geqslant \varphi^{(2)}(M) \geqslant \ldots \qquad (M \text{ in } \overline{B}) \tag{178}$$

such that $\varphi^{(n)}(P) \to u(P)$.

As we saw above, a sequence $\psi_n(M)$ of upper functions exists such that $\psi_n(P) \to u(P)$. We put:

$$\varphi^{(n)}(M) = \min\left[\psi_1(M), \psi_2(M), \ldots, \psi_n(M)\right]. \tag{179}$$

We saw above that the functions $\varphi^{(n)}(M)$ are continuous upper functions. As n increases, the number of functions $\psi_S(M)$ from which the minimum is composed is increased, so that the $\varphi^{(n)}(M)$ satisfy condition (178). Since $u(M)$ is the strict lower bound of the upper functions, it follows, using (179), that $u(P) \leqslant \varphi^{(n)}(P) \leqslant \psi_n(P)$, and hence it follows, since $\psi_n(P) \to u(P)$, that $\varphi^{(n)}(P) \to u(P)$. This proves the lemma.

Note. Let K be any circle with centre P lying inside B. We construct functions $\varphi^{(n)}(M)$ as indicated in Theorem III′. Since (178) holds on the circumference of K, similar inequalities hold throughout the closed circular domain K. Outside K, $\varphi_K^{(n)}(M)$ coincides with $\varphi^{(n)}(M)$, so that (178) is also satisfied by

$$\varphi_K^{(1)}(M) \geqslant \varphi_K^{(2)}(M) \geqslant \dots \qquad (M \text{ in } \bar{B}).$$

Furthermore, $u(M) \leqslant \varphi_K^{(n)}(M) \leqslant \varphi^{(n)}(M)$, and since $\varphi^{(n)}(P) \to u(P)$, we have $\varphi_K^{(n)}(P) \to u(P)$. We can therefore assume that *the functions $\varphi^{(n)}(M)$ featured in the lemma are harmonic inside any fixed circle with centre P that belongs to B.*

We pass on to the proof of the theorem. It is sufficient to show that $u(M)$ is harmonic inside any circle K contained in B. Let P be the centre of this circle. In accordance with the lemma and the remark on it, we construct functions $\varphi^{(n)}(M)$, harmonic inside K. These functions have the limit $u(P)$ at the point P. By Carnac's theorem, these functions tend to a harmonic function at all points inside K:

$$\varphi^{(n)}(M) \to v(M) \qquad (M \text{ inside } K),$$

the convergence being uniform in any closed circle K' with centre P that lies inside K. Let us show that $v(M) = u(M)$ inside K. This will prove our theorem. We use *reductio ad absurdum.* Let $v(P_1) \neq u(P_1)$ at some interior point P_1 of K. Since $\varphi^{(n)}(P_1) \to v(P_1)$, and $u(P_1)$ is the strict lower bound of values of the upper functions at P_1, we must have $v(P_1) > u(P_1)$. Hence an upper function $w(M)$ must exist such that $w(P_1) > v(P_1)$. Let K' be a circle with centre P such that P_1 lies on its circumference. We form the upper functions:

$$\varrho^{(n)}(M) = \min\left[w(M),\ \varphi^{(n)}(M)\right] \quad \text{and} \quad \varrho_{K'}^{(n)}(M),$$

where

$$\varrho_{K'}^{(n)}(M) \leqslant \varrho^{(n)}(M).$$

Since $\varphi^{(n)}(M)$ converges to $v(M)$ uniformly in the closed circle K', we can say that $\varrho^{(n)}(M)$ is also uniformly convergent to the same limit function:

$$\varrho(M) = \min\left[w(M),\ v(M)\right].$$

Hence the convergence is uniform on the circumference of K' and we can say that the functions $\varrho_K^{(n)}(M)$, harmonic inside K', are uniformly convergent in the closed circle K' to some harmonic function $\varrho_{K'}(M)$. Since $w(P_1) < v(P_1)$, we have $\varrho(P_1) < v(P_1)$, and in general we have $\varrho(M) \leqslant v(M)$ at every point of the circumference of K'. Therefore, by the mean value theorem for harmonic functions, $\varrho_{K'}(P) < v(P)$. But $v(P) = u(P)$, so that $\varrho_{K'}(P) < u(P)$. At P, the function $\varrho_{K'}(M)$ is the limit of the upper functions $\varrho_K^{(n)}(M)$, and the inequality $\varrho_{K'}(P) < u(P)$ contradicts the fact that $u(P)$ is the strict lower bound of values of the upper functions at P. We have thus proved the theorem that $u(M)$ is harmonic inside B.

The proof is precisely the same in the three-dimensional case. Hence, for any bounded function $\omega(N)$ defined on the boundary l, we can construct by the method indicated a function $u(M)$ which is harmonic inside B. Instead of forming the strict lower bound $u(M)$ of the upper functions, we could have formed the strict lower bound $u_0(M)$ of the lower functions. If $\omega(N)$ is continuous on the boundary l, it can be shown that $u_0(M)$ coincides with $u(M)$. In what follows, we shall always speak of the strict lower bound of the upper functions.

As already mentioned, the whole of the construction can be carried over without change to the three-dimensional case. The function $u(M)$ is termed the generalized solution of the Dirichlet problem with boundary values $\omega(N)$. The meaning of this will be explained in the next section.

Note. Given continuity of the function $\omega(N)$ on the boundary l, the above generalized solution of Dirichlet's problem $u(M)$ can be formed by a further method, which will now be described. We continue the function $\omega(N)$ on to the entire plane, whilst preserving its continuity. Suppose further that B_n $(n = 1, 2, \ldots)$ is a sequence of domains which, together with their boundaries l_n, lie inside B and tend to B, so that every point M inside B lies inside all the B_n as from a certain n. The B_n can be constructed, say, from a finite number of circles. Suppose we know the solution of Dirichlet's problem for the B_n with continuous values on l_n.

Let $u_n(M)$ be the solution of Dirichlet's problem for B_n, the boundary values on l_n being given as the continuation of $\omega(N)$ mentioned above

It can be shown that, as n tends to infinity the functions $u_n(M)$ tend to the generalized solution $u(M)$ constructed above for Dirichlet's problem, the convergence being uniform in every closed domain contained in B. We thus find that the limit $u_n(M)$ does not depend either on the method of continuation of $\omega(N)$ or on the choice of domains B_n. Only the properties mentioned above for the domains are important. The proof of these statements can be found in an article by M. V. Keldysh *(Uspekhi matematicheskikh nauk,* t. VIII).

218. Investigation of boundary values. We have so far made no assumptions regarding the boundary of the domain B. We now impose a condition, the statement of which will refer to a fixed point N_0 of the boundary of B. The set of boundary points of B will be denoted by l.

CONDITION I. *There exists a function $w(M)$, continuous in $\bar{B}$ and superharmonic inside B, such that $w(N_0) = 0$ and $w(M) > 0$ at the remaining points of B.* We now prove the following theorem:

THEOREM. *If this condition is satisfied and the boundary function $\omega(N)$ is continuous at N_0, then $u(M)$ tends to $\omega(N_0)$ as M approaches N_0 from inside the domain.*

We write β_η for the set of the points of $\bar{B}$ whose distance from N_0 does not exceed $\eta > 0$. Let ε be any given positive number. Since $\omega(N)$ is continuous at N_0, there exists a positive η such that, for all points belonging to β_η of the boundary of B, we have

$$\omega(N_0) - \varepsilon \leqslant \omega(N) \leqslant \omega(N_0) + \varepsilon \qquad (N \text{ on } l \text{ and in } \beta_\eta). \quad (180)$$

We construct a function continuous in $\bar{B}$ and subharmonic inside B:

$$\varphi_1(M) = \omega(N_0) - \varepsilon - Cw(M), \quad (181)$$

where C is a positive constant which we shall now choose. By (180) and since $w(M) \geqslant 0$, we have: $\varphi_1(N) \leqslant \omega(N)$ at points of l belonging to β_η. We choose C so large that, outside β_η, we have the same inequality at points of l, i.e.:

$$\omega(N_0) - \varepsilon - Cw(N) \leqslant \omega(N) \qquad (N \text{ on } l \text{ and outside } \beta_\eta). \quad (182)$$

At all points of $\bar{B}$ whose distance from N_0 is not less than η, the function $w(M)$ attains a least positive value, which we write as m_η. This follows at once from the fact that the points in question form

a closed set, and the function $w(M)$ is continuous and positive on this set [II, 89]. To satisfy inequality (182), it is sufficient to take

$$C \geqslant \frac{\omega(N_0) - \varepsilon - a}{m_\eta},$$

where a is the number appearing in inequality (177). With this choice of C, function (181) will be a lower function. Similarly, with sufficiently large C, the function

$$\psi_1(M) = \omega(N_0) + \varepsilon + Cw(M) \tag{183}$$

will be an upper function. It follows from $w(N_0) = 0$ that

$$\varphi_1(N_0) = \omega(N_0) - \varepsilon,$$

and, by virtue of the continuity of $\varphi_1(M)$ in $\overline{B}$, a small positive δ_1 can be found such that, in β_{δ_1},

$$\varphi_1(M) \geqslant \omega(N_0) - 2\varepsilon \qquad (M \text{ in } \beta_{\delta_1}).$$

Let $\psi(M)$ be any upper function. We have $\psi(M) \geqslant \varphi_1(M)$ for all points M of $\overline{B}$, so that it follows from the last inequality that:

$$\psi(M) \geqslant \omega(N_0) - 2\varepsilon \qquad (M \text{ in } \beta_{\delta_1}).$$

The strict lower bound of $\psi(M)$ must also satisfy this inequality, i.e.

$$u(M) \geqslant \omega(N_0) - 2\varepsilon \qquad (M \text{ inside } B \text{ and in } \beta_{\delta_1}). \tag{184}$$

Similarly, it follows from (183) that

$$\psi_1(N_0) = \omega(N_0) + \varepsilon,$$

i.e., since $\psi_1(M)$ is continuous, there exists a small positive δ_2 such that, in β_{δ_2}:

$$\psi_1(M) \leqslant \omega(N_0) + 2\varepsilon \qquad (M \text{ in } \beta_{\delta_2})$$

and certainly

$$u(M) \leqslant \omega(N_0) + 2\varepsilon \qquad (M \text{ inside } B \text{ and in } \beta_{\delta_2}). \tag{185}$$

Let δ be the lesser of the numbers δ_1 and δ_2. We have, by (184) and (185):

$$\omega(N_0) - 2\varepsilon \leqslant u(M) \leqslant \omega(N_0) + 2\varepsilon \qquad (M \text{ inside } B \text{ and in } \beta_\delta). \tag{186}$$

It now follows, since ε is arbitrary, that $u(M)$ tends to $\omega(N_0)$ as $M \to N_0$ from inside the domain, and the theorem is proved. The proof is also suitable for the two- and three-dimensional case. If $\omega(N)$

is continuous at every point of the boundary and Condition I is fulfilled at every point, $u(M)$ is continuous in the closed domain $\bar{B}$ and takes the values $\omega(N)$ at boundary points.

DEFINITION. *If, given any choice of function $\omega(N)$ continuous on l, the function $u(M)$ tends to $\omega(N_0)$ as $M \to N_0$, the point N_0 is called a regular point of the boundary. Points not possessing this property are described as irregular points of the boundary.*

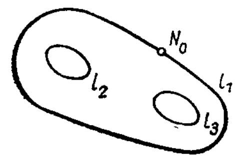

FIG. 18

It follows from the theorem proved above that Condition I is sufficient for N_0 to be a regular point.

We now indicate for the three-dimensional case a simple sufficient condition of a geometrical nature for a boundary point to be regular. Suppose that N_0 on the boundary has the following property: there exists a sphere which contains no points of $\bar{B}$ except for N_0. Let M_1 be the centre of this sphere and R its radius. We write r for the distance $|M_1M|$, and form the function:

$$w(M) = \frac{1}{R} - \frac{1}{r}.$$

This function obviously satisfies all the requirements of Condition I, being harmonic inside B.

We now take the plane case, and let the boundary of B consist of a finite number of simple closed curves, having the equations $x = x(t)$ $y = y(t)$, where $x(t)$ and $y(t)$ are continuous periodic functions of the parameter t (Fig. 18). Suppose first that the point N_0 is on the exterior contour l_1 (Fig. 18). We take it as the origin $z = 0$ and choose the scale so that $\bar{B}$ lies inside the circle $|z| < 1$. We form the function

$$F(z) = -\frac{1}{\log z}.$$

When z moves in B, it cannot make a circuit round the origin, and $F(z)$ is single-valued in $\bar{B}$, regular inside B and continuous in $\bar{B}$, whilst $F(0) = 0$.

On putting $z = \varrho e^{i\varphi}$, we obtain for the real part of $F(z)$:

$$w(z) = -\frac{\log \varrho}{(\log \varrho)^2 + \varphi^2},$$

where $\log \varrho < 0$. This harmonic function satisfies all the conditions indicated above.

In particular, outside β_ε we have

$$w(z) > -\frac{\log R}{(\log \varepsilon)^2 + \varphi_0^2},$$

where φ_0 is the greatest value of φ in $\overline{B}$ and R is the greatest distance from the origin to points of $\overline{B}$.

Suppose now that N_0 is on the interior contour l_2. We choose a point a inside l_2 and carry out the conformal transformation:

$$z' = \frac{1}{z-a}.$$

The contour l_2 becomes an exterior contour, and we can form a function $w(M)$ by the above method for the point N_0. On returning to the former variable z, we obtain the required function. Hence, if $\omega(N)$ is continuous at all points of the contour l considered, $u(M)$ is continuous up to the contour and is equal to $\omega(N)$ on the contour.

Suppose now that N_0 is a point of discontinuity of $\omega(N)$, where $\omega(N)$ has limits, though these are different (a discontinuity of the first kind), when N tends to N_0 along the contour from either sides. Let these limits be $\omega_1(N_0)$ and $\omega_2(N_0)$, and let $\omega_1(N_0) < \omega_2(N_0)$. On arguing precisely as above, instead of (184) we obtain:

$$u(M) \geqslant \omega_1(N_0) - 2\varepsilon$$

and instead of (185):

$$u(M) \leqslant \omega_2(N_0) + 2\varepsilon.$$

As M approaches N_0 from inside B, the function $u(M)$ can have a variety of different limits. But, in view of the above inequalities and the fact that ε is arbitrary, we have for any of these limits u_0:

$$\omega_1(N_0) \leqslant u_0 \leqslant \omega_2(N_0). \tag{187}$$

If $\omega(N)$ is a bounded function, i.e. satisfies condition (177), $u(M)$ also satisfies the condition, as we have seen. Thus $u(M)$ is a bounded harmonic function, taking the boundary values $\omega(N)$ at all the points of continuity of the function.

Let us return to the three-dimensional case. A fairly simple closed surface can be constructed having irregular points. This fact was discovered by Lebesgue, and then, independently of him, by Uryson. A more detailed treatment of the subject of boundary values can be found in an article by Keldysh *(Uspekhi matematicheskikh nauk,* VIII).

Let us mention a further example in the case of a plane. Let B be the domain consisting of a circle with centre at the origin, but excluding the centre. The set l of boundary points consists of the circumference and the centre. Let $\omega(N) = 0$ on the circumference and $\omega(N) = 1$ at the centre. Such an $\omega(N)$ is continuous on l. The harmonic function $u(M)$ obviously tends to zero as M approaches points of the circumference. We show that $u(M)$ cannot tend to unity on approaching the centre. If this were so, $u(M)$ would be harmonic inside the entire circle, if we take its value at the centre equal to unity [203]. But this contradicts the theorem on the mean value of a harmonic function at the centre of the circle. The origin is therefore an irregular point of the boundary.

It may easily be shown that $u(M) = 0$ in this present case. For, $u(M)$ is bounded, so that it has a limit as M approaches the centre [203], and if this limit is taken equal to the value of $u(M)$ at the centre, $u(M)$ will be harmonic everywhere inside the circle [203] and will vanish on the circumference, i.e. $u(M) \equiv 0$.

We remark further that a condition for regularity different to I can be laid down, in which we are concerned only with a neighbourhood of the point N_0; this condition can be shown to be equivalent to I.

CONDITION II. *For some neighbourhood β_η of the point N_0 there exists a function $w_\eta(M)$, continuous in β_η up to the boundary, superharmonic inside β_η and such that $w_\eta(N_0) = 0$ and $w_\eta(M) > 0$ at the remaining points of β_η.*

It can be shown that, in the three-dimensional case, N_0 satisfies Condition II if it is the vertex of a circular cone, all the points of which, sufficiently close to N_0, lie outside $\bar{B}$ (except for N_0). Hence such points are regular. (See I. G. Petrovskii, *Lectures on Partial Differential Equations* (Lektsii ob uravneniyakh s chastnymi proizvodnymi)).

We shall always assume in future, unless there are special provisos, that the contours or surfaces bounding domains are such that all their points are regular. This is the case e.g. for Lyapunov surfaces. We have constructed the solution of Dirichlet's problem for these with the aid of potential theory and integral equations.

If the continuous function $\omega(N)$ is given on the boundary, and all the points of the boundary are regular, the harmonic function $u(M)$ constructed is continuous up to the boundary and takes the values $\omega(N)$ on the boundary. We know that only one such function can exist. If there are irregular points on the boundary, the harmonic function $u(M)$ is bounded inside the domain and takes the values $\omega(N)$ at all regular points of the boundary. It can be shown that only

one function can exist with these properties. The proof of this important assertion can be found in the above-mentioned article by M. V. Keldysh *(Uspekhi matematicheskikh nauk,* t. VIII, 1941).

219. Laplace's equation in n-dimensional space. We have so far considered Laplace's equation on a plane and in three-dimensional space.

The results are readily extended to the case of n-dimensional space, where the equation has the form:

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0 .$$

We shall give the fundamental results regarding the solutions of this equation. Functions having continuous derivatives up to the second order and satisfying this equation are described as harmonic. The basic singular solution has the form:

$$\frac{C}{r^{n-2}} \quad (n > 2) ,$$

where the constant C is chosen equal to $1/(n-2)\omega_n$, where ω_n is the surface area of the unit sphere in n-dimensional space, i.e. the basic singular solution has the form

$$\varphi_0 (r) = \frac{1}{(n-2)\, \omega_n \, r^{n-2}} \quad (n > 2).$$

The volume of an n-dimensional sphere of radius r is given by [II, 99]:

$$v_n = \frac{(2\pi)^{\frac{n}{2}}}{n(n-2)\dots 2} r^n \qquad \text{for } n \text{ even},$$

$$v_n = \frac{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}}}{n(n-2)\dots 1} r^n \qquad \text{for } n \text{ odd},$$

and these may easily be seen to be expressible in the unified form:

$$v_n = \frac{2\,(\sqrt{\pi})^n}{n\Gamma\left(\frac{n}{2}\right)} r^n,$$

whence, on differentiating with respect to r and putting $r = 1$:

$$\omega_n = \frac{2\,(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)} .$$

We have, for a function harmonic in a domain D with surface S [II, 194]:

$$u(M) = \int_S \left(\varphi_0(r)\frac{\partial u}{\partial n} - u\frac{\partial \varphi_0(r)}{\partial n}\right) dS,$$

where, as in future, only one sign of integration is written. Here, r is the distance of the variable point of integration from the surface S to M. The fundamental properties of harmonic functions are valid, including a mean value theorem for the harmonic function at the centre of a sphere, and the uniqueness of the solution of Dirichlet's problem.

The formula for the solution of Dirichlet's problem for a sphere of radius R is:

$$u(M) = \frac{R^n(R^2-\varrho^2)}{\omega_n}\int_S f(N)\frac{dS}{(R^2+\varrho^2-2R\varrho\cos\theta)^{\frac{n}{2}}}, \qquad (188)$$

where ϱ is the distance from the centre O of the sphere to M, N is the variable point on the sphere and θ is the angle between ON and OM.

The method of upper and lower functions for the solution of Dirichlet's problem can be carried over without change to n-dimensional space; here, the condition still holds that the points of the surface be regular, as we proved above.

220. Green's function for the Laplace operator. We can define Green's function for a partial differential equation in the same way as for an ordinary differential equation. We start with the definition of Green's function for Laplace's equation with one of the following homogeneous boundary conditions:

$$u|_S = 0, \qquad (189)$$

$$\frac{\partial u}{\partial n} + p(N)u\Big|_S = 0 \qquad (p(N) > 0), \qquad (190)$$

in the three-dimensional case. We can form Green's function both for a finite domain D_i lying inside S, and for an infinite domain D_e outside S. We start with the finite domain D_i. Green's function $G(P; Q)$ must be a function of a pair of points $(P; Q)$; as a function of P, it must have continuous derivatives up to the second order everywhere inside D_i except at Q, and must satisfy Laplace's equation and the boundary condition on the boundary. Further, as a function of P, $G(P; Q)$ must have a singularity at Q corresponding to a finite charge

(or mass), concentrated at the point Q. On taking into account the factor 4π, appearing in the formula [II, 201]:

$$\Delta\left[\iiint_{D_i} \frac{\mu(M)}{r}\,\mathrm{d}\tau_M\right] = -4\pi\mu(M_0) \qquad (r = |M_0M|), \tag{191}$$

we define Green's function as follows for conditions (189) or (190):

DEFINITION. *Green's function of Laplace's operator, corresponding to boundary condition* (189) *or* (190), *is defined as the function* $G(P;Q)$ *satisfying the following conditions* : (1) *it is harmonic inside* D_i *except at* Q; (2) *it satisfies the boundary condition* (189) *or* (190); (3) *it can be written as*

$$G(P;\ Q) = G(x, y, z;\ \xi, \eta, \zeta) = \frac{1}{4\pi r} + g(P;\ Q), \tag{192}$$

where $r = |PQ|$ *and* $g(P;Q)$ *is harmonic everywhere inside* D_i.

The construction of Green's function amounts to finding its regular part $g(P;Q)$. In the case of boundary condition (189) $g(P;Q)$, harmonic inside D_i, must have the boundary values on S:

$$g(N;\ Q)\Big|_S = -\frac{1}{4\pi r} \qquad (r = |NQ|). \tag{193}$$

In the case of (190), the boundary conditions for $g(P;Q)$ become:

$$\left(\frac{\partial g(N;\ Q)}{\partial n}\right)_i + p(N)\,g(N;\ Q)\Big|_S = -\frac{1}{4\pi}\left[\frac{\partial \frac{1}{r}}{\partial n} + \frac{p(N)}{r}\right]_S. \tag{194}$$

Hence, the construction of Green's function amounts to the solution of the first or third boundary problem for Laplace's equation, and we can assume that the existence of Green's function is established if S is a Lyapunov surface.

For the exterior domain D_e, we add to the definition of Green's function the condition that it be regular at infinity, i.e. that $G(P;Q)$, given any fixed Q at a finite distance, must tend to zero as P tends to infinity.

Let D_i be any bounded domain and Γ the set of its boundary points. A generalized solution of Dirichlet's problem with boundary condition (193) exists in D_i. Formula (192) here defines the generalized Green function for domain D_i with boundary condition (189). If N_0 is a regular point of the boundary, $G(P;Q) \to 0$ as $P \to N_0$. The converse can be proved: if $G(P;Q) \to 0$ as $P \to N_0$, then N_0 is a regular point of the boundary.

In the case of a plane the definition of Green's function is very similar, except that we have to replace (192) by the formula:

$$G(P;\ Q)=\frac{1}{2\pi}\log\frac{1}{r}+g(P;\ Q). \tag{195}$$

It follows from (192) and (195) that Green's function becomes infinite when P and Q coincide, Green's function being positive for P sufficiently close to Q. The point Q is called the pole of Green's function. In future, we shall consider Green's function only with boundary condition (189). We show that $G(P; Q)$ is a continuous function of the points P and Q if these points do not coincide. On taking (192) into account, we can say that a proof of the continuity of $G(P; Q)$ can be reduced to a proof of the continuity of $g(P; Q)$. Let us consider the difference $g(P'; Q') - g(P''; Q'')$; on adding and subtracting $g(P'; Q'')$, we obtain

$$\begin{aligned}g(P';\ Q')-g(P'';\ Q'')\,|\leqslant\\ \leqslant|g(P';\ Q')-g(P';\ Q'')|+|g(P';\ Q'')-g(P'';\ Q'')|.\end{aligned}$$

The difference $g(P'; Q'') - g(P''; Q'')$ is the difference of the values of $g(P; Q'')$ at points P' and P'', and obviously tends to zero as $P'' \to P'$. The difference $g(P'; Q') - g(P'; Q'')$ is the value at P' of the harmonic function $g(P; Q') - g(P; Q'')$ with boundary values $(1/r' - 1/r'')/4\pi$ on S, where r' and r'' are the distances of the variable point N on S from Q' and Q''.

If Q'' is sufficiently close to Q', the absolute value of the difference $(1/r' - 1/r'')$ is as small as desired as N varies on S. But the harmonic function $g(P; Q') - g(P; Q'')$ takes its least and greatest values on the boundary S, and we can assert that $g(P'; Q') - g(P'; Q'') \to 0$ as $Q'' \to Q'$. This proves the continuity of $g(P; Q)$, and hence of $G(P; Q)$.

The function $G(P; Q)$ is positive in the neighbourhood of the point Q and vanishes on S, so that *it is positive inside the domain D_i*. The same argument is suitable for D_e in three-dimensional space. Let us bring in a further simple inequality for $G(P; Q)$. The function $g(P; Q)$ has negative boundary values (193) on S. Hence $g(P; Q) < 0$ in the closed domain D_i, so that

$$0<G(P;\ Q)<\frac{1}{4\pi r}\quad\text{inside}\quad D_i\qquad(r=|PQ|). \tag{196}$$

The same inequality holds for D_e.

Let us now carry out the arguments for the plane case. Let d be the diameter of the finite domain B on the plane, i.e. the greatest distance between any two points of the closed domain $\bar{B}$. The harmonic function $g(P; Q) + (1/2\pi) \log (1/d)$ takes on the boundary l the values $(1/2\pi) \log (r/d)$, which are negative for any position of the pole Q inside B. We therefore have:
$g(P; Q) + (1/2\pi) \log (1/d) < 0$, i.e. $g(P; Q) < -(1/2\pi) \log (1/d)$ inside B. This gives us:

$$G(P;\ Q) < \frac{1}{2\pi} \log \frac{1}{r} - \frac{1}{2\pi} \log \frac{1}{d},$$

i.e. we have an inequality of the form:

$$0 < G(P;\ Q) < a \log \frac{1}{r} + b \qquad (\text{inside } B), \tag{197}$$

where a and b are constants. Inequalities (196) and (197) give us estimates for Green's function that depend on the distance r between points P and Q.

221. Properties of Green's function. Let us consider Green's function in D_i, writing r, as above, for the distance of the variable points of space from the point Q. We define the function:

$$v(P) = \begin{cases} g(P;\ Q) & \text{inside } S \\ -\dfrac{1}{4\pi r} & \text{outside } S. \end{cases} \tag{198}$$

It is harmonic both inside D_i and inside D_e, and vanishes at infinity. It has derivatives of any order in D_e, continuous up to S. We can regard $v(P)$ in D_e as the solution of Neumann's problem with the boundary values:

$$f(N) = -\frac{1}{4\pi} \frac{\partial}{\partial n} \left(\frac{1}{r}\right) \tag{199}$$

and we can therefore express $v(P)$ in D_e as the potential of a simple layer with continuous density:

$$v(P) = \iint_S \frac{\mu(N)}{r'} \mathrm{d}S \qquad (r' = |NP|). \tag{200}$$

The values of this potential on S are equal to $(-1/4\pi r')$, where $r' = |NQ|$, i.e. the same as for $g(P;Q)$. It is clear from this that formula

(200) for $v(P)$, defined by (198), holds throughout space, i.e.

$$g(P;\ Q) = \iint\limits_S \frac{\mu(N)}{r'}\, \mathrm{d}S \qquad (P \text{ in } D_i), \tag{201}$$

so that $g(P; Q)$ has regular normal derivatives on S in D_i. The same can obviously be said for $G(P; Q)$.

We remark, in connection with boundary condition (199), that the function $1/r$ is defined, given any position of the point Q inside D_i, not only on S, but also in the space close to S, and has derivatives of all orders. The right-hand side of (199) obviously satisfies on S the Lipschitz condition:

$$|f(N_2) - f(N_1)| \leqslant a\, r_{1,2} \qquad (r_{1,2} = |N_1 N_2|),$$

and we can say that $\mu(N)$ also satisfies the Lipschitz condition [196], so that $G(P; Q)$ has continuous first order derivatives as far as S [198].

We now prove the symmetry of Green's function:

$$G(P;\ Q) = G(Q;\ P). \tag{202}$$

It may be remarked here, by what has been proved above, that $G(P; Q)$ has regular normal derivatives on S. Inside D_i, it has continuous derivatives everywhere except at Q. We now apply the formula

$$\iiint\limits_{D_i'} (u\,\Delta v - v\,\Delta u)\ \mathrm{d}\tau = \iint\limits_{S'} \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right) \mathrm{d}S$$

to the functions $u = G(P; Q_1)$ and $v = G(P; Q_2)$, taking as the domain of integration D_i', the domain D_i with the exclusion of two spheres of small radius ε with centres Q_1 and Q_2. We are justified in applying the formula by what has been said above. The triple integral over this domain vanishes, since Green's function satisfies Laplace's equation outside the poles. The integral over S vanishes by virtue of the boundary condition (no matter what), so that we arrive at the equation:

$$\iint\limits_{S_1} \left[G(P;\ Q_1)\,\frac{\partial G(P;\ Q_2)}{\partial n} - G(P;\ Q_2)\,\frac{\partial G(P;\ Q_1)}{\partial n}\right] \mathrm{d}S + \iint\limits_{S_2} [\quad]\, \mathrm{d}S = 0,$$

where S_1 and S_2 are the surfaces of the above-mentioned spheres. The function $G(P; Q_1)$ has no singularities at Q_2, whilst $G(P; Q_2)$ tends to infinity of order $1/r$ at Q_2. Observing that the product of $1/\varepsilon$ and the surface area of the sphere $4\pi\varepsilon^2$ tends to zero as $\varepsilon \to 0$, we see that the only terms in the last formula which do not tend to zero as $\varepsilon \to 0$ are those which contain the normal derivative of $G(P; Q_1)$ in the

neighbourhood of the point where $G = +\infty$. There are two of these terms, and we obtain by taking their sum:

$$\frac{1}{4\pi}\iint\limits_{S_2} G(P;\ Q_1)\frac{\partial\frac{1}{r_2}}{\partial n}\,\mathrm{d}S - \frac{1}{4\pi}\iint\limits_{S_1} G(P;\ Q_2)\frac{\partial\frac{1}{r_1}}{\partial n}\,\mathrm{d}S + \eta = 0,$$

where $\eta \to 0$ as $\varepsilon \to 0$, r_1 is the distance of the variable point P from Q_1 and r_2 is the distance of the variable point P from Q_2. We have the outward normal in Green's formula, i.e. the normal must be directed inwards to the spheres in the last formulae, i.e. opposite to the radius, and we have:

$$\frac{1}{4\pi\varepsilon^2}\iint\limits_{S_2} G(P;\ Q_1)\,\mathrm{d}S - \frac{1}{4\pi\varepsilon^2}\iint\limits_{S_1} G(P;\ Q_2)\,\mathrm{d}S + \eta = 0.$$

On applying the mean value theorem to the integrals, we can write:

$$G(P_2;\ Q_1) - G(P_1;\ Q_2) + \eta = 0,$$

where P_2 is a point on S_2 and P_1 on S_1. On passing to the limit as $\varepsilon \to 0$, we get:

$$G(Q_2;\ Q_1) = G(Q_1;\ Q_2),$$

which proves that Green's function is symmetrical.

For a sphere, Green's function has the form [II, 198]:

$$G(P;\ Q) = \frac{1}{4\pi}\left(\frac{1}{r} - \frac{R}{\varrho r_1}\right), \tag{203}$$

where ϱ is the distance from Q to the centre, r_1 the distance from P to Q', symmetrical with Q with respect to the sphere, and R is the radius of the sphere. On taking (x, y, z) and (ξ, η, ζ) as the coordinates of P and Q, we can write:

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}; \quad \varrho = \sqrt{\xi^2 + \eta^2 + \zeta^2};$$

$$r_1 = \sqrt{\left(x - \frac{R^2\xi}{\varrho^2}\right)^2 + \left(y - \frac{R^2\eta}{\varrho^2}\right) + \left(z - \frac{R^2\zeta}{\varrho^2}\right)^2}.$$

On differentiating (203), say with respect to x, and observing that

$$\frac{|x-\xi|}{r} < 1 \quad \text{and} \quad \frac{\left|x - \frac{R^2\xi}{\varrho^2}\right|}{r_1} \leqslant 1,$$

we obtain the inequality:

$$\left|\frac{\partial G(P;\ Q)}{\partial x}\right| \leqslant \frac{1}{4\pi}\left(\frac{1}{r^2} + \frac{R}{\varrho r_1^2}\right).$$

Since $r_1 > r$ and $R/\varrho > 1$ for interior points P of the sphere, we get:

$$\left| \frac{\partial G(P;\,Q)}{\partial x} \right| \leqslant \frac{1}{2\pi r^2}. \tag{204}$$

Similar inequalities hold for the other partial derivatives.

Let $u(M)$ be the solution of Dirichlet's interior problem for the domain D_i bounded by the surface S, with boundary values $f(N)$. If we know that $u(M)$ has a regular normal derivative, we can apply (91) in D_i, putting $v = g(P; Q)$. We now obtain [cf. II, 198]:

$$u(Q) = -\iint_S f(N) \frac{\partial G(N;\,Q)}{\partial n}\, \mathrm{d}S_N. \tag{205}$$

Lyapunov proved that this formula gives the solution of Dirichlet's problem with any choice of continuous function $f(N)$ for the boundary values. He also provided the first strict proof of the symmetry of Green's function. These results, together with the results on potential theory that we mentioned earlier, may be found in Lyapunov's *On certain questions regarding Dirichlet's problem* (O nekotorykh voprosakh, svyazannykh s zadachei Dirikhle) (1898), which has already been quoted.

222. Green's function in the case of a plane. The discussion of Green's function for a plane has certain special features as compared with the spatial case. We shall consider Green's function for the bounded domain B_i with contour l and boundary condition (189) on l.

As in [221], we define a function $v(P)$ on the plane:

$$v(P) = \begin{cases} g(P;\,Q) & \text{inside } l \\ -\dfrac{1}{2\pi} \log \dfrac{1}{r} & \text{outside } l. \end{cases} \tag{206}$$

As in [221], we construct the simple layer potential:

$$v_1(P) = \int_l \mu(s) \log \frac{1}{r'}\, \mathrm{d}s, \tag{207}$$

where r' is the distance from P to the variable point N on l, with boundary values of the normal derivative in the domain B_e exterior to l

$$f(N) = -\frac{1}{2\pi} \frac{\partial}{\partial n} \log \frac{1}{r} = -\frac{\cos(r', n)}{2\pi r'}, \tag{208}$$

where r' is the direction QN and n is the direction of the normal to l outwards from the closed contour l. We now form the function, harmonic in B_e:

$$w(P) = \int_l \mu(s) \log \frac{1}{r'} \, ds + \frac{1}{2\pi} \log \frac{1}{r} \qquad \left(\begin{matrix} r = |PQ| \\ Q \text{ inside } l \end{matrix} \right), \tag{209}$$

having a regular normal derivative on l equal to zero. Inside B_e, we draw any closed contour l' enclosing l, and apply Green's formula with $u = w(P)$ and $v = 1$ to the domain bounded by l and l'. We have:

$$\int_l \frac{\partial w(P)}{\partial n} \, ds - \int_{l'} \frac{\partial w(P)}{\partial n} \, ds = 0,$$

where n is in both cases the outward normal with respect to the closed contour. Hence, since $\partial w(P)/\partial n = 0$ on l:

$$\int_{l'} \frac{\partial w(P)}{\partial n} \, ds = 0. \tag{210}$$

But

$$\frac{\partial w(P)}{\partial n} = \int_l \mu(s) \frac{\cos(r', n)}{r'} \, ds + \frac{\cos(r, n)}{2\pi r'},$$

where $\mathbf{r}'$ is the direction $\overline{PN}$. On integrating over l', changing the order of integration [18] and recalling that P and N lie inside l', we obtain by (210):

$$2\pi \int_l \mu(s) \, ds + 1 = 0.$$

We can now rewrite (209) as

$$w(P) = \int_l \mu(s) \log \frac{r}{r'} \, ds \qquad (r = |QP|; \; r' = |NP|). \tag{211}$$

As P moves away to infinity, the ratio r/r' tends uniformly to unity, i.e. given any positive ε, there exists a positive M such that $|1 - r/r'| \leqslant \varepsilon$ for any position of N on l provided $r > M$. Hence function (211), harmonic in B_e, has a regular normal derivative on l equal to zero, and tends to zero as P moves away to infinity. We can apply to such a function the formula

$$\iint_{B_e} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dS = \int_l w \left(\frac{\partial w}{\partial n} \right)_e ds,$$

from which it follows that $w(P) = 0$ in B_e, i.e.

$$\int_l \mu(s) \log \frac{1}{r'} ds = -\frac{1}{2\pi} \log \frac{1}{r} \quad \text{in} \quad B_e.$$

Hence it follows directly, as in [221], that the simple layer potential (207) coincides throughout the plane with the function $w(P)$ defined by equation (206), and we can say that $g(P; Q)$ has a regular normal derivative on l. Further, as in [221], we can assert that $g(P; Q)$ has continuous first order derivatives up to l in B_i. The proof that $G(P; Q)$ is symmetric is precisely the same as in [221]. For a circle of radius R, Green's function has the form:

$$G(P;\ Q) = \frac{1}{2\pi} \log \frac{\varrho r_1}{Rr}, \tag{212}$$

where the notation is the same as in [221]. This leads to the inequalities:

$$\left| \frac{\partial G(P;\ Q)}{\partial x} \right| \leqslant \frac{1}{\pi r}; \quad \left| \frac{\partial G(P;\ Q)}{\partial y} \right| \leqslant \frac{1}{\pi r}. \tag{213}$$

A formula analogous to (205) holds for the solution of Dirichlet's problem in B_i.

Green's function of Laplace's operator for a plane singly-connected domain with boundary condition (189) is closely connected with the function carrying out the conformal transformation of this domain into the circle $|w| \leqslant 1$ [III$_2$, 35]. Let B be a singly-connected domain with contour l and $z_0 = \xi + \eta i$ an interior point of the domain. Further, let $w = f(z)$ be a function conformally transforming B into the unit circle, where $f(z_0) = 0$, i.e. the point $z = z_0$ becomes the centre of the circle.

Since the transformation is one-sheeted, it follows that $f(z)$ has a simple zero at $z = z_0$:

$$f(z) = (z - z_0)\left[a_0 + a_1(z - z_0) + \ldots\right] \quad (a_0 \neq 0). \tag{214}$$

We form the function

$$G(x, y;\ \xi, \eta) = -\frac{1}{2\pi} \log |f(z)|. \tag{215}$$

It may easily be shown that this is in fact Green's function for the domain B with pole (ξ, η). For, $\log |f(z)|$ is the real part of $\log f(z)$, and consequently satisfies Laplace's equation. By (214), the infinite part of the function (215) at the point (ξ, η) will be $(1/2\pi) \log (1/|z - z_0|)$, and finally, the contour l of domain B becomes

the circumference of the unit circle, i.e. $|f(z)| = 1$ on l, whilst function (215) vanishes on this contour.

Let $H(x, y; \xi, \eta)$ denote the harmonic conjugate to function (215). We have:

$$G + iH = -\frac{1}{2\pi}\log f(z), \tag{216}$$

so that we can write $f(z)$ in terms of Green's function and the conjugate to the latter:

$$f(z) = e^{-2\pi(G+Hi)}.$$

The function H is determined up to an added constant, so that we have an arbitrary constant factor of unit modulus in the right-hand side of the last formula; this corresponds to an arbitrary rotation of the unit circle $|w| \leqslant 1$ about the origin.

Suppose that the contour l of the domain B has the following property: the angle $\theta(s)$, formed by the tangent to l and any fixed direction, satisfies, as a function of the arc length s, the Lipschitz condition:

$$|\theta(s_2) - \theta(s_1)| \leqslant b\,|s_2 - s_1|^{\beta}, \tag{217}$$

where b and β are positive constants. It can be shown that the derivative $f'(z)$ is now continuous up to l, and there exist two positive constants m and M such that

$$m \leqslant |f'(z)| \leqslant M. \tag{218}$$

These constants clearly depend on the choice of the point z_0 which becomes the origin on the w plane. We fix z_0 and now construct a more general conformal transformation of B into the circle $|w| \leqslant 1$, on condition that the arbitrary interior point z' of B becomes the origin. For this, we must first carry out the conformal transformation $w = f(z)$, then transform the circle $|w| \leqslant 1$ into itself so that the point $f(z')$ becomes the origin. This latter transformation is a rational fractional transformation, and we finally obtain, on discarding the constant factor with unit modulus:

$$-2\pi G(z; z') = \mathcal{R}\left[\log\frac{f(z) - f(z')}{1 - f(z)\overline{f(z')}}\right],$$

where, as usual, $\mathcal{R}$ denotes the real part, and $G(z; z')$ is Green's function for the domain B with pole at z'. On differentiating with respect to x, where any direction can be taken for x, we obtain:

$$-2\pi\frac{\partial G(z; z')}{\partial x} = \mathcal{R}\left[\frac{f'(z)}{f(z) - f(z')} + \frac{\overline{f(z')}f'(z)}{1 - f(z)\overline{f(z')}}\right] =$$

$$= \mathcal{R}\left\{\frac{f'(z)\left[1 - |f(z')|^2\right]}{[f(z) - f(z')]\left[1 - f(z)\overline{f(z')}\right]}\right\},$$

or, on replacing the real part by the modulus:

$$2\pi\left|\frac{\partial G(z\,;\,z')}{\partial x}\right| \leqslant \frac{|f'(z)|\,|1-|f(z')|^2|}{|f(z)-f(z')|\,|1-f(z)\overline{f(z')}|}\,,$$

which gives us, in view of the fact that $|f(z)| < 1$ and $|f(z')| < 1$:

$$2\pi\left|\frac{\partial G(z\,;\,z')}{\partial x}\right| < \frac{|f'(z)|\,|1-|f(z')|^2|}{|f(z)-f(z')|\,(1-|f(z')|)} < \frac{2\,|f'(z)|}{|f(z)-f(z')|}\,. \qquad (218_1)$$

Let $z = \varphi(w)$ be the inverse of $w = f(z)$. It is defined in the circle $|w| \leqslant 1$. It follows from (218) that $|\varphi'(w)| \leqslant 1/m$, and we obtain:

$$|\varphi(w)-\varphi(w')| = \left|\int_w^{w'} \varphi'(\tau)\,\mathrm{d}\tau\right| \leqslant \frac{1}{m}\,|w-w'|,$$

where the integration can be carried out over a straight segment. This last inequality gives: $|f(z) - f(z')| \geqslant m\,|z - z'|$, and we obtain with the aid of (218_1) and the last inequality:

$$\left|\frac{\partial G(z\,;\,z')}{\partial x}\right| \leqslant \frac{2M}{2\pi m\,|z-z'|} = \frac{M}{\pi m r}\,, \qquad (219)$$

where r is the distance between points z and z'. Therefore, given our assumption regarding the contour l, an inequality is obtained for the derivative of Green's function with respect to any direction that depends only on the distance r.

If the domain B is multiply-connected, and each of the closed contours bounding it satisfies the above condition, an inequality of the form (219) can again be obtained. The above proof of (219), as also the proof for a multiply-connected region, which will be omitted, was communicated to me by Goluzin.

223. Examples. We now turn to some examples of constructing Green's function, and start with the function for the circle $|z| < 1$. We have already had a function transforming this circle into itself, on condition that some point a inside the circle becomes the origin. This function can be written as [III_2, 31]:

$$w = \frac{\mathrm{e}^{i\varphi}}{\overline{a}}\cdot\frac{z-a}{z-a'}\,,$$

where $\overline{a}$ is the complex number conjugate to a, and a' is the point symmetrical to a with respect to the circumference, i.e. $a' = \overline{a}^{-1}$. On writing r_1 and r_2 for the distances of the variable point z' from a and a', we at once obtain the following expression for Green's function for the circle:

$$G(z\,;\,a) = -\frac{1}{2\pi}\log\left|\frac{\mathrm{e}^{i\varphi}}{\overline{a}}\cdot\frac{z-a}{z-a'}\right| = -\frac{1}{2\pi}\log\frac{r_1}{r_2} + \frac{1}{2\pi}\log\sqrt{\xi^2+\eta^2}\,.$$

Now let B be the rectangle with corners $(0, 0)$, $(0, a)$, (a, b), $(0, b)$. Putting $\omega_1 = 2a$ and $\omega_2 = 2bi$, we form Weierstrass' function $\sigma(z; \omega_1, \omega_2)$. We have

seen that the function transforming our rectangle into the unit circle, such that the point $z = \xi + \eta i$ becomes the origin, has the form [III_2, 188]:

$$f(z) = e^{i\varphi} \frac{\sigma(z - \xi - \eta i)\,\sigma(z + \xi + \eta i)}{\sigma(z - \xi + \eta i)\,\sigma(z + \xi - \eta i)}.$$

We therefore have the following expression for Green's function for a rectangle:

$$G(z;\ a) = -\frac{1}{2\pi} \log \left| \frac{\sigma(z - \xi - \eta i)\,\sigma(z + \xi + \eta i)}{\sigma(z - \xi + \eta i)\,\sigma(z + \xi - \eta i)} \right|.$$

The theory of functions of a complex variable can also be used to construct Green's function for a multiply-connected region; we confine ourselves here, as above, to boundary condition (189) on l. Let B be for instance the doubly-connected domain bounded by the external contour l_1 and the internal contour l_2, and let $G(z; a)$ be Green's function for this domain. We form the function $H(z; a)$ the harmonic conjugate to $G(z; a)$, and the function of a complex variable $\varphi(z) = G(z; a) + H(z; a)\,i$. At the point $z = a$, which is the pole of Green's function, $\varphi(z)$ has a logarithmic singularity, i.e. it can be written in the neighbourhood of this point as the sum of $-(1/2\pi) \log (z - a)$ and a term regular at this point. But $\varphi(z)$ will moreover acquire a purely imaginary added term γi on a circuit round a closed contour round l_2, whilst $f(z) = e^{-2\pi\varphi(z)}$ will acquire the factor $e^{2\pi\gamma i}$, of unit modulus. Furthermore, this last function will have a simple zero at $z = a$, whilst its modulus is unity on contours l_1 and l_2 inasmuch as Green's function $G(z; a)$ vanishes on these contours. The formation of Green's function therefore amounts to the formation of an analytic function $f(z)$ which has a single-valued modulus inside the multiply-connected domain B and which is equal to unity on the contours, whilst $z = a$ is its unique simple zero.

Let us take as an example the case of a ring, bounded by two concentric circles. We take the centre of the circles as the origin and the radii as $h^{-1/2}$ and $h^{1/2}$, where $0 < h < 1$. This can always be done with the aid of a suitably chosen similitude transformation. We replace z by a new variable v in accordance with $z = e^{i\pi v}$ and consider along with h the pure imaginary $\tau = ci$ $(c > 0)$, defined by $h = e^{\pi\tau i}$. Our ring corresponds on the v plane to the strip formed by the straight lines $y = \pm c/2$ parallel to the real axis and two straight lines parallel to the imaginary axis whose distance apart is equal to two.

Regarded as a function of v, $f(z)$ must be analytic throughout the strip. Passage from v to $(v + 2)$ is equivalent to a circuit round the origin in the ring, in which $f(z)$ must acquire a factor of unit modulus. On the boundaries $y = \pm 2/c$ of the strip the condition $|f(z)| = 1$ must be fulfilled, and if $z = a$ is the pole of Green's function in the z plane, $f(z)$, as a function of v, must have simple zeros at the points β given by $a = e^{\pi\beta i}$. These points must exhaust all the zeros of $f(z)$ inside the strip. We can take a as a real positive number. This can always be achieved by a simple rotation of the ring about the origin. It may be shown that the function

$$f(z) = z^{\frac{\log a}{\log h}} \frac{\vartheta_1\left(\frac{v}{2} - \frac{\beta}{2}\right)}{\vartheta_0\left(\frac{v}{2} + \frac{\beta}{2}\right)} = e^{-\frac{\pi\beta}{\tau} vi} \frac{\vartheta_1\left(\frac{v}{2} - \frac{\beta}{2}\right)}{\vartheta_0\left(\frac{v}{2} + \frac{\beta}{2}\right)}$$

satisfies all the conditions laid down above. In this formula, $\theta_0(v)$ and $\theta_1(v)$ are functions which we defined in [III_2, 176], and for clarity we have used β to denote the pure imaginary solution of the equation $a = e^{\pi\beta i}$. To prove all the properties of $f(z)$, we have to use tables (109) and (110) of [III_2, 177], and also the fact that, with real h, the functions $\theta_k(v)$ have imaginary conjugate values for imaginary conjugate values of v. Having $f(z)$, we find Green's function from the formula

$$G(z;\ a) = -\frac{1}{2\pi}\log|f(z)|.$$

We remark that Green's function for the circle $|z| < 1$ with the boundary condition

$$\frac{\partial u}{\partial n} + hu = 0$$

has a fairly complicated form, viz:

$$G(z;\ a) = \frac{1}{2\pi h} + \frac{1}{2\pi}\log\sqrt{\frac{1 - 2rr_1\cos(\varphi-\varphi_1) + r^2 r_1^2}{r_1^2 - 2rr_1\cos(\theta-\theta_1) + r^2}} +$$

$$+ \frac{1}{\pi}\mathcal{R}\left[(rr_1 e^{i(\varphi-\varphi_1)})^{-h}\int\limits_0^{rr_1 e^{(\varphi-\varphi_1)i}} \frac{z'^{h-1}}{1-z'}\,dz'\right],$$

where (r, φ) and (r_1, φ_1) denote the polar coordinates of the points z and a, and $\mathcal{R}$ denotes the real part. With $h = \infty$, the last term has to be thrown discarded and we arrive, as is readily seen, at the expression given above for Green's function with boundary condition (189).

224. Green's function and the non-homogeneous equation. We take the non-homogeneous equation

$$\Delta u(P) = -\varphi(P) \tag{220}$$

in the domain D_i bounded by the surface S. We assume that $\varphi(P)$ is continuous in D_i as far as S and has continuous first order derivatives inside D_i. We seek the solution of (220) which is continuous as far as S and satisfies the boundary condition

$$u|_S = 0. \tag{221}$$

There can only be one such solution. This follows at once from the fact that the difference of two solutions of (220) with condition (221) must satisfy Laplace's equation and condition (221), i.e. must be identically zero. We show that *the required solution has the form:*

$$u(P) = \iiint\limits_{D_i} G(P;\ Q)\,\varphi(Q)\,d\tau_Q \tag{222}$$

or alternatively:

$$u(x, y, z) = \iiint\limits_{D_i} G(x, y, z;\ \xi, \eta, \zeta)\,\varphi(\xi, \eta, \zeta)\,d\xi\,d\eta\,d\zeta. \qquad (223)$$

We can write, in view of (192):

$$u(P) = \frac{1}{4\pi}\iiint\limits_{D_i} \varphi(Q)\frac{1}{r}\,d\tau + \iiint\limits_{D_i} g(P;\ Q)\,\varphi(Q)\,d\tau. \qquad (224)$$

The first term has continuous derivatives up to the second order inside D_i, and its Laplace operator is equal to $[-\varphi(P)]$ [II, 201]. We show that the second term can be differentiated with respect to coordinates (x, y, z) of the point P as many times as desired under the integral. It will follow from this that it is harmonic inside D_i, since $g(P; Q)$ is a harmonic function of P. A remark must first be made. Let the boundary values $f(N; a)$ of a harmonic function depend on a parameter a. The harmonic function $u(P; a)$ itself now depends on a. If we have $f(N; a) \to f(N; a_0)$ uniformly on S as $a \to a_0$, then $u(P; a) \to u(P; a_0)$ uniformly in the closed domain D_i [201].

The function $g(P; Q)$ is a harmonic function of the point Q (ξ, η, ζ) [220] with the boundary values $(-1/4\pi r)$, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$. We assume that P lies inside D_i.

The function

$$\frac{g(x + \Delta x, y, z;\ \xi, \eta, \zeta) - g(x, y, z;\ \xi, \eta, \zeta)}{\Delta x} \qquad (225)$$

is a harmonic function of the point (ξ, η, ζ) with boundary values

$$-\frac{1}{4\pi\,\Delta x}\left[\frac{1}{\sqrt{(x + \Delta x - \xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} - \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}\right].$$

As $\Delta x \to 0$, these boundary values tend uniformly on S to

$$-\frac{1}{4\pi}\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \qquad (226)$$

and it follows immediately from this that (225) tends uniformly in the closed domain D_i to a harmonic function of the point (ξ, η, ζ) with boundary values (226). Similarly arguments can be applied for the other derivatives of any order. Hence the function $g(P; Q)$ has continuous derivatives of all orders with respect to the coordinates of the point P, when P lies inside D_i. Hence it follows at once that the second

term of (224) can be differentiated any number of times with respect to (x, y, z) under the integral sign.

It remains to prove that the function $u(M)$, defined by (222), satisfies boundary condition (221). This follows in essence from the fact that $G(P; Q)$, as a function of P, satisfies this condition. The inadequacy of such an argument lies in the fact that Q can be as close as desired to S during the integration, whereas, on the other hand, it is P that must tend to S for condition (221) to hold. The behaviour of $G(P; Q)$ is not clear in this case.

Let us give a strict proof of the fact that function (222) satisfies condition (221). Let D_i' be the part of D_i lying outside the sphere with centre N_0 and radius d_1, and D_i'' the part of D_i inside this sphere. If ε is a given positive number, in view of inequality (196), we can take d_1 so small that the integral appearing in (222) and taken over D_i'' has an absolute value less than $\varepsilon/2$ for any position of P inside our sphere. When integrating over D_i', the point Q belongs to D_i', whilst the point P is assumed to lie in a small neighbourhood of N_0, say inside the sphere with centre N_0 and radius $d_1/2$. The distance $|PQ|$ is now greater than $d_1/2$, and it follows from [220] that $G(P; Q)$ is a continuous function of the pair of points $(P; Q)$.

Therefore, we can pass to the limit in the integral over D_i' with respect to P as $P \to N_0$, and this limit is zero, since $G(P; Q)$ satisfies condition (221). Hence the integral over D_i' has an absolute value less than $\varepsilon/2$ if P is sufficiently close to N_0, i.e. the whole of the integral appearing in (222) has an absolute value less than ε if P is sufficiently close to N_0. Hence it follows, since ε is arbitrary, that this integral satisfies condition (221). Our assertion that (222) gives the solution of (220) satisfying condition (221) is therefore fully proved.

Note 1. When proving the existence of continuous second order derivatives for the volume potential and Poisson's formula, it is sufficient to assume that the density satisfies a Lipschitz condition in D_i instead of the existence of continuous first order derivatives (see e.g. N. M. Günther, *La Théorie du potentiel*). Hence our statement that formula (222) gives the solution of problem (220), (221), holds if $\varphi(P)$ satisfies the Lipschitz condition:

$$|\varphi(P_2) - \varphi(P_1)| \leqslant b r_{1,2}^{\beta} \qquad (r_{1,2} = |P_1 P_2|). \tag{227}$$

If $\varphi(P)$ is merely continuous in the closed domain D_i, we can no longer say that the first term of the right-hand side of (224) has continuous derivatives up to the second order and satisfies equation (220). But

the proof remains in force that the second term of the formula is a harmonic function inside D_i and that $u(M)$, defined by (222), satisfies condition (221).

Since the volume potential with continuous density is a generalized solution of equation (220) [160], we can say that (222) gives a generalized solution of (220) satisfying condition (221) when $\varphi(P)$ is continuous in the closed domain D_i.

Let us show that such a solution is unique. We suppose that there exist two continuous generalized solutions $u_1(M)$ and $u_2(M)$ of (220), satisfying condition (221). We have:

$$\iiint\limits_{D_i} u_1\,\Delta\sigma\,d\tau = -\iiint\limits_{D_i} \varphi\sigma\,d\tau\,;$$

$$\iiint\limits_{D_i} u_2\,\Delta\sigma\,d\tau = -\iiint\limits_{D_i} \varphi\sigma\,d\tau\,,$$

where σ is any function with continuous derivatives up to the second order in D_i, which vanish at all points sufficiently close to S. On subtracting term by term, we get:

$$\iiint\limits_{D_i} (u_2 - u_1)\,\Delta\sigma\,d\tau = 0\,,$$

whence it follows that $(u_2 - u_1)$ is a harmonic function inside D_i [160]. Since $(u_2 - u_1)$ is continuous as far as S and vanishes on S, it follows that u_2 is identically equal to u_1 in D_i.

Therefore, given any continuous function $\varphi(P)$, (222) gives the unique generalized solution of (220) satisfying condition (221). This solution has continuous first order derivatives in D_i [II, 200].

Note 2. Suppose we are given a function $u(P)$, continuous in the closed domain $\overline{D}_i$, satisfying condition (221) and having continuous derivatives up to the second order inside D_i, and such that Laplace's operator $\Delta u(P)$ is continuous as far as S. On substituting this function in the left-hand side of (220), we obtain a function $\varphi(P)$, continuous in the closed domain $\overline{D}_i$. The function $u(P)$ is obviously both a generalized solution and an ordinary solution of equation (220), satisfying condition (221), so that $u(P)$ must be expressible by (222) in terms of $\varphi(P)$.

Everything that has been said can be applied to the two-dimensional case, when equation (220) takes the form

$$\Delta u\,(x,\,y) = -\,\varphi\,(x,\,y)\,. \tag{228}$$

Its solution in the domain B with contour l, satisfying the boundary condition

$$u|_l = 0\,, \tag{229}$$

is given by

$$u(x,y) = \iint\limits_B G(x, y;\ \xi, \eta)\,\varphi(\xi, \eta)\,d\xi\,d\eta. \tag{230}$$

225. Eigenvalues and eigenfunctions. The fundamental property proved above for Green's function in respect of the non-homogeneous equation (220) lies at the basis of the application of Green's function to the solution of the boundary problem for the equation

$$\Delta v + \lambda v = 0 \qquad \text{(inside } D_i\text{)}, \tag{231}$$

with the boundary condition

$$v|_S = 0\,; \tag{232}$$

this has relevance for the solution of boundary value problems for the wave equation and the heat conduction equation, which will be discussed in detail below.

On taking λv over to the right, we can show as in [173] that the problem (231), (232) is equivalent to the integral equation

$$v(P) = \lambda \iiint\limits_{D_i} G(P;\ Q)\,v(Q)\,d\tau \tag{233}$$

with a symmetrical kernel. The kernel becomes infinite when P and Q coincide, but the whole of the theory of [18] is applicable to it because, by (192), the polarity of the kernel is of order $1/r$:

$$G(P;\ Q) = \frac{K(P;\ Q)}{r}\,, \tag{234}$$

where $K(P; Q)$ is a continuous function.

Let us write (233) in the form:

$$v(P) = \frac{\lambda}{4\pi}\iiint\limits_{D_i} v(Q)\,\frac{1}{r}\,d\tau + \lambda\iiint\limits_{D_i} g(P;\ Q)\,v(Q)\,d\tau$$

$$(r = |PQ|)\,. \tag{235}$$

If $v(P)$ is a continuous solution of this equation, the first term on the right, being the potential of mass distributed over D_i with continuous density, has continuous first derivatives inside D_i, whilst the second term on the right has continuous derivatives of any order inside D_i, as we have seen above; thus $v(P)$ has continuous first order derivatives

inside D_i. But in this case, as we know from [II, 201], the first term on the right-hand side has continuous derivatives up to the second order inside the domain. Hence, by what has been said above, $v(P)$ also has continuous second order derivatives. On applying Poisson's formula, we obtain by (235): $\Delta v = -\lambda v$ inside D_i. Boundary condition (232) is obtained, as we saw in [224]. Conversely, we saw in [224] that (233) follows at once from (231) and (232). We have therefore proved the equivalence of equation (231) with boundary condition (232) to the integral equation (233). We have (234) for this integral equation, whence the inequality:

$$\iiint\limits_{D_i} G^2(P;\ Q)\,\mathrm{d}\tau \leqslant C, \tag{236}$$

where C is a constant follows at once.

Let λ_k and $v_k(P)$ be the eigenvalues and eigenfunctions of (233), or, what amounts to the same thing, of problem (231), (232):

$$\Delta v_k + \lambda_k v_k = 0 \qquad (\text{inside } D_i)\,; \tag{237$_1$}$$

$$v_k|_S = 0. \tag{237$_2$}$$

We can assume that the $v_k(P)$ form an orthonormal system in D_i:

$$\iiint\limits_{D_i} v_l(P)\,v_m(P)\,\mathrm{d}\tau = \begin{cases} 0 & \text{for} \quad l \neq m \\ 1 & \text{for} \quad l = m. \end{cases} \tag{238}$$

Let $\omega(P)$ and its derivatives up to the second order be continuous in D_i as far as S, and let the function satisfy condition (232). We can write it in the form [224]:

$$\omega(P) = -\iiint\limits_{D_i} G(P;\ Q)\,\Delta\omega(Q)\,\mathrm{d}\tau, \tag{239}$$

and, on applying the fundamental expansion theorem of [22], we can say that $\omega(P)$ can be expanded as a Fourier series in the eigenfunctions

$$\omega(P) = \sum_{k=1}^{\infty} c_k v_k(P), \tag{240}$$

this series being regularly convergent in the closed domain D_i. The coefficients are defined in the usual way:

$$c_k = \iiint\limits_{D_i} \omega(P)\,v_k(P)\,\mathrm{d}\tau. \tag{241}$$

We therefore have:

THEOREM. *Every function $\omega(P)$ which is continuous and has continuous derivatives up to the second order in the closed domain D_i and satisfies condition* (232) *can be expanded in a Fourier series in the eigenfunctions $v_k(P)$ which is regularly convergent in the closed domain D_i.*

We shall prove later that the number of eigenvalues λ_k is infinite. This fact has been used when writing series (240). It follows from the uniform convergence of series (240) that, if $\omega(P)$ satisfies the conditions indicated in the theorem, the closure equation holds:

$$\iiint\limits_{D_i} \omega^2(P)\, d\tau = \sum_{k=1}^{\infty} c_k^2. \tag{242}$$

We prove below that this equation also holds for any function continuous in the closed domain $\overline{D}_i$. It may readily be shown that, if the Fourier series

$$\sum_{k=1}^{\infty} a_k v_k(P) \tag{243}$$

of a function $\omega_1(P)$ continuous in the closed domain $\overline{D}_i$ is uniformly convergent in $\overline{D}_i$, its sum is equal to $\omega_1(P)$. On writing $\omega_2(P)$ for the sum of series (243), the function $\omega_2(P) - \omega_1(P)$ will be continuous in $\overline{D}_i$ and orthogonal to all the eigenfunctions $v_k(P)$. It is thus orthogonal to the kernel, i.e.

$$\iiint\limits_{D_i} G(P;\ Q)\left[\omega_2(Q) - \omega_1(Q)\right] d\tau = 0 \qquad (P \text{ in } \overline{D}_i).$$

Hence it is clear that the generalized solution of the equation

$$\Delta u(P) = \omega_1(P) - \omega_2(P)$$

with condition (232) is $u(P) = 0$, so that $\omega_2(P)$ coincides with $\omega_1(P)$.

It follows directly from the last arguments that the kernel $G(P, Q)$ iscomplete [cf. 175], and hence that there is an infinite set of eigenvalues λ_k [25]. We now show that the closure equation (242) holds for any continuous function $\omega(P)$ in $\overline{D}_i$. Such a function is necessarily bounded, i.e. there exists a positive M such that $|\omega(P)| \leqslant M$. Let ε be a given positive number. We choose a closed domain D_i' lying inside D_i such that the volume $(D_i - D_i')$ is less that $\varepsilon/(32M^2)$. We draw inside $(D_i - D_i')$ a closed surface S', that contains D_i' inside itself, and define the function $\varphi(P)$ such that it is equal to $\omega(P)$ in the closed domain D_i and vanishes on S' and outside S'. This function can be extended throughout space in such a way as to be continuous and satisfy $|\varphi(P)| \leqslant M$ [157]. Let $\varphi_n(P)$ be the mean functions for $\varphi(P)$. They

have derivatives of all orders for sufficiently large n, that vanish on the surface S and satisfy $|\varphi_n(P)| \leqslant M$. The functions $\varphi_n(P)$ tend uniformly in $\overline{D}$ to $\omega(P)$, and we can fix so large an n that

$$\iiint\limits_{D_i'} [\omega(P) - \varphi_n(P)]^2 \, d\tau \leqslant \frac{\varepsilon}{4}.$$

From what has been said, we have the closure equation for functions $\varphi_n(P)$, i.e. there exists an N such that

$$\iiint\limits_{D_i} [\varphi_n(P) - s_m(\varphi_n)]^2 \, d\tau \leqslant \frac{\varepsilon}{8} \quad \text{for} \quad m \geqslant N,$$

where $s_m(\varphi_n)$ is the segment of the Fourier series of $\varphi_n(P)$. Since $(a+b)^2 \leqslant 2(a^2+b^2)$, we can write:

$$\begin{aligned}
\iiint\limits_{D_i} [\omega(P) - s_m(\varphi_n)]^2 \, d\tau &= \\
&= \iiint\limits_{D_i} \{[\omega(P) - \varphi_n(P)] + [\varphi_n(P) - s_m(\varphi_n)]\}^2 \, d\tau \leqslant \\
&\leqslant 2\iiint\limits_{D_i} [\omega(P) - \varphi_n(P)]^2 \, d\tau + 2\iiint\limits_{D_i} [\varphi_n(P) - s_m(\varphi_n)]^2 \, d\tau \leqslant \\
&\leqslant 2\iiint\limits_{D_i} [\omega(P) - \varphi_n(P)]^2 \, d\tau + \frac{\varepsilon}{4}.
\end{aligned}$$

We have further:

$$\begin{aligned}
2\iiint\limits_{D_i} [\omega(P) - \varphi_n(P)]^2 \, d\tau &= 2\iiint\limits_{D_i - D_i'} [\omega(P) - \varphi_n(P)]^2 \, d\tau + \\
&+ 2\iiint\limits_{D'} [\omega(P) - \varphi_n(P)^2] \, d\tau \leqslant \\
&\leqslant 2\iiint\limits_{D_i - D_i'} [\omega(P) - \varphi_n(P)]^2 \, d\tau + \frac{\varepsilon}{2}.
\end{aligned}$$

We use for the last integral the inequality:

$$|\omega(P) - \varphi_n(P)|^2 \leqslant 4M^2,$$

and obtain:

$$2\iiint\limits_{D_i - D_i'} [\omega(P) - \varphi_n(P)]^2 \, d\tau \leqslant 8M^2 \cdot \text{volume } (D_i - D_i') < \frac{\varepsilon}{4},$$

after which the previous inequalities give:

$$\iiint\limits_{D_i} [\omega(P) - s_m(\varphi_n)]^2 \, d\tau < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for} \quad m \geqslant N$$

and all the more [3]:

$$\iiint\limits_{D_i} [\omega(P) - s_m(\omega)]^2 \,\mathrm{d}\tau < \varepsilon \quad \text{for} \quad m \geqslant N,$$

whence it follows, since ε is arbitrary, that the closure equation holds for $\omega(P)$.

We remark further that it follows immediately from (236) that the series

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2}$$

is convergent [3]. It may readily be shown that the closure equation also holds for unbounded functions of the type indicated in [3], and in particular, for Green's function $G(P; Q)$.

226. The normal derivative of an eigenfunction. It is important for what follows to investigate the behaviour of the derivatives of the $v_k(P)$ on approaching S.

THEOREM. *The functions $v_k(P)$ have a regular normal derivative on S.*

We form the potential of a three-dimensional mass distribution:

$$u(P) = \frac{\lambda_k}{4\pi} \iiint\limits_{D_i} \frac{v_k(Q)}{r} \,\mathrm{d}\tau \quad (r = |PQ|).$$

It is defined throughout space, is continuous, has continuous first order derivatives, vanishes at infinity, is a harmonic function inside D_e, has continuous derivatives up to the second order inside D_i and satisfies the equation inside D_i:

$$\Delta u = -\lambda_k v_k. \tag{244}$$

We can form the potential of a simple layer:

$$v(P) = \iint\limits_{S} \frac{\mu(N)}{r} \,\mathrm{d}S,$$

satisfying the boundary condition on S:

$$\left(\frac{\partial v(N)}{\partial n}\right)_e = \frac{\partial u(N)}{\partial n},$$

where it must be borne in mind that $u(P)$ has first order derivatives which are continuous throughout space. We form the function:

$$w(P) = u(P) - v(P).$$

It is harmonic inside D_e, vanishes at infinity and has a zero regular normal derivative from outside on S. Green's formula [200] is applicable to the function $w(P)$ in D , from which it follows that $w(P) = 0$ in D_e, and hence on S. Inside D_i, the function $w(P)$ satisfies, by (244):

$$\Delta w = \Delta u - \Delta v = -\lambda_k v_k,$$

whence it follows that $w(P)$ coincides with $v_k(P)$ in D_i, i.e.

$$v_k(P) = u(P) - v(P) \qquad (P \text{ of } D_i),$$

where $u(P)$ and $v(P)$ are defined above. An immediate consequence of this is that *the eigenfunctions $v_k(P)$ have a regular normal derivative on S*. On taking this into account, we can apply Green's formula to the (non-harmonic) function $v_k(P)$:

$$\iiint\limits_{D_i} \left[\left(\frac{\partial v_k}{\partial x}\right)^2 + \left(\frac{\partial v_k}{\partial y}\right)^2 + \left(\frac{\partial v_k}{\partial z}\right)^2\right] d\tau =$$

$$= \iint\limits_S v_k \left(\frac{\partial v_k}{\partial n}\right)_i dS - \iiint\limits_{D_i} v_k \,\Delta v_k \, d\tau .$$

Using the equation $\Delta v_k = -\lambda_k v_k$ and condition (237_2), as also the normalization of the $v_k(P)$, i.e.

$$\iiint\limits_{D_i} v_k^2(P)\, d\tau = 1 ,$$

we obtain the formula

$$\lambda_k = \iiint\limits_{D_i} \left[\left(\frac{\partial v_k}{\partial x}\right)^2 + \left(\frac{\partial v_k}{\partial y}\right)^2 + \left(\frac{\partial v_k}{\partial z}\right)^2\right] d\tau , \tag{245}$$

from which it follows that all the λ_k are positive. This last result can be obtained more simply. It follows at once from a theorem that says that equation (231) with $\lambda > 0$ and condition (232) has only a zero solution. We shall prove this theorem in [234].

In the work mentioned in [208], Smolitskii investigated the derivatives of various orders of the eigenfunctions on the assumption of a sufficiently smooth surface.

He showed that, if S is of class S_{l+5}, $v_k(P)$ belongs to the class Lip $\beta(l-2, C_l\,\lambda_k^{(l+1)/2})$, where β is any number such that $0 < \beta < 1$, and the choice of β determines the choice of C_l.

In the case of a plane, the proof that an eigenfunction has a regular normal derivative can be performed by modifying the above proof, just as we modified in [222] the proof that Green's function has a regular normal derivative.

227. Extremal properties of the eigenvalues and eigenfunctions. The extremal properties of the eigenvalues λ_n and eigenfunctions $v_n(P)$ can be discovered precisely as in [186]: as we have seen, they are the eigenvalues and eigenfunctions of the integral equation (233) with a symmetric kernel possessing weak polarity, by (234). We assume that the λ_n (which are positive) are arranged in non-decreasing order, i.e. $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ We know that λ_1 is the least value of

$$\iiint\limits_{D_i}\iiint\limits_{D_i} G(P;\ Q)\,\omega(P)\,\omega(Q)\,d\tau_P\,d\tau_Q \tag{246}$$

in the class of continuous functions $\omega(P)$ satisfying the condition:

$$\iiint\limits_{D_i}\Big[\iiint\limits_{D_i} G(P;\ Q)\,\omega(Q)\,d\tau_Q\Big]^2 d\tau_P = 1\,,$$

and this least value is attained for $\omega(P) = v_1(P)$. The subscripts of the $d\tau$ indicate the point which is the variable of integration. The order of integration in (246) with respect to the points P and Q is of no importance [cf. 17].

To obtain the following eigenvalues and eigenfunctions, we have to add the orthogonality condition:

$$\iiint\limits_{D_i} \omega(P)\,v_k(P)\,d\tau = 0 \qquad (k = 1, 2, \ldots, n-1).$$

On introducing the class A of functions expressible in terms of a kernel:

$$v(P) = \iiint\limits_{D_i} G(P;\ Q)\,\omega(Q)\,d\tau_Q\,,$$

where $\omega(Q)$ is any function continuous in $\overline{D}_i$, we can formulate the above problem as the problem of the minimum of the integral

$$\iiint\limits_{D_i} v(P)\,\omega(P)\,d\tau \quad \text{subject to the condition} \quad \iiint\limits_{D_i} v^2(P)\,d\tau = 1$$

in the above-mentioned class A of functions $v(P)$.

This class of functions is the class of generalized solutions of Poisson's equation

$$\Delta v(P) = -\,\omega(P)\,,$$

equal to zero on S for any functions $\omega(P)$ continuous in $\overline{D}_i$ and we can speak, finally, of the minimum of the integral

$$-\iiint\limits_{D_i} v\,\Delta v\,d\tau \tag{247}$$

in class A, where Δv is the generalized Laplace operator. The orthogonality conditions given above reduce, as in [186], to orthogonality conditions for $v(P)$:

$$\iiint\limits_{D_i} v(P)\, v_k(P)\, d\tau = 0 \qquad (k = 1, 2, \ldots, n-1). \tag{248}$$

The functions $v(P)$ of class A have continuous first order derivatives inside D_i, and we can show by repeating word for word the arguments of [226] that $v(P)$ has a regular normal derivative on S.

We now define a class of functions A_1 forming part of class A. Class A_1 is the set of functions $v(P)$ that possess the following properties: functions $v(P)$ are continuous in the closed domain $\bar{D}_i$ and vanish on S, and they have continuous derivatives up to the second order inside D_i, their Laplace operator Δv being continuous as far as S. All the eigenfunctions $v_n(P)$ belong to class A_1. If $v(P)$ belongs to class A_1, we can apply Green's formula to integral (247), and in view of the fact that $v(P) = 0$ on S, we can write instead of (247):

$$\iiint\limits_{D_i} (v_x^2 + v_y^2 + v_z^2)\, d\tau. \tag{249$_1$}$$

We can therefore say that λ_1 is the least value of this integral in class A_1 subject to the condition

$$\iiint\limits_{D_i} v^2\, d\tau = 1, \tag{249$_2$}$$

and this least value is attained with $v(P) = v_1(P)$. To obtain the following eigenvalues and eigenfunctions, we have to add the above-mentioned orthogonality conditions (248) to the eigenfunctions already obtained. It can be shown that Green's formula:

$$\iiint\limits_{D_i} (v_x^2 + v_y^2 + v_z^2)\, d\tau = -\iiint\limits_{D_i} v\, \Delta v\, d\tau$$

where Δv is the generalized Laplace operator, holds for any function v of class A.

The above-mentioned extremal properties of the eigenvalues and eigenfunctions thus hold for the whole of class A. We shall prove in Vol. V that these extremal properties also hold in a much wider class of functions.

228. Helmholtz's equation and the radiation principle. Let us take the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u \tag{250}$$

and seek its solution as a steady sinusoidal state of given frequency:

$$u = e^{i\omega t} v. \tag{251}$$

We obtain Helmholtz's equation for v:

$$\Delta v + k^2 v = 0 \qquad (k = \omega : a), \tag{252}$$

which looks more complicated than Laplace's equation. Let us start by seeing the conditions that must be satisfied by the solution of this equation at infinity. This condition has already been mentioned in [III$_2$, 154], where it was termed the radiation principle. We shall give a strictly mathematical statement of this condition in the present section. Suppose we have a steady state outside some surface S. We draw a sphere S_ϱ with centre at a point M lying outside S, and with sufficiently large radius, so that S lies inside S_ϱ, and apply Kirchhoff's formula [II, 202]:

$$u(M;\ t) = \frac{1}{4\pi} \iint\limits_{S+S_\varrho} \left[\frac{1}{r}\left[\frac{\partial u}{\partial n}\right] + \frac{1}{ar}\left[\frac{\partial u}{\partial t}\right]\frac{\partial r}{\partial n} - [u]\frac{\partial \frac{1}{r}}{\partial n}\right] dS$$

to the solution (251). In this formula, the integration is performed over S and S_ϱ. We have for solution (251):

$$[u] = e^{i\omega\left(t - \frac{r}{a}\right)} v; \quad \left[\frac{\partial u}{\partial n}\right] = e^{i\omega\left(t - \frac{r}{a}\right)} \frac{\partial v}{\partial n}; \quad \left[\frac{\partial u}{\partial t}\right] = i\omega\, e^{i\omega\left(t - \frac{r}{a}\right)} v,$$

and we obtain when integrating over S_ϱ an integral of the form

$$\iint\limits_{S_\varrho} \frac{e^{-ikr}}{r}\left(\frac{\partial v}{\partial r} + ikv\right) dS + \iint\limits_{S_\varrho} \frac{v}{r^2} e^{-ikr}\, dS, \tag{253}$$

where we have to put $r = \varrho$ under the integral sign. It is natural to require that the last expression tends to zero as $\varrho \to \infty$ (the absence of a source of vibration at infinity). An elementary surface area of the sphere contains the factor ϱ^2, and the above condition is fulfilled if we subject v to two requirements:

$$rv \text{ is bounded and } r\left(\frac{\partial v}{\partial r} + ikv\right) \to 0$$

as $r \to \infty$; these conditions have to be satisfied for any choice of the origin of the radius vectors r and uniformly with respect to the direction

of these radius vectors. We shall make use of the following notation below. We write $O(r^a)$ for a magnitude x such that the ratio $x : r^a$ remains bounded as $r \to \infty$, and $o(r^a)$ for an x such that $x : r^a \to 0$ as $r \to \infty$, the convergence being uniform with respect to the direction of the radius vector r, independently of the choice of its initial point. The foregoing conditions can be written as:

$$v = O(r^{-1})\,; \tag{254}$$

$$\frac{\partial v}{\partial r} + ikv = o(r^{-1})\,. \tag{255}$$

These conditions in fact represent the mathematical statement of the radiation principle in the three-dimensional case. The conditions take a similar form in the two-dimensional case:

$$v = O\left(r^{-\frac{1}{2}}\right); \tag{256}$$

$$\frac{\partial v}{\partial r} + ikv = o\left(r^{-\frac{1}{2}}\right). \tag{257}$$

In the three-dimensional case

$$v(P) = \frac{e^{-ikr}}{r} \tag{258}$$

is the basic singular solution satisfying the radiation principle, where r is the distance measured from some fixed point O to the variable point P. On differentiating solution (258) with respect to r, we see that it satisfies a stronger condition than (255), namely we can put $O(r^{-2})$ on the right instead of $o(r^{-1})$. We assume here that the distances are measured in (255) from the same point O. We now verify (254) and (255), on the assumption that the distances are measured from another point O_1, where we use the notation $O_1 P = \varrho$. The fact that ϱv is bounded is an immediate consequence of the fact that $\varrho : r \to 1$. We get (255) by simple differentiation of solution (258) with respect to ϱ via r. We have here:

$$\frac{\partial r}{\partial \varrho} = \cos\gamma\,,$$

where γ is the angle between the directions of r and ϱ; using the formula for the square of the side OO_1 of triangle OO_1P, we have:

$$\cos\gamma = 1 + O(r^{-2})\,. \tag{259}$$

In the plane case, the basic solution satisfying the radiation principle is $H_0^{(2)}(kr)$, where $H_0^{(2)}(z)$ is the second Hankel function. We can prove

this simply by using the asymptotic expression for a Hankel function and the equation

$$\frac{\mathrm{d}}{\mathrm{d}z} H_0^{(2)}(z) = - H_1^{(2)}(z). \tag{260}$$

Condition (257) will now be satisfied in the strengthened form, i.e. with $O(r^{-3/2})$ on the right instead of $o(r^{-1/2})$. We multiply this solution $H_0^{(2)}(kr)$ by a constant factor such that the singularity at $r = 0$ reduces to $\log 1/r$. We thus obtain the solution:

$$v = \frac{\pi}{2i} H_0^{(2)}(kr). \tag{261}$$

It can be shown, as above, that the radiation principle will also be satisfied by the solutions:

$$H_m^{(2)}(kr) \cos m\varphi; \quad H_m^{(2)}(kr) \sin m\varphi \qquad (m = 1, 2, 3, \ldots). \tag{262}$$

229. Uniqueness theorem. A uniqueness theorem can be proved provided the radiation principle holds: *if a function v satisfies outside a closed contour l both equations* (252), *the radiation principle at infinity and a homogeneous boundary condition on the contour l, say $v_l = 0$ or $\partial v/\partial n \,|_l = 0$, it is identically zero.*

We apply the formula

$$\iint\limits_{B_1} (u_1 \Delta u_2 - u_2 \Delta u_1)\, \mathrm{d}S = \int\limits_{l} \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n}\right) \mathrm{d}s \tag{263}$$

to the domain B_1 bounded from inside by contour l and from outside by a circle S_r with centre at a fixed point and sufficiently large radius, and we put $u_1 = v$ and $u_2 = \bar{v}$, where $\bar{v}$ is the complex conjugate of v. Suppose v is continuous as far as l and has a regular normal derivative. The double integral vanishes by (252), and the integral over l vanishes by virtue of the boundary condition. The integral over S_r remains, and the directions of n and r coincide on this contour. Condition (257) enables us to substitute

$$\frac{\partial v}{\partial r} = - ikv + o\left(r^{-\frac{1}{2}}\right); \quad \frac{\partial \bar{v}}{\partial r} = ik\bar{v} + o\left(r^{-\frac{1}{2}}\right),$$

and hence we arrive at the equation:

$$2ik \int\limits_{S_r} |v|^2 \,\mathrm{d}s + \int\limits_{S_r} v \cdot o\left(r^{-\frac{1}{2}}\right) \mathrm{d}s + \int\limits_{S_r} \bar{v} \cdot o\left(r^{-\frac{1}{2}}\right) \mathrm{d}s = 0.$$

Since $\sqrt{r}v$ and $\sqrt{r}\bar{v}$ are bounded as $r \to \infty$, the last two terms tend to zero, and we have, on introducing the polar angle φ on the circumference S_r:

$$\int_0^{2\pi} |\sqrt{r}v|^2 \, d\varphi \to 0. \tag{264}$$

We now apply Green's formula to the solution v and to the first of solutions (262). The double integral vanishes as before and there remain the integrals over l and S_r, so that the size of the integral over S_r does not depend on r. Both our solutions satisfy the radiation principle; solutions (262) satisfy condition (257) in the strengthened form, as also does $H_0^{(2)}(kr)$. On using (257), as above, we find that the integral over S_r tends to zero, and since its size does not depend on r, it is simply equal to zero, i.e.

$$H_m^{(2)}(kr) \int_{S_r} \frac{\partial v}{\partial r} \cos m\varphi \, d\varphi - \frac{dH_m^{(2)}(kr)}{dr} \int_{S_r} v \cos m\varphi \, d\varphi = 0.$$

If we put

$$f_m(r) = \int_{S_r} v \cos m\varphi \, d\varphi,$$

this gives

$$H_m^{(2)}(kr) f_m'(r) = \frac{dH_m^{(2)}(kr)}{dr} f_m(r),$$

whence $f_m(r) = c_m H_m^{(2)}(kr)$, where c_m is a constant.

Similarly, we obtain for

$$g_m(r) = \int_{S_r} v \sin m\varphi \, d\varphi$$

the expression $g_m(r) = d_m H_m^{(2)}(kr)$, where d_m is also a constant. The closure equation [3] and formula (264) show that, with fixed m and as $r \to \infty$:

$$c_m \sqrt{r} H_m^{(2)}(kr) \quad \text{and} \quad d_m \sqrt{r} H_m^{(2)}(kr) \to 0.$$

But it follows from the asymptotic expression for $H_m^{(2)}(kr)$ that $\sqrt{r}H_m^{(2)}(kr)$ remains greater in modulus than some positive number for large r, whence it follows that $c_m = d_m = 0$, i.e. $f_m(r) = g_m(r) = 0$; and now it follows, by virtue of the closure equation, that v vanishes on the circumferences S_r.

If l is a circle, we find by taking circles concentric with l as S_r that v vanishes identically outside l, which is what we required to prove. With a general contour, the above arguments show that v vanishes in the neighbourhood of a point at infinity. We shall prove

later [230] that $v(x, y)$ must be an analytic function, as for Laplace's equation, and it will follow from the principle of analytic continuation that the vanishing of v at a point at infinity implies that $v = 0$ everywhere outside l. A uniqueness theorem can be proved for the three-dimensional case in essentially the same way.

230. The principle of limiting amplitude and the principle of limiting absorption. It can be shown as in the previous section that the radiation condition produces a unique solution for the equation

$$\Delta v + k^2 v = - F(P) \qquad (k > 0), \tag{264$_1$}$$

defined throughout space. We shall assume that $F(P)$ is continuously differentiable, is a function of a point of three-dimensional Euclidean space, is defined throughout the space and vanishes outside some finite domain D. Our solution is now given by

$$v(P) = \frac{1}{4\pi} \iiint\limits_D \frac{e^{-ikr}}{r} F(Q)\, dF_Q \qquad (r = |PQ|). \tag{264$_2$}$$

We can also arrive at this by considering the non-stationary problem of the forced oscillations due to the action of a periodic force. In fact, it follows at once from Kirchhoff's formula [II, 202] that

$$v(P) = \lim_{t \to +\infty} u(P, t)\, e^{-ikt},$$

where $u(P, t)$ is a solution of the wave equation

$$\Delta u - u_{tt} = - F(P)\, e^{ikt},$$

satisfying the zero initial condition. We therefore say of the solution $v(P)$ that it is "the limiting amplitude of the (transient) periodic vibration building up for large t under the action of a periodic force." This principle for originating solutions of equation (264_1) is called the limiting amplitude principle.

Another principle for producing solutions of equation (264_1), called the limiting absorption principle (see V. S. Ignatovskii, *Ann. d. Phys.* 18, 1905), consists of the following process: a complex parameter $-i\varepsilon$ ($\varepsilon > 0$) (the "absorption") is introduced into the equation:

$$\Delta v_\varepsilon + (k^2 - i\varepsilon)\, v_\varepsilon = - F(P)$$

and we take the solution $v_\varepsilon(P)$ that tends to zero at infinity (there is only one such solution):

$$v_\varepsilon(P) = \frac{1}{4\pi} \iiint\limits_D \frac{e^{-ir(a_\varepsilon - ib_\varepsilon)}}{r} F(Q)\, d\tau_Q,$$

where $a_\varepsilon - ib_\varepsilon = \sqrt{k^2 - i\varepsilon}$ ($a_\varepsilon > 0, b_\varepsilon > 0$), where $b_\varepsilon \to 0$ as $\varepsilon \to +0$. As $\varepsilon \to +0$ the function $v_\varepsilon(P)$ has a limit, which coincides with the $v(P)$ given by (264_2).

The three principles, given here and in [228], for producing solutions in the elementary case under discussion, all lead to the same solution (264_2), as we

have seen. They can naturally be expected to apply to more general boundary value problems for elliptic equations in infinite domains. The principles have different fields of application, however. Thus the radiation principle is suitable for the cases when equation (264_1) is considered throughout space or in a domain E containing the point at infinity as an interior point [229]. If E is, say, the strip $0 \leqslant z \leqslant 1$, equation ($264_1$) has no one solution that vanishes for $z = 0$ and $z = 1$ and satisfies the radiation condition in the form (254) and (255) ($F(Q) \not\equiv 0$). If these conditions are slightly modified, however, the problem does have a unique solution (see A. G. Sveshinkova, *DAN SSSR*, 1950, 73, no. 5, The radiation principle [Printsip izlucheniya]).

If the other two principles are applied without any modifications to this example, we can conclude that the "radiation conditions" must depend on the form of the domain E at infinity. Certain considerations of a physical nature indicate that the limiting absorption principle is not always applicable in the above form if E narrows down rapidly enough at infinity.

In general, the question of the applicability of the principles stated here has not been sufficiently investigated. Rellich's work may be mentioned in this connection (*Jahresber. der Deutsch Math. Verein.*, Bd. 53, 57), where the form of the "radiation conditions" is considered for equation (264_1) in unbounded domains of various types; we may also quote A. Ya. Povzner's The expansion of functions in eigenfunctions of the operator $-\Delta u + cu$ (O razlozhenii funktsii po sobstvennym funktsiyam operatora $-\Delta u + cu$) (*Mat. sbornik*, t. 32 (74): 1, 1953, 107—156), which provides a basis for the limiting absorption principle for the equation

$$\Delta u + q(P)\, u + k^2 u = -F(P)$$

in unbounded three-dimensional space, and O. A. Ladyzhenskaya's note [The limiting amplitude principle (O printsipe predel'noi amplitudy)] to be published shortly in *Uspekhi Matematicheskikh Nauk*. This note is concerned with the limiting amplitude principle for the above equation.

It is clear from the works of Povzner and Ladyzhenskaya that the limiting absorption principle has a wider field of application than the limiting amplitude principle, at any rate in the form in which it is stated above.

In fact, the limit of solutions vanishing at infinity of the equation

$$\Delta v_\varepsilon + q(P)\, v_\varepsilon + (k^2 - i\varepsilon)\, v_\varepsilon = -F(P),$$

exists as $\varepsilon \to +0$ provided k^2 is not an eigenvalue of the operator $\Delta v + q(P)v$; the limit of solutions of the corresponding transient problem, $\lim\limits_{t\to+\infty} u(P, t)\, e^{-ikt}$, may not exist if the operator has at any rate one eigenvalue. It should be noted that c is called an eigenvalue of the operator $\Delta v + q(P)v$ if the equation $\Delta v + q(P)v + cv = 0$ has a non-zero solution, the square of the modulus of which is integrable over the entire space.

231. Boundary value problems for Helmholtz's equation. The solution (258) of equation (252) has polarity $1/r$ at $r = 0$, and this makes it possible for us to construct a potential theory for equation (252) analogous to the Newtonian potential theory for Laplace's equation. If r is the distance from a variable point N of surface S to a point P,

we obtain in the three-dimensional case the following analogues to the potentials of a simple and double layer:

$$\left.\begin{aligned} v(P) &= \iint\limits_S \mu(N)\frac{e^{-ikr}}{r}\,dS \\ w(P) &= \iint\limits_S \mu(N)\frac{\partial}{\partial n}\left(\frac{e^{-ikr}}{r}\right)dS, \end{aligned}\right\} \tag{265}$$

where n is the outward normal to S at the variable point N. On isolating the polar term $1/r$ from the kernel, we get the usual potentials, in which a passage to the limit as P approaches the surface is carried out in accordance with the formulae of [193] and [195]. The kernel that remains no longer has a singularity at $r = 0$, and we can pass to the limit under the integral sign. Formulae are thus obtained which are entirely analogous to the formulae of [193] and [195]:

$$\left.\begin{aligned} \left(\frac{\partial v(N_0)}{\partial n_0}\right)_i &= 2\pi\mu(N_0) + \iint\limits_S \mu(N)\frac{\partial}{\partial n_0}\left(\frac{e^{-ikr_0}}{r_0}\right)dS \\ & \qquad\qquad\qquad\qquad (r_0 = |N_0N|) \\ \left(\frac{\partial v(N_0)}{\partial n_0}\right)_e &= -2\pi\mu(N_0) + \iint\limits_S \mu(N)\frac{\partial}{\partial n_0}\left(\frac{e^{-ikr_0}}{r_0}\right)dS \end{aligned}\right\} \tag{266}$$

and

$$\left.\begin{aligned} w_i(N_0) &= 2\pi\mu(N_0) + \iint\limits_S \mu(N)\frac{\partial}{\partial n}\left(\frac{e^{-ikr_0}}{r_0}\right)dS, \\ w_e(N_0) &= -2\pi\mu(N_0) + \iint\limits_S \mu(N)\frac{\partial}{\partial n}\left(\frac{e^{-ikr_0}}{r_0}\right)dS, \end{aligned}\right\} \tag{267}$$

the kernel of the integral in (266) being the value of the derivative with respect to the normal direction n_0 at the point N_0, whilst in (267) the derivative is with respect to the direction of the normal n at the point N when P coincides with N_0.

In the plane case we have the simple and double layer potentials:

$$\left.\begin{aligned} v(P) &= \int\limits_l \mu(N)\frac{\pi}{2i}H_0^{(2)}(kr)\,ds, \\ w(P) &= \int\limits_l \mu(N)\frac{\partial}{\partial n}\left[\frac{\pi}{2i}H_0^{(2)}(kr)\right]ds, \end{aligned}\right\} \tag{268}$$

and formulae similar to (266) and (267) hold for them, the factor 2π on the right being replaced by π. These last potentials satisfy equation

(252), and by virtue of the special choice of kernel, each element of the integrals and the integrals themselves satisfy the radiation principle.

We introduce the kernel

$$K(N_0, N;\ k) = \frac{\partial}{\partial n}\left(\frac{e^{-ikr_0}}{r_0}\right) = -\frac{e^{-ikr_0}(ikr_0+1)}{2\pi r_0^2}\cos\varphi,$$

where φ is the angle between the direction n and $\overline{N_0N}$. The transposed kernel will be

$$K(N, N_0;\ k) = \frac{\partial}{\partial n_0}\left(\frac{e^{-ikr_0}}{r_0}\right) = \frac{e^{-ikr_0}(ikr_0+1)}{2\pi r_0}\cos\psi,$$

where ψ is the angle between the normal n_0 at N_0 and $\overline{N_0N}$. The Dirichlet and Neumann problems can be stated in the same way as for Laplace's equation.

The interior Dirichlet problem consists in seeking inside l the solution of (252) which satisfies on S the boundary condition:

$$u|_S = f(N_0).$$

The exterior problem can be similarly stated; here, the radiation principle must be satisfied at infinity. In the case of the Neumann problem, we have the boundary condition:

$$\left.\frac{\partial u}{\partial n}\right|_S = f(N_0).$$

It follows from the uniqueness theorem [229] that the exterior problems can only have one solution. The uniqueness is not true for all k in the case of the interior problems.

The number k^2 is called *an eigenvalue of the interior Dirichlet problem* if there exists inside l a solution of (252) satisfying on l the homogeneous boundary condition $u|_S = 0$. The eigenvalues of the interior Neumann problem are similarly defined.

If we seek the solution of the exterior Dirichlet problem as the potential of a double layer, and that of the interior Neumann problem as the potential of a simple layer, we arrive at the adjoint integral equations:

$$\mu(N_0) + \iint_S \mu(N) K(P, N;\ k)\, dS = -\frac{1}{2\pi} f(N_0), \tag{269}$$

$$\mu(N_0) + \iint_S \mu(N) K(N, P;\ k)\, dS = -\frac{1}{2\pi} f(N_0), \tag{270}$$

where N_0 is a variable point of S. Suppose that k^2 is not an eigenvalue of the interior Neumann problem. We show that the homogeneous

equation (269) now has only a zero solution. We use reductio ad absurdum: let it have a non-zero solution. The homogeneous equation (270) now also must have a non-zero solution $\mu_0(N)$. On forming the potential of a simple layer with density $\mu_0(N)$, we obtain the solution of equation (252) with the homogeneous boundary condition $\partial u/\partial n|_l=0$. But this solution must vanish inside l because k^2 is not an eigenvalue of the interior Neumann problem. Since it is continuous, the simple layer potential must vanish on l also; and now, by the uniqueness theorem, it must likewise vanish outside l. By a formula analogous to (54), $\mu_0(N)$ must here be identically zero. This contradiction shows that, *if k^2 is not an eigenvalue of the interior Neumann problem, homogeneous equation* (269) *has only a zero solution, so that the non-homogeneous equation is soluble for any* $f(N_0)$, i.e. *the exterior Dirichlet problem has a solution in the form of a double layer potential for any* $f(N_0)$. Similarly, *if k^2 is not an eigenvalue of the interior Dirichlet problem, the exterior Neumann problem has a solution in the form of a simple layer potential.*

In a recently published book, *Boundary Value Problems of the Theory of Vibrations and Integral Equations* (Granichnye zadachi teorii kolebanii i integral'nye uravneniya), Kupradze has investigated in detail the steady state in electrodynamics and theory of elasticity, and in particular the problem of diffraction, which we shall discuss in the next section. This book also deals with the cases when k is an eigenvalue of the interior Dirichlet or Neumann problem, and shows how solutions of the exterior problems are obtained in such cases.

Let us now prove that every solution $v(P)$ of equation (252) with continuous derivatives up to the second order inside some domain D is an analytic function of the coordinates. All we need do is show that $v(P)$ is analytic inside some sphere S_0 with centre at any interior point P_0 of D.

We try to write $v(P)$ inside D as the potential of a double layer (265). An integral equation is obtained for the density $\mu(N)$ of the layer:

$$\mu(N_0) = \frac{1}{2\pi} f(N_0) - \frac{1}{2\pi} \iint\limits_{S_0} \mu(N) \frac{\partial}{\partial n_0}\left(\frac{e^{-ikr_0}}{r_0}\right) dS, \qquad (271)$$

where $f(N)$ are the values of $v(P)$ on the sphere S_0. The radius of this sphere can be taken so small that the equation

$$\mu(N_0) = -\frac{1}{2\pi} \iint\limits_{S_0} \mu(N) \frac{\partial}{\partial n_0}\left(\frac{e^{-ikr_0}}{r_0}\right) dS \qquad (272)$$

has only a zero solution. Let us prove this. Let λ_1 be the first eigenvalue of the equation $\Delta u + \lambda u = 0$ with the boundary condition $u\,|_{S_0} = 0$ when S_0 is a sphere with unit radius. It may readily be seen, by performing a similitude transformation, that the first eigenvalue for a sphere of radius R is equal to $(\lambda_1 : R)$, and the number R can be taken so small that $(\lambda_1 : R)$ is greater than k^2. The interior Dirichlet problem for the equation $\Delta u + k^2 u = 0$ with a homogeneous boundary condition now has only a zero solution. Integral equation (272) is the equation for the density of the double layer potential which gives the solution of the homogeneous interior Dirichlet problem just mentioned. Since this problem has only a zero solution, we find by arguing exactly as in [207] that equation (272) has only a zero solution for a sphere of radius R. With this choice, we can assert that (271) has a solution, and we have:

$$v(P) = \iint\limits_{S_0} \mu(N) \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{-ikr}}{r}\right) \mathrm{d}S \qquad (P \text{ inside } S_0;\ r = |PN|)$$

or [II, 197]

$$v(P) = -\iint\limits_{S_0} \mu(N)\, \frac{\mathrm{e}^{-ikr}(ikr+1)}{r^2} \cdot \frac{R^2 + r^2 - \varrho^2}{2Rr}\, \mathrm{d}S\,, \tag{273}$$

where R is the radius of S_0 and $\varrho = |\,P_0\,P\,|$. The integrand is an analytic function of the coordinates (x, y, z) of the point P inside S_0, and it follows from (273) that $v(P)$ is also analytic in (x, y, z) [cf. 159]. A similar proof holds for the plane case, where use is made of the singular solution indicated below.

We can form Green's function for equation (252) in precisely the same way as for Laplace's equation. In the three-dimensional case the basic singular solution of this equation can be written as $(\cos kr)/r$. Green's function, corresponding to the condition

$$v\,|_S = 0\,, \tag{274}$$

must be sought in the form:

$$G_1(P, Q\,;\ k^2) = \frac{\cos kr}{4\pi r} + g_1(P, Q\,;\ k^2) \qquad (r = |PQ|)\,, \tag{275}$$

where $g_1(P; Q; k^2)$ satisfies (252) inside D, and the boundary condition on S:

$$g_1(P, Q\,;\ k^2)\,|_S = -\left.\frac{\cos kr}{4\pi r}\right|_S. \tag{276}$$

If k^2 is not an eigenvalue of (252) with boundary condition (274), such a function can be formed.

In the plane case the solutions of (252) depending only on the distance $r = | PQ |$ have the form $Z_0(kr)$, where $Z_0(z)$ is any solution of the Bessel equation corresponding to the zero subscript:

$$Z_0''(z) + \frac{1}{z} Z_0'(z) + Z_0(z) = 0. \tag{277}$$

We take as a solution of this equation Neumann's function [III, 214]:

$$N_0(z) = \frac{2}{\pi} J_0(z)\left(\log\frac{z}{2} + C\right) - \\ - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + \ldots + 1\right). \tag{278}$$

The basic singular solution having polarity $(1/2\pi)\log(1/r)$ will be

$$-\frac{1}{4} N_0(kr). \tag{279}$$

It follows from (278) that function (279) will contain, in addition to the polar term mentioned, terms of the form $r^{2n}\log r$ $(n = 1, 2, \ldots)$ that include $\log r$. These terms tend to zero as $r \to 0$. It may easily be seen by direct differentiation that their first order derivatives with respect to the coordinates also tend to zero, and function (279) will have continuous first order derivatives at Q. Suppose that k is not an eigenvalue of (252) with a boundary condition of the form (274). Green's function $G_1(P; Q; k^2)$ is easily formed for equation (252) with these values of k.

We shall seek this Green's function in the form:

$$G_1(P, Q;\ k^2) = -\frac{1}{4} N_0(kr) + g_1(P, Q;\ k^2). \tag{280}$$

Since the first term on the right-hand side satisfies the equation and has the required polarity, the problem amounts to finding the term $g_1(P; Q; k^2)$ such that it has no polarity, satisfies (252), and on the contour l satisfies the following non-homogeneous boundary condition:

$$g_1|_l = \frac{1}{4} N_0(kr).$$

Recalling that k is not an eigenvalue, a unique function g_1 is obtained that satisfies these conditions.

Returning to the three-dimensional case, a formula connected with equation (252) must be mentioned. Let $v_0(r)$ be any singular solution

of this equation with polarity $1/r$, and let $v(P)$ be any function having continuous derivatives up to the second order in domain D_i and as far as S. By arguing precisely as in [II, 193], we get the equation:

$$v(P) = \frac{1}{4\pi}\iint\limits_S \left[v_0(r)\frac{\partial v}{\partial n} - v\frac{\partial v_0(r)}{\partial n}\right] \mathrm{d}S - \frac{1}{4\pi}\iiint\limits_{D_i} \frac{\Delta v + k^2 v}{r}\,\mathrm{d}\tau,$$

where P is inside D and r is the distance from P to the point of integration. If v satisfies equation (252), the triple integral is zero. Let us apply this formula to the case when S is a sphere of radius R and P is its centre, and let us choose $v_0(r)$ such that $v_0(R) = 0$:

$$v_0(r) = \frac{\cos kr}{r} - \cot kR \frac{\sin kr}{r}.$$

We obtain as a result:

$$\frac{\sin kR}{kR} v(P) = \frac{1}{4\pi R^2}\iint\limits_S v\,\mathrm{d}S,$$

and the right-hand side represents the mean value of v over the sphere S. This formula generalizes the mean value property of harmonic functions.

232. Diffraction of electromagnetic waves. A more difficult problem is presented by the diffraction of sinusoidal electromagnetic waves incident on a body with dielectric constant ε and conductivity σ. Let us take the plane problem. Let l be the contour of the body, so that we have space outside l. Let B_i and B_e denote the parts of the plane lying inside and outside l. The problem amounts mathematically to finding the function $E(x, y)$ satisfying the equations:

$$\Delta E + k_i^2 E = 0 \quad (\text{in } B_i); \quad \Delta E + k_e^2 E = 0 \quad (\text{in } B_e),$$

where

$$k_i^2 = \frac{\omega^2\varepsilon^2 - \omega\sigma i}{c^2}; \quad k_e^2 = \frac{\omega^2}{c^2}; \tag{281}$$

ω is the frequency of the incident wave and c is the velocity of light in vacuo. The function E represents in B_i the complement along the Z axis of the electric intensity vector produced as a result of the incident excitation $e^{i\omega t}A(x, y)$; in B_e, E is the sum of the incident wave A and the wave obtained as a result of diffraction at the contour l, so that the difference $(E - A)$ must satisfy the radiation principle. The given function A must satisfy throughout the plane the equation:

$$\Delta A + k_e^2 A = 0. \tag{282}$$

The boundary conditions are given by the continuity of E and $\partial E/\partial n$ on passing through the contour.

We apply Green's formula (263) to the domain B_i, the functions $E(Q)$ and

$$G(P\,;\,Q) = \frac{\pi}{2i} H_0^{(2)}(k_e r) \qquad (r = |PQ|), \tag{283}$$

on the assumption that P lies inside B_i. Here, we isolate the point P by a small circle γ, and denote the remaining part of B_i by B_i':

$$\iint\limits_{B_i'} (E\,\Delta G - G\,\Delta E)\,\mathrm{d}S = \int\limits_{l}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s + \int\limits_{\gamma}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s.$$

The first of equations (281) and the analogous equation for function (283) give:

$$E\,\Delta G - G\,\Delta E = (k_e^2 - k_i^2)\,G(P\,;\,Q)\,E(Q).$$

Recalling that $G(P; Q)$ has polarity $\log 1/r$ at $r = 0$, and indefinitely contracting the circle γ, we arrive at [cf. II, 186]:

$$2\pi E(P) = (k_i^2 - k_e^2)\iint\limits_{B_i} G(P\,;\,Q)\,E(Q)\,\mathrm{d}S + \int\limits_{l}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s. \tag{284$_1$}$$

Let S_ϱ be a circle with centre at the origin and sufficiently large radius ϱ and B_e' the part of B_e lying inside S_ϱ. On applying Green's formula to B_e' and assuming that P lies in B_i, we get:

$$0 = \int\limits_{S_\varrho}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s + \int\limits_{l}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s. \tag{284$_2$}$$

If we now take P to be situated in B_e, and apply the formula to B_i, we get:

$$0 = (k_i^2 - k_e^2)\iint\limits_{B_i} G(P\,;\,Q)\,E(Q)\,\mathrm{d}S + \int\limits_{l}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s. \tag{284$_3$}$$

Finally, if P lies in B_e and we apply the formula to B_e', we get:

$$2\pi E(P) = \int\limits_{l}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s + \int\limits_{S_\varrho}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s. \tag{284$_4$}$$

The directions of the outward normals to l are opposed in (284$_1$) and (284$_2$). The same is true for (284$_3$) and (284$_4$). On adding (284$_1$) and (284$_2$), and also (284$_3$) and (284$_4$), term by term and remembering that E and $\partial E/\partial n$ are continuous on passing through the contour l, we obtain the same formula for the cases when P lies in B_i or in B_e:

$$2\pi E(P) = (k_i^2 - k_e^2)\iint\limits_{B_i} G(P\,;\,Q)\,E(Q)\,\mathrm{d}S + \int\limits_{S_\varrho}\left(E\frac{\partial G}{\partial n} - G\frac{\partial E}{\partial n}\right)\mathrm{d}s, \tag{285}$$

and it remains for us to pass to the limit as ϱ tends to infinity. On applying Green's formula to the circular domain bounded by S_ϱ, and to the functions A and G, and taking P to be an interior point of S_ϱ, we obtain:

$$2\pi A(P) = \int\limits_{S_\varrho}\left(A\frac{\partial G}{\partial n} - G\frac{\partial A}{\partial n}\right)\mathrm{d}s$$

so that the line integral appearing in (285) is equal to the expression:

$$2\pi A(P) + \int_{S_\varrho} \left\{ [E(Q) - A(Q)] \frac{\partial G(P, Q)}{\partial n} - G(P;\ Q) \frac{\partial}{\partial n} [E(Q) - A(Q)] \right\} ds. \tag{286}$$

We shall use the fact that the difference $(E - A)$ satisfies the radiation principle to show, at the end of the next section, that our last integral must tend to zero, and (285) gives us:

$$E(P) = \frac{k_i^2 - k_e^2}{2\pi} \iint_{B_i} G(P;\ Q) E(Q)\, dS + A(P). \tag{287}$$

If P is taken to lie in B_i, the equation written is an ordinary integral equation. By finding $E(P)$ from it, where P belongs to B_i, and substituting the solution obtained in the right-hand side of (287), we obtain an explicit expression for $E(P)$ for the case when P belongs to B_e. We have obtained (287) on the assumption that a solution of the problem exists. Strictly speaking, we need to make an investigation of (287) and show that it has a solution if P belongs to B_i, and that this solution is also a solution of our diffraction problem. This is in fact done in the works quoted at the end of the next section. A further point to be mentioned is that the integration is carried out in (287) over the whole of the domain B and not over the contour l.

233. The magnetic intensity vector. Except for one modification, we have the same equations and conditions for the magnetic intensity vector $H(x, y)$. Instead of $\partial H/\partial n$ being continuous on passing through l, $(1/k)\ \partial H/\partial n$ must be continuous, where $k = k_i$ in B_i and $k = k_e$ in B_e. In addition, the difference $(H - B)$, where $B(x, y)$ is a given function satisfying equation (282), must be subject to the radiation principle. We have equation (284) as before. On taking into account the required continuity of $(1/k)\ \partial H/\partial n$, we multiply (284_1) by $1/k_i^2$, (284_2) by $1/k_e^2$ and add. We do the same for (284_3) and (284_4). On then passing to the limit, as above, we obtain:

$$\frac{H(P)}{k_i^2} = \frac{k_i^2 - k_e^2}{2\pi k_i^2} \iint_{B_i} G(P;\ Q) H(Q)\, dS + \frac{1}{2\pi}\left(\frac{1}{k_i^2} - \frac{1}{k_e^2}\right) \int_l H(Q) \frac{\partial G(P;\ Q)}{\partial n}\, ds + \frac{B(P)}{k_e^2},$$

$(P \text{ in } B_i)$

$$\frac{H(P)}{k_e^2} = \frac{k_i^2 - k_e^2}{2\pi k_i^2} \iint_{B_i} G(P;\ Q) H(Q)\, dS + \frac{1}{2\pi}\left(\frac{1}{k_i^2} - \frac{1}{k_e^2}\right) \int_l H(Q) \frac{\partial G(P;\ Q)}{\partial n}\, ds + \frac{B(P)}{k_e^2}.$$

$(P \text{ in } B_e)$

In view of the polarity of the functions $G(P; Q)$ when P and Q coincide, the line integral written behaves like the potential of a double layer when P approaches the contour, and we obtain for the case when P lies on the contour:

$$\frac{1}{2}\left(\frac{1}{k_i^2}+\frac{1}{k_e^2}\right) H(P)=\ldots,$$

where the right-hand side is the same as in the previous equations. We can rewrite the three previous equations as the single formula:

$$\frac{1}{k^2} H(P)=\frac{k_i^2-k_e^2}{2\pi k_i^2}\iint\limits_{B_i} G(P;\ Q)\, H(Q)\,\mathrm{d}S+$$

$$+\frac{1}{2\pi}\left(\frac{1}{k_i^2}-\frac{1}{k_e^2}\right)\int\limits_{l} H(Q)\,\frac{\partial G(P;\ Q)}{\partial n}\,\mathrm{d}s+\frac{B(P)}{k_e^2}, \qquad (288)$$

where $k^2=k_i^2$ (if P is inside B_i); $k^2=k_e^2$ (if P is inside B_e) and

$$\frac{1}{k^2}=\frac{1}{2}\left(\frac{1}{k_i^2}+\frac{1}{k_e^2}\right) \qquad (\text{if } P \text{ is on } l).$$

If P lies in the closed domain B_i, (288) is a weighted integral equation [49], and the usual Fredholm theory is applicable to it. It can be shown that it has a unique solution and that this solution is in fact the answer to our diffraction problem. It should be noted that, if we can solve the above-mentioned integral equation, i.e. if we know $H(Q)$ in the closed domain B_i, (288) gives us $H(P)$ in B_e.

We now show that the integral in (286) tends to zero as $\varrho\to\infty$. We have from the asymptotic expressions for $H_0^{(2)}(z)$ and $H_1^{(2)}(z)$:

$$\frac{\partial G(P;\ Q)}{\partial r}=-ik_e G(P;\ Q)+O\left(r^{-\frac{3}{2}}\right) \qquad (r=|PQ|).$$

We have to take P as fixed in future and Q as being in C_ϱ. We have:

$$\frac{\partial G(P;\ Q)}{\partial \varrho}=\frac{\partial G}{\partial r}\cdot\frac{\partial r}{\partial \varrho}=\frac{\partial G}{\partial r}\cos\gamma,$$

where we have expression (259) for $\cos\gamma$.

Further, we obviously have

$$O(r^a)=O(\varrho^a),$$

so that

$$\frac{\partial G(P;\ Q)}{\partial \varrho}=-ik_e G(P;\ Q)+O\left(\varrho^{-\frac{3}{2}}\right).$$

The integral in (286) can be rewritten as

$$J=\int\limits_{S_\varrho}\left\{(E-A)\left[-ik_e G+O\left(\varrho^{-\frac{3}{2}}\right)\right]-G\left[-ik_e(E-A)+o\left(r^{-\frac{1}{2}}\right)\right]\right\}\mathrm{d}s,$$

or

$$J=\int\limits_{S_\varrho}\left[(E-A)\cdot O\left(\varrho^{-\frac{3}{2}}\right)+G\cdot o\left(\varrho^{-\frac{1}{2}}\right)\right]\mathrm{d}s=\int\limits_{S_\varrho}\left[o\left(r^{-1}\right)+o\left(r^{-1}\right)\right]\mathrm{d}s,$$

whence it follows immediately that $J \to 0$. An investigation of the diffraction problem can be found in the following works:

1. V. D. Kupradze. *Boundary Value Problems of the Theory of Vibrations and Integral Equations* (Granichnye zadachi teorii kolebanniï i integral'nye uravneniya).

2. Sternberg: Anwendung der Integralgleichungen in der elektromagnetischen Lichttheorie (*Compositio Matematica,* vol. 3, f. 2, 1936).

3. Freudental: Ueber Beugungsprobleme der elektromagnetischen Lichttheorie (*Compositio Matematica,* vol. 6, f. 2, 1938).

234. The uniqueness of the solution of Dirichlet's problem for elliptic equations. We shall prove an auxiliary proposition on matrices before turning to the subject indicated in the heading.

LEMMA. *Let A and B be two real square symmetric matrices, all the characteristic roots of A being positive. If all the characteristic roots of B are not positive, the trace of the product AB will be a non-positive number:* $\mathrm{Sp}(AB) \leqslant 0$. *If all the characteristic roots of B are non-negative,* $\mathrm{Sp}(AB) \geqslant 0$.

Let U be the matrix of the orthogonal transformation reducing B to the diagonal form, i.e. $UBU^{-1} = [\mu_1, \mu_2, \ldots, \mu_n]$, where μ_i are the characteristic roots of matrix B. We know that

$$\mathrm{Sp}\,(AB) = \mathrm{Sp}\,(UABU^{-1}) = \mathrm{Sp}\,(UAU^{-1}UBU^{-1})$$

[III_1, 27] and that the real symmetric matrix $A' = UAU^{-1}$ has the same characteristic roots as A. All these are positive by hypothesis, so that the quadratic form

$$\sum_{i,k=1}^{n} \{A'\}_{ik}\, \xi_i\, \xi_k$$

is positive definite. On putting $\xi_s = 1$ and the remaining ξ_i equal to zero, it is seen that $\{A'\}_{ss} > 0$ $(s = 1, 2, \ldots, n)$. On the other hand,

$$\mathrm{Sp}\,(AB) = \mathrm{Sp}\,\big(A'\,[\mu_1, \mu_2, \ldots, \mu_n]\big) = \sum_{s=1}^{n} \{A'\}_{ss}\, \mu_s,$$

whence it follows directly that, if all the $\mu_s \leqslant 0$, then $\mathrm{Sp}(AB) \leqslant 0$, whilst if all the $\mu_s \geqslant 0$, then $\mathrm{Sp}(AB) \geqslant 0$, and the lemma is proved.

A further fact must be mentioned, which we shall require below. Let $u(P) = U(x_1, x_2, \ldots, x_n)$ be a real continuous function defined inside a domain D of space $(x_1, x_2, \ldots, x_n)$ and having continuous derivatives up to the second order inside D. Suppose that $u(P)$ has

a maximum at some interior point P_0 of D. In this case the real symmetric form

$$\sum_{i,k=1}^{n} u_{x_i x_k}(P_0)\,\xi_i\,\xi_k$$

cannot take positive values [cf. III_1, 35], i.e. all the characteristic roots of the real symmetric matrix $u_{x_i x_k}(P_0)$ are non-positive. Similarly, they are non-negative at a minimum point.

We take the linear elliptic equation:

$$L(u) = \sum_{i,k=1}^{n} a_{ik}\,u_{x_i x_k} + \sum_{i=1}^{n} b_i\,u_{x_i} + cu = f, \qquad (289)$$

where a_{ik}, b_i, c and f are taken to be continuous functions in some finite domain $\overline{D}$. We assume that the quadratic form

$$\sum_{i,k=1}^{n} a_{ik}\,\xi_i\,\xi_k$$

is positive definite in $\overline{D}$. We remark that the sum

$$\sum_{i,k=1}^{n} a_{ik}\,u_{x_i x_k} = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ik}\,u_{x_i x_k}\right) \qquad (290)$$

is the trace of the product of the real symmetric matrices $\| a_{ik} \|$ and $\| u_{x_i x_k} \|$, the function $u(P)$ being assumed to have continuous derivatives up to the second order inside D. The uniqueness of the solution of Dirichlet's problem for equation (289) is based on the following theorem.

THEOREM. *If $c < 0$ inside D, every solution of the homogeneous equation*

$$L(u) = 0 \qquad (291)$$

lacks either a positive maximum or negative minimum inside D.

We use *reductio ad absurdum*. Let $u(P)$ have a maximum $u(P_0) > 0$ at a point P_0 inside D. All the first order derivatives must vanish at this point, and equation (291) gives

$$\sum_{i,k=1}^{n} a_{ik}\,u_{x_i x_k} = -\,cu \qquad (\text{at the point } P_0). \qquad (292)$$

It follows from the condition for a maximum that all the characteristic roots of the matrix $\| u_{x_i x_k} \|$ are non-positive, and by the lemma proved above, the left-hand side of (292) is $\leqslant 0$, whilst the right-hand side is positive, since $u(P_0) > 0$ by hypothesis, and $c(P_0) < 0$. This

contradiction proves the theorem. The impossibility of a negative minimum may be proved in a similar way, or this case can be reduced to the previous one by replacing u by $(-u)$.

It can similarly be shown that, if $f(P) \geqslant 0$ inside D, the solution of the non-homogeneous equation cannot (289), have a positive maximum inside D; and if $f(P) \leqslant 0$, it cannot have a negative minimum.

It follows immediately from the above theorem that, if $c < 0$ inside D, the solution of the Dirichlet problem for equation (289) is unique. For, let $u_1(P)$ and $u_2(P)$ be two solutions of equation (289) inside D, which are continuous in the closed domain $\overline{D}$ and satisfy on the boundary S of D the same boundary condition

$$u\,|_S = \varphi\,(P)\,. \tag{293}$$

The difference $v(P) = u_1(P) - u_2(P)$ must now satisfy the homogeneous equation $L(v) = 0$ and vanish on S. It follows from this, by our theorem, that $v(P) \equiv 0$ in $\overline{D}$ i.e. $u_1(P) \equiv u_2(P)$ in $\overline{D}$. For, if this were not the case, $v(P)$ would have to have positive maxima or negative minima (or both) inside D, and this is impossible by virtue of the theorem. The uniqueness of the solution of the Dirichlet problem can be proved with the assumption that $c \leqslant 0$ inside D. In this case we replace the above-mentioned function $v(P)$ by a new function $w(P)$, where

$$v = (\alpha - e^{-\beta x_1})\, w, \tag{294}$$

and the numbers α and β, which are defined below, are such that the difference $\alpha - e^{-\beta x_1}$ is positive in $\overline{D}$. On substituting (294) in the equation $L(v) = 0$, we obtain an equation for w of the form

$$\sum_{i,k=1}^{n} a_{ik}\, w_{x_i x_k} + \sum_{i=1}^{n} b'_i\, w_{x_i} + c'\, w = 0\,, \tag{295}$$

where the b'_i, like the b_i, are continuous in $\overline{D}$ and

$$c' = c + e^{-\beta x_1}\, \frac{b_1\, \beta - a_{11}\, \beta^2}{\alpha - e^{-\beta x_1}}\,, \tag{296}$$

and, by (294), $w = 0$ on S.

Since $a_{11} > 0$ in D, we can choose β so as to have $b_1\beta - a_{11}\,\beta^2 < 0$ in $\overline{D}$; then we choose α so large that $\alpha - e^{-\beta x_1} > 0$ in $\overline{D}$. We now have $c' < 0$ inside D in equation (295), and we can apply our theorem to the function $w(P)$, which vanishes on S; this gives us $w(P) \equiv 0$ in $\overline{D}$. But it now follows from (294) that $v(P) = 0$ in $\overline{D}$.

The assumption that $c \leqslant 0$, is essential for the uniqueness of the solution of the Dirichlet problem. It is easy to quote an example

when, with $c > 0$, the homogeneous equation $L(v) = 0$ has a solution which vanishes on the boundary S but is not identically zero. Let us take as our example the equation

$$v_{x_1x_1} + v_{x_2x_2} + 2k^2v = 0 \tag{297}$$

and consider it in the square $0 \leqslant x_1 \leqslant \pi$, $0 \leqslant x_2 \leqslant \pi$. If k is an integer, (297) has the solution

$$v = \sin kx_1 \sin kx_2,$$

which vanishes on the boundary of the above square. We recall that the equation

$$\Delta v + \lambda v = 0 \tag{298}$$

has in general an infinite set of positive eigenvalues $\lambda = \lambda_1$; $\lambda = \lambda_2$, ..., such that, with $\lambda = \lambda_k$, the equation has solutions which are not identically zero and which vanish on the boundary S of the given domain.

Note. The above results can be used to obtain certain inequalities for the solution of the Dirichlet problem. We shall indicate the simplest of them.

Let $u(P)$ be a solution of the problem

$$L(u) = f \text{ inside } D; \qquad u|_S = 0. \tag{299}$$

We take $c < 0$ in $\overline{D}$. Let μ denote the least value of $|c|$ and M the greatest value of $|f|$ in $\overline{D}$.

We introduce a function v by putting $v = u + k$, where k is a constant. We have: $L(v) = L(u) + ck = f + ck$, so that $v(P)$ is a solution of the problem

$$L(v) = f + ck \text{ inside } D; \qquad v|_S = k. \tag{300}$$

Suppose first that $k = M/\mu$. Then $f + ck < 0$ in $\overline{D}$, and the solutions of the equation $L(v) = f + ck$ cannot have negative minima inside D. But $v(P)$ is equal on S to the positive constant M/μ, whence it follows that $v(P)$ cannot be negative, i.e. $v = u + M/\mu \geqslant 0$ or $u \geqslant -M/\mu$. Similarly, by putting $k = -M/\mu$, we get $u \leqslant M/\mu$. Finally, we can say that the solution of problem (299) (if it exists) must satisfy in $\overline{D}$ the inequality:

$$|u| \leqslant \frac{M}{\mu}.$$

We now take the problem

$$L(u) = 0 \text{ inside } D; \qquad u|_S = \varphi \tag{301}$$

and write N for the greatest value of $|\varphi|$ on S. On again putting $v = u + k$, we arrive at the problem

$$L(v) = ck \text{ inside } D; \qquad v|_S = \varphi + k. \tag{302}$$

We put $k = N$. Since we are assuming $c < 0$ inside D, the function v cannot have negative minima inside D. We obviously have $\varphi + N \geqslant 0$ for the boundary values of $v(P)$. Hence it follows, as above, that $v = u + N \geqslant 0$, i.e. $u \geqslant -N$. Similarly, by setting $k = -N$, we get $u \leqslant N$, i.e. for problem (301): $|u| \leqslant N$ in $\overline{D}$.

235. The equation $\Delta v - \lambda v = 0$. We consider the equation

$$\Delta v - \lambda v = 0, \tag{303}$$

where λ is a given positive number, and pose the interior Dirichlet problem with the boundary condition

$$v|_S = f(N). \tag{304}$$

The solutions of (303) can have neither positive maxima nor negative minima inside D_i [233], and hence it follows that the solution of this Dirichlet problem is unique.

If the function $f(N)$ satisfies the inequality $-a \leqslant f(N) \leqslant b$, where a and b are positive numbers, the same inequality must be satisfied in D_i by the solution of the Dirichlet problem.

Let us first take the non-homogeneous equation

$$\Delta v - \lambda v = -\varphi(P) \qquad (\text{inside } D_i) \tag{305}$$

with the homogeneous boundary condition

$$v|_S = 0. \tag{306}$$

We assume that $\varphi(P)$ is continuous in the closed domain $\overline{D}_i$ and has continuous derivatives inside D_i. Problem (305), (306) is equivalent to the integral equation [cf. 224]:

$$v(P) = -\lambda \iiint_{D_i} G(P; Q)\, v(Q)\, d\tau + \iiint_{D_i} G(P; Q)\, \varphi(Q)\, d\tau, \tag{307}$$

where $G(P; Q)$ is Green's function for Laplace's equation with boundary condition (306). Since $(-\lambda)$ is a negative number, whilst all the eigenvalues of the kernel $G(P; Q)$ are positive, equation (307) has a unique solution whatever the function φ and this solution is the solution of problem (305), (306).

We now turn to the solution of the Dirichlet problem (303) and (304). Let $w(P)$ be the solution of the Dirichlet problem for Laplace's equation with boundary condition (304). The function

$$u(P) = v(P) - w(P) \tag{308}$$

must satisfy the equation

$$\Delta u - \lambda u = \lambda w$$

and the boundary condition

$$u|_S = 0.$$

We have just proved the existence of the solution of this problem. Knowing $u(P)$, we can find the solution $v(P)$ of the Dirichlet problem in accordance with (308).

The basic singular solution of (303) is

$$v_0(P) = \frac{e^{-\sqrt{\lambda}r}}{r}, \tag{309}$$

where r is the distance of a point P from some fixed point Q. A potential theory, similar to that of [230], can be constructed on the basis of this solution. We shall not dwell on this, and pass on to determining Green's function.

Green's function $G_1(P, Q; \lambda)$ of equation (303) with boundary condition (306) is a function of the point P, continuous inside D_i and as far as S, except at the point Q, having continuous derivatives up to the second order everywhere inside D_i except at Q, satisfying equation (303) inside D_i and boundary condition (306) on S, and of the form:

$$G_1(P, Q; \lambda) = \frac{e^{-\sqrt{\lambda}r}}{4\pi r} + g_1(P, Q; \lambda), \tag{310}$$

where $g_1(P; Q; \lambda)$ has continuous derivatives up to the second order everywhere inside D_i. The function $g_1(P, Q; \lambda)$ is the solution of Dirichlet's problem for equation (303) with the boundary condition:

$$g_1(P, Q; \lambda)\Big|_S = -\frac{e^{-\sqrt{\lambda}r}}{4\pi r}\Big|_S. \tag{311}$$

It can be shown, precisely as in [220], that $g_1(P, Q; \lambda)$ is a continuous function of the pair of points P and Q, and that the inequality holds inside D_i:

$$0 < G_1(P, Q; \lambda) < \frac{e^{-\sqrt{\lambda}r}}{4\pi r} \qquad (\text{inside } D_i;\ r = |PQ|). \tag{312}$$

Furthermore, the symmetry of $G_1(P, Q; \lambda)$ can be proved in the same way as in [221].

The solution of equation (305) with condition (306) can be expressed by

$$v(P) = \iiint\limits_{D_i} G_1(P, Q; \lambda)\, \varphi(Q)\, \mathrm{d}\tau. \tag{313}$$

This is proved as in [224]. The integral

$$\iiint\limits_{D_i} g_1(P, Q; \lambda)\, \varphi(Q)\, \mathrm{d}\tau$$

satisfies inside D_i the homogeneous equation (303) [224]. The integral with the singular part can be written as

$$\iiint\limits_{D_i} \frac{\mathrm{e}^{-\sqrt{\lambda} r}}{4\pi r}\varphi(Q)\, \mathrm{d}\tau_Q = \iiint\limits_{D_i} \frac{\varphi(Q)}{4\pi r}\mathrm{d}\tau + \iiint\limits_{D_i} \left(\frac{\mathrm{e}^{-\sqrt{\lambda}\, r}}{4\pi r} - \frac{1}{4\pi r}\right)\varphi(Q)\, \mathrm{d}\tau.$$

Poisson's formula is applicable to the first term, whilst the polarity is isolated in the second term and double differentiation is possible under the integral sign. Hence it follows at once that application of the operator $(\Delta - \lambda)$ to (313) gives $[-\varphi(P)]$. Boundary condition (306) for function (313) may be verified as in [224].

A different approach to the concept of Green's function namely the one used in [172], is possible. We take the non-homogeneous equation (305) and assume that $\varphi(P)$ vanishes everywhere except on a sphere D_ε with centre Q and small radius ε, where

$$\iiint\limits_{D_\varepsilon} \varphi(P)\, \mathrm{d}\tau = 1. \tag{314}$$

Turning to integral equation (307), we can write its solution in the form [8]:

$$v(P) = \iiint\limits_{D_i} R(P, Q'; \lambda)\, \varphi(Q')\, \mathrm{d}\tau_{Q'}, \tag{315}$$

where $R(P, Q; \lambda)$ is the resolvent of equation (307). On taking into account the definition of $\varphi(Q')$, we can anticipate that the left-hand side of (315) tends to $G_1(P, Q; \lambda)$, and the right to $R(P, Q; \lambda)$ as ε tends to zero, so that

$$G_1(P, Q; \lambda) = R(P, Q; \lambda),$$

i.e. Green's function $G_1(P, Q; \lambda)$ is the resolvent of integral equation (307).

This leads naturally to the relationship:

$$G_1(P, Q; \lambda) = G(P, Q) - \lambda \iiint\limits_{D_i} G(P, Q')\, G_1(Q', Q; \lambda))\, \mathrm{d}\tau_Q, \tag{316}$$

which can easily be proved on the basis of the fact that the difference $H(P, Q) = G_1(P, Q; \lambda) - G(P, Q)$ satisfies the equation $\Delta H(P, Q) = \lambda G_1(P, Q; \lambda)$ and the boundary condition (306) and retains its continuity at the point Q. But we have had an expression for the resolvent as a series in eigenfunctions of the kernel [30], which gives us in the present case:

$$G_1(P, Q; \lambda) = G(P, Q) - \lambda \sum_{k=1}^{\infty} \frac{v_k(P)\, v_k(Q)}{\lambda_k(\lambda_k + \lambda)},$$

where λ_k and $v_k(P)$ are the eigenvalues and eigenfunctions of the kernel $G(P, Q)$, i.e. of equation (231) with condition (232). Comparison with (316) gives us

$$\iiint_{D_i} G(P;\, Q')\, G_1(Q', Q; \lambda)\, d\tau_{Q'} = \sum_{k=1}^{\infty} \frac{v_k(P)\, v_k(Q)}{\lambda_k(\lambda_k + \lambda)}. \tag{317}$$

We have no strict basis for what has been said above. A proof will now be given of (317), which we shall require below.

We first of all recall that the series

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2}. \tag{318}$$

where λ_k are the eigenvalues of the kernel $G(P; Q)$, is convergent.

We find the Fourier coefficients of the function $G_1(Q', Q; \lambda)$ with respect to the eigenfunctions of the kernel $G(P, Q)$:

$$h_k = \iiint_{D_i} G_1(Q', Q; \lambda)\, v_k(Q')\, d\tau_{Q'}.$$

On substituting $v_k(Q') = -\Delta\, v_k(Q')/\lambda_k$, we obtain

$$\lambda_k h_k = -\iiint_{D_i} G_1(Q', Q; \lambda)\, \Delta v_k(Q')\, d\tau_{Q'}.$$

It follows from the last two formulae that

$$(\lambda_k + \lambda)\, h_k = -\iiint_{D_i} G_1(Q', Q; \lambda)\left[\Delta v_k(Q') - \lambda v_k(Q')\right] d\tau_{Q'}. \tag{319}$$

On taking into account the symmetry of $G_1(Q', Q; \lambda)$ and the fact that (313) gives the solution of equation (305) that satisfies condition (306), we can say that the right-hand side of (319) is equal to $v_k(Q)$. In the present case the role of $\varphi(Q)$ of (313) is played by

$$-\left[\Delta v_k(Q') - \lambda v_k(Q')\right] = (\lambda + \lambda_k)\, v_k(Q').$$

This function has continuous derivatives inside D_i, and if we take it as the right-hand side of equation (305), the solution of this equ tion

that satisfies condition (306) (such a solution is unique) will be $v(P) = v_k(P)$. Formula (319) gives:

$$h_k = \frac{v_k(Q)}{\lambda_k + \lambda}. \tag{320}$$

The right-hand side of (317) is therefore the Fourier series of the left-hand side, this latter being a function expressible in terms of a kernel. The series on the right-hand side of (317) is regularly convergent with respect to P for any fixed Q. This follows from the inequalities:

$$\sum_{k=1}^{\infty} \frac{v_k^2(P)}{\lambda_k^2} = \iiint_{D_i} G^2(P;\, Q)\, \mathrm{d}\tau \leqslant C; \qquad \sum_{k=m}^{m+p} h_k^2 \leqslant \varepsilon$$

precisely as in [22]. The first of these formulae expresses the closure equation for the function $G(P, Q)$ [225]. We remark further that the left-hand side of (317) is a function of points P and Q continuous in the closed domain $\overline{D}_i$. The proof of this follows the same lines as the proof in [II, 200] that the volume potential and its first order derivatives are continuous. We remark that the term with greatest polarity in the integrand of the left-hand side of (317) is equal to

$$\frac{\mathrm{e}^{-\lambda r}}{rr'},$$

where r and r' are the distances of Q' from P and Q.

The validity of (317) follows from what has been said above. When P and Q coincide, we obtain the formula:

$$\sum_{k=1}^{\infty} \frac{v_k^2(P)}{\lambda_k(\lambda_k + \lambda)} = \iiint_{D_i} G(P;\, Q')\, G_1(Q',\, P;\, \lambda)\, \mathrm{d}\tau_{Q'}, \tag{321}$$

the series being uniformly convergent in the closed domain D_i, since the right-hand side is a continuous function of P by what has been said above [23]. Integration of (321) over D_i gives:

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k(\lambda_k + \lambda)} = \iiint_{D_i} \psi(P,\, \lambda)\, \mathrm{d}\tau, \tag{322}$$

where

$$\psi(P,\, \lambda) = \iiint_{D_i} G(P;\, Q)\, G_1(Q,\, P;\, \lambda)\, \mathrm{d}\tau_Q. \tag{323}$$

We shall use (322) when investigating the numbers λ_k.

236. An asymptotic expression for the eigenvalues. As a preliminary, we must discuss certain properties of the function $\psi(P, \lambda)$. In view of the inequalities for G and G_1, we have:

$$|\psi(P, \lambda)| \leqslant \iiint\limits_{D_i} \frac{e^{-\sqrt{\lambda} r}}{16\pi^2 r^2} \, d\tau_Q \qquad (r = |PQ|). \tag{324}$$

On integrating over the whole of space and introducing spherical coordinates with centre P, we obtain:

$$|\psi(P, \lambda)| \leqslant \frac{1}{16\pi^2} \int\limits_0^\infty \int\limits_0^\pi \int\limits_0^{2\pi} e^{-\sqrt{\lambda} r} \sin\theta \, dr \, d\theta \, d\varphi = \frac{1}{4\pi\sqrt{\lambda}}. \tag{325}$$

We now show that the product $\sqrt{\lambda}\,\psi(P, \lambda)$ tends uniformly to $1/4\pi$ as $\lambda \to +\infty$ in any closed domain D' contained inside D_i:

$$\sqrt{\lambda}\,\psi(P, \lambda) \to \frac{1}{4\pi} \text{ uniformly in } D'. \tag{326}$$

On taking into account the boundary values of $g(P, Q)$ and $g_1(P, Q; \lambda)$ on S, we obtain the inequalities:

$$0 \geqslant g(P; Q) \geqslant -\frac{1}{4r'}; \quad 0 \geqslant g_1(P, Q; \lambda) \geqslant -\frac{e^{-\sqrt{\lambda} r'}}{4\pi r'} \; (P \text{ in } D'),$$

where r' is the distance from the boundary of D' to S. We have:

$$\sqrt{\lambda}\,\psi(P, \lambda) = \sqrt{\lambda} \iiint\limits_{D_i} \left[\frac{1}{4\pi r} + g(P; Q)\right]\left[\frac{e^{-\sqrt{\lambda} r}}{4\pi r} + g_1(P, Q; \lambda)\right] d\tau_Q.$$

We remove the brackets and split the integral into four terms:

$$\left|\sqrt{\lambda} \iiint\limits_{D_i} g(P; Q)\, g_1(P, Q; \lambda)\, d\tau_Q\right| \leqslant \sqrt{\lambda}\frac{e^{-\sqrt{\lambda} r'}}{16\pi r'}, \text{ volume of } D_i,$$

whence it is clear that the integral on the left tends uniformly to zero when $\lambda \to +\infty$ if P belongs to D'. We have further:

$$\left|\sqrt{\lambda} \iiint\limits_{D_i} \frac{1}{4\pi r} g_1(P, Q; \lambda)\, d\tau_Q\right| \leqslant \sqrt{\lambda}\frac{e^{-\sqrt{\lambda} r'}}{16\pi^2 r'} \iiint\limits_{D_i} \frac{d\tau_Q}{r},$$

and the integral on the right does not exceed a certain constant for any position of P in D_i, whence it follows that the integral on the

left tends uniformly to zero. We can consider in precisely the same way the integral:

$$\sqrt{\lambda}\iiint_{D_i} \frac{e^{-\sqrt{\lambda} r}}{4\pi r} g(P; Q)\, d\tau_Q .$$

It remains to consider the integral:

$$\sqrt{\lambda}\iiint_{D} \frac{e^{-\sqrt{\lambda} r}}{16\pi^2 r^2}\, d\tau_Q \tag{327}$$

and to show that it tends uniformly to $1/4\pi$ if P belongs to D'. Let D_0 and D_1 be spheres with centre P and radii equal to r' and the diameter d of the domain D_i. We have:

$$\sqrt{\lambda}\iiint_{D_0} \frac{e^{-\sqrt{\lambda} r}}{16\pi^2 r^2}\, d\tau_Q \leqslant \sqrt{\lambda}\iiint_{D_i} \frac{e^{-\sqrt{\lambda} r}}{16\pi^2 r^2}\, d\tau_Q \leqslant \sqrt{\lambda}\iiint_{D_1} \frac{e^{-\sqrt{\lambda} r}}{16\pi^2 r^2}\, d\tau_Q .$$

We express the integrals over D_0 and D_1 in spherical coordinates with centre P and introduce the new variable $\varrho = \sqrt{\lambda}\, r$. Hence we arrive at the inequality:

$$\frac{1}{4\pi}\int_0^{\sqrt{\lambda} r} e^{-\varrho}\, d\varrho \leqslant \sqrt{\lambda}\iiint_{D_i} \frac{e^{-\sqrt{\lambda} r}}{16\pi^2 r^2}\, d\tau \leqslant \frac{1}{4\pi}\int_0^{\sqrt{\lambda}\, l} e^{-\varrho}\, d\varrho .$$

The extreme terms tend to $1/4\pi$ as $\lambda \to +\infty$, and they do not depend on the position of the point P in D'. It follows at once from this that integral (327) tends uniformly to $1/4\pi$ in D', and our assertion (326) is therefore proved. On taking (325) into account, we can take D' so close to D_i that the integral of $\sqrt{\lambda}\psi(P, \lambda)$ over $(D_i - D')$ is less than $\varepsilon/2$, where ε is a given positive number. On the other hand, by (326), we have for sufficiently large λ:

$$\left|\iiint_{D'} \sqrt{\lambda}\, \psi(P, \lambda)\, d\tau - \frac{v'}{4\pi}\right| \leqslant \frac{\varepsilon}{2},$$

where v' is the volume of D', whence

$$\left|\iiint_{D_i} \sqrt{\lambda}\, \psi(P, \lambda)\, d\tau - \frac{v}{4\pi}\right| \leqslant \frac{1}{4\pi}(v - v') + \varepsilon ,$$

where v is the volume of D_i. Hence it follows that

$$\lim_{\lambda \to +\infty} \iiint_{D_i} \sqrt{\lambda}\, \psi(P, \lambda)\, d\tau = \frac{v}{4\pi},$$

and, by (322):

$$\lim_{\lambda\to+\infty} \sqrt{\lambda} \sum_{k=1}^{\infty} \frac{1}{\lambda_k(\lambda_k+\lambda)} = \frac{v}{4\pi}. \tag{328}$$

The employment of this formula for deducing an asymptotic expression for λ_k is based on the following theorem:

THEOREM. *If the series*

$$s(\lambda) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k+\lambda} \qquad (c_k > 0; \quad \lambda_k > 0), \tag{329}$$

where $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \lambda_n \to \infty$, *is convergent for* $\lambda > 0$ *and*

$$\lim_{\lambda\to+\infty} \sqrt{\lambda}\, s(\lambda) = H, \tag{330}$$

hen

$$\lim_{\lambda\to+\infty} \frac{1}{\sqrt{\lambda}} \sum_{\lambda_k\leqslant\lambda} c_k = \frac{2H}{\pi}, \tag{331}$$

the summation in the last sum being over the k *for which* $\lambda_k \leqslant \lambda$.

We apply this theorem to series (328). In this case $c_k = 1/\lambda_k$ and $H = v/4\pi$, and we get:

$$\lim_{\lambda\to+\infty} \frac{1}{\sqrt{\lambda}} \sum_{\lambda_k\leqslant\lambda} \frac{1}{\lambda_k} = \frac{v}{2\pi^2}$$

or what amounts to the same thing:

$$\sum_{\lambda_k\leqslant\lambda} \frac{1}{\lambda_k} = \frac{v}{2\pi^2}\sqrt{\lambda} + \varepsilon(\lambda)\sqrt{\lambda}, \tag{332}$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to +\infty$. If we take $\lambda = \lambda_n$, we obtain:

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} = \frac{v}{2\pi^2}\sqrt{\lambda_n} + \varepsilon_n\sqrt{\lambda_n} \qquad (\varepsilon_n \to 0). \tag{333}$$

We use the notation:

$$\sigma_r = \sum_{k=1}^{n} \frac{1}{\lambda_k}$$

and write $\varphi(\lambda)$ for the left-hand side of (332). This latter is a non-decreasing function of λ:

$$\varphi(\lambda) = 0 \text{ for } \lambda < \lambda_1 \text{ and } \varphi(\lambda) = \sigma_m \text{ for } \lambda_m \leqslant \lambda < \lambda_{m+1}. \tag{334}$$

We deduce an asymptotic expression for λ_n with large n. We have:

$$n = \sum_{k=1}^{n} \lambda_k \cdot \frac{1}{\lambda_k} = \sigma_1 (\lambda_1 - \lambda_2) + \sigma_2 (\lambda_2 - \lambda_3) + \ldots + \\ + \sigma_{n-1} (\lambda_{n-1} - \lambda_n) + \sigma_n \lambda_n. \qquad (335)$$

The non-decreasing function

$$\varphi(\lambda) = \frac{v}{2\pi^2} \sqrt{\lambda} + \varepsilon(\lambda) \sqrt{\lambda} \qquad (336)$$

is integrable over any finite interval, and hence the second term on the right is also an integrable function. We have by (332) and (334):

$$\int_0^{\lambda_n} \varphi(\lambda)\, \mathrm{d}\lambda = \sigma_1 (\lambda_2 - \lambda_1) + \sigma_2 (\lambda_3 - \lambda_2) + \ldots + \sigma_{n-1} (\lambda_n - \lambda_{n-1}) = \\ = \frac{v}{3\pi^2} \lambda_n^{3/2} + \int_0^{\lambda_n} \varepsilon(\lambda) \sqrt{\lambda}\, \mathrm{d}\lambda. \qquad (337)$$

It may easily be shown that

$$\frac{1}{\lambda_n^{3/2}} \int_0^{\lambda_n} \varepsilon(\lambda) \sqrt{\lambda}\, \mathrm{d}\lambda \to 0 \quad \text{as} \quad n \to \infty. \qquad (338)$$

Let δ be a given positive number. We fix p so large that $|\varepsilon(\lambda)| \leqslant \delta$ for $\lambda \geqslant \lambda_p$. We have:

$$\left| \int_0^{\lambda_n} \varepsilon(\lambda) \sqrt{\lambda}\, \mathrm{d}\lambda \right| \leqslant \int_0^{\lambda_p} |\varepsilon(\lambda)| \sqrt{\lambda}\, \mathrm{d}\lambda + 2\delta \frac{(\lambda_n^{3/2} - \lambda_p^{3/2})}{3} \qquad (n > p),$$

whence it follows that

$$\left| \frac{1}{\lambda_n^{3/2}} \int_0^{\lambda_n} \varepsilon(\lambda) \sqrt{\lambda}\, \mathrm{d}\lambda \right| \leqslant \frac{2}{3}\delta + \left[\frac{1}{\lambda_n^{3/2}} \int_0^{\lambda_p} |\varepsilon(\lambda)| \sqrt{\lambda}\, \mathrm{d}\lambda - \frac{2\delta \lambda_p^{3/2}}{3\lambda_n^{3/2}} \right].$$

With sufficiently large n the absolute value of the square bracket is $\leqslant \delta/3$, i.e.

$$\left| \frac{1}{\lambda_n^{3/2}} \int_0^{\lambda_n} \varepsilon(\lambda) \sqrt{\lambda}\, \mathrm{d}\lambda \right| \leqslant \delta \text{ for large } n,$$

whence (338) follows. We thus obtain, by (337):

$$\sigma_1 (\lambda_2 - \lambda_1) + \sigma_2 (\lambda_3 - \lambda_2) + \ldots + \sigma_{n-1} (\lambda_n - \lambda_{n-1}) = \frac{v}{3\pi^2} \lambda_n^{3/2} + \varepsilon_n' \lambda_n^{3/2},$$

where $\varepsilon_n' \to 0$ as $n \to \infty$. On substituting this in (335) and using (333), we get:

$$n = \frac{v}{6\pi^2} \lambda_n^{3/2} + \varepsilon_n'' \lambda_n^{3/2}, \tag{339}$$

where $\varepsilon_n'' \to 0$. Hence

$$\lambda_n = \left(\frac{6\pi^2 n}{v}\right)^{\frac{2}{3}} \left(1 + \frac{6\pi^2}{v} \varepsilon_n''\right)^{-\frac{2}{3}};$$

and finally,

$$\lambda_n = \left(\frac{6\pi^2 n}{v}\right)^{\frac{2}{3}} + \varepsilon_n''' n^{\frac{2}{3}} \qquad (\varepsilon_n''' \to 0). \tag{340}$$

In the case of a plane domain the result has the form:

$$\lambda_n = \frac{4\pi n}{S} + \varepsilon_n''' n, \tag{341}$$

where S is the area of the domain.
Thus it all comes down to proving the theorem on series (329).

Carleman used the method described above in his article "Ueber die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen" (*Berichte der Sächsisch. Akad. der Wiss. zu Leipzig. Math. Phys. Klass*, Bd. LXXXVIII, 1936) to obtain asymptotic expressions for the eigenvalues of equations of a general type.

His results are as follows. Suppose we have the expression:

$$L(u) = \sum_{p,q=1}^{3} a_{pq} \frac{\partial^2 u}{\partial x_p \, \partial x_q} + \sum_{p=1}^{3} a_p \frac{\partial u}{\partial x_p} + au \qquad (a_{qp} = a_{pq}),$$

where a_{pq}, a_p and a are given real continuous functions in the closed domain D of space (x_1, x_2, x_3). Further, let the quadratic form

$$\sum_{p,q=1}^{3} a_{pq} \xi_p \xi_q$$

of variables ξ_k be positive definite if the point (x_1, x_2, x_3) lies in the closed domain D. We consider the boundary value problem:

$$L(u) + \lambda u = 0$$

with condition (302). It has an infinite set of eigenvalues, which may in fact be complex. There is only a finite number of eigenvalues in any bounded part of the plane, and if these are arranged in order of non-decreasing modulus, the formula holds:

$$\lim_{n \to \infty} \frac{n}{\lambda_n^{3/2}} = \frac{1}{6\pi^2} \iiint\limits_D \frac{dv}{\sqrt{\Delta}},$$

where Δ is the determinant consisting of the elements a_{pq}. We remark that, since the quadratic form is positive, $\Delta > 0$ and the right-hand side of the last formula is real.

In Courant and Hilbert's *Methoden der Mathematischen Physik*, Vol. I, the asymptotic expressions obtained above for the eigenvalues λ_n with large n for the equation $\Delta u + \lambda u = 0$ are established with the aid of the extremal properties of the eigenvalues. We explained this method in [188] for the case of a single independent variable. It becomes more difficult to apply in the case of the equation $\Delta u + \lambda u = 0$.

237. Proof of the auxiliary theorem. Before turning to the proof of the theorem of the previous section, we must establish some auxiliary formulae and prove a number of lemmas.

We introduce the following notation:

$$\varphi(\lambda) = \sum_{\lambda_k \leqslant \lambda} c_k\,; \tag{342}$$

$$\sigma_n = \varphi(\lambda_n) = \sum_{k=1}^{n} c_k. \tag{343}$$

The summation in (342) is carried out over the k for which $\lambda_k \leqslant \lambda$. The function $\varphi(\lambda)$ is a non-decreasing, non-negative function of λ:

$$\varphi(\lambda) = 0 \quad \text{for} \quad \lambda < \lambda_1\,; \quad \varphi(\lambda) = \sigma_m \quad \text{for} \quad \lambda_m \leqslant \lambda < \lambda_{m+1}. \tag{344}$$

Since $\lambda_k + \lambda \leqslant 2\lambda$ for $\lambda_k \leqslant \lambda$, we can write:

$$\varphi(\lambda) \leqslant 2\lambda \sum_{\lambda_k \leqslant \lambda} \frac{c_k}{\lambda_k + \lambda}$$

and we obtain on taking (330) into account:

$$\varphi(\lambda) = O\left(\sqrt{\lambda}\right), \tag{345}$$

i.e. the ratio $\varphi(\lambda) : \sqrt{\lambda}$ remains bounded as $\lambda \to \infty$. We have further:

$$\sum_{k=1}^{n} \frac{c_k}{\lambda_k + \lambda} = \sum_{k=1}^{n-1} \sigma_k \left(\frac{1}{\lambda_k + \lambda} - \frac{1}{\lambda_{k+1} + \lambda} \right) + \frac{\sigma_n}{\lambda_n + \lambda},$$

where

$$\frac{1}{\lambda_k + \lambda} - \frac{1}{\lambda_{k+1} + \lambda} = \int_{\lambda_k}^{\lambda_{k+1}} \frac{dx}{(x + \lambda)^2},$$

and we can write, on taking (334) into account:

$$\sum_{k=1}^{n} \frac{c_k}{\lambda_k + \lambda} = \int_0^{\lambda_n} \frac{\varphi(x)}{(x + \lambda)^2}\, dx + \frac{\sigma_n}{\lambda_n + \lambda}.$$

But $\sigma_n = \varphi(\lambda_n)$, and it follows from (345) that $\sigma_n : (\lambda_n + \lambda) \to 0$ as $n \to \infty$, so that the last formula gives:

$$s(\lambda) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k + \lambda} = \int_0^{\infty} \frac{\varphi(x)}{(x+\lambda)^2}\, dx. \tag{346}$$

It follows at once from (345) that the integrand is of order $1/x^{3/2}$ as $x \to \infty$. For the sake of brevity we introduce the following notation. If $\psi(\lambda) = a\,\lambda^b + \varepsilon(\lambda)\,\lambda^b$, where $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$, we shall write: $\psi(\lambda) \sim a\,\lambda^b$ [cf. III_2, 106]. Let us prove two lemmas:

LEMMA I. *If $f(\lambda)$ is defined for all sufficiently large positive λ, has a continuous derivative, $\lambda\, f'(\lambda)$ does not decrease as λ increases and $f(\lambda) \sim a\,\lambda^q$ $(q > 0)$, then $f'(\lambda) \sim aq\,\lambda^{q-1}$.*

We shall first prove this lemma for $a = 1$ and $q = 1$. We have $f(\lambda) \sim \lambda$ and we want to prove that $f'(\lambda) \sim 1$, i.e. that $f'(\lambda) \to 1$ as $\lambda \to \infty$.

Suppose that the reverse is true. If $f'(\lambda)$ does not tend to unity, there exists a sequence of λ_n such that $\lambda_n \to \infty$ and $f'(\lambda_n) \to h$, where the number h differs from unity. Suppose that say $h > 1$. Let γ be a positive number. Since $\lambda\, f'(\lambda)$ is a non-decreasing function, we can write:

$$\frac{f(\lambda_n + \gamma\lambda_n) - f(\lambda_n)}{\gamma\lambda_n} = \frac{1}{\gamma\lambda_n} \int_{\lambda_n}^{\lambda_n+\gamma\lambda_n} f'(\lambda)\, d\lambda$$

$$\geqslant \frac{\lambda_n f'(\lambda_n)}{\gamma\lambda_n} \int_{\lambda_n}^{\lambda_n+\gamma\lambda_n} \frac{d\lambda}{\lambda} = \frac{f'(\lambda_n)}{\gamma} \log(1+\gamma).$$

The right-hand side tends to the number $h/\gamma \log(1+\gamma)$, which is greater than unity if we take γ sufficiently close to zero. But it follows at once from $f(\lambda) \sim \lambda$ that we must have:

$$\frac{f(\lambda_n + \gamma\lambda_n) - f(\lambda_n)}{\gamma\lambda_n} \to 1.$$

This contradiction in fact proves the lemma for $a = q = 1$. We turn to the general case. Putting $\mu = \lambda^q$, we replace $f(\lambda)$ by the new function: $f_1(\mu) = (1/a)\, f(\mu^{1/q})$. We have:

$$f_1(\mu) \sim \mu\,(\mu \to \infty); \quad \mu f_1'(\mu) = \frac{1}{aq}\,\mu^{\frac{1}{q}} f'\left(\mu^{\frac{1}{q}}\right) = \frac{1}{aq}\,\lambda f'(\lambda).$$

Therefore $\mu f_1'(\mu)$ is a non-decreasing function, and we can apply to $f_1(\mu)$ the lemma for $a = q = 1$, whence it follows that

$$f_1'(\mu) \sim 1, \quad \text{i.e.} \quad \frac{1}{aq} \mu^{\frac{1}{q}-1} f'\left(\mu^{\frac{1}{q}}\right) \sim 1,$$

so that $f'(\lambda) \sim aq\, \lambda^{q-1}$, which proves the lemma. This present proof remains in force even with $h = \infty$.

We consider the integral

$$K_p = \int_0^\infty \frac{u^{p+\frac{1}{2}}}{(u+1)^{2p+2}}\, du \qquad (p = 1, 2, \ldots). \tag{347}$$

We carry out the change of variables: $u = x/(1-x)$ and transform the integral to the form [III$_2$, 72]:

$$K_p = \int_0^1 x^{p+\frac{1}{2}} (1-x)^{p-\frac{1}{2}} dx = \frac{\Gamma\left(p+\frac{3}{2}\right) \Gamma\left(p+\frac{1}{2}\right)}{\Gamma(2p+2)}. \tag{348}$$

LEMMA II. *Let*

$$K_{p,1} = \int_0^{1-a} \frac{u^{p+\frac{1}{2}}}{(u+1)^{2p+2}}\, du; \qquad K_{p,2} = \int_{1-a}^{1+a} \frac{u^{p+\frac{1}{2}}}{(u+1)^{2p+2}}\, dp;$$

$$K_{p,3} = \int_{1+a}^\infty \frac{u^{p+\frac{1}{2}}}{(u+1)^{2p+2}}\, du,$$

where $0 < a < 1$. *Here,*

$$K_{p,1} \leqslant \delta_p' K_p; \quad K_{p,2} \geqslant (1-\delta_p'') K_p; \quad K_{p,3} \leqslant \delta_p''' K_p, \tag{349}$$

where δ_p', δ_p'' *and* δ_p''', *which depend on the choice of* a, *tend to zero as* $p \to \infty$.

We have had Stirling's formula [III$_2$, 75]:

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} [1 + \varepsilon(z)] \qquad (\varepsilon(z) \to 0 \quad \text{as} \quad z \to +\infty).$$

We obtain by applying it to the right-hand side of (348):

$$K_p = \sqrt{2\pi}\, 2^{-\frac{3}{2}} 2^{-2p} \frac{\left(p+\frac{3}{2}\right)^{p+1} \left(p+\frac{1}{2}\right)^p}{(p+1)^{2p-\frac{3}{3}}} (1+\varepsilon_p),$$

where $\varepsilon_p \to 0$ as $p \to \infty$. The fraction written, multiplied by $\sqrt{p}$, tends to unity as $p \to \infty$, and we can write:

$$K_p = Ap^{-\frac{1}{2}} 2^{-2p} (1+\varepsilon_p') \qquad (\varepsilon_p' \to 0), \tag{350}$$

where $A = \sqrt{2\pi}\, 2^{-3/2}$. The function $u : (u+1)^2$ has a maximum equal to $1/4$ at $u = 1$, whence it follows that

$$K_{p,1} \leqslant k^p \int_0^{1-\alpha} \frac{u^{\frac{1}{2}}}{(u+1)^2}\, \mathrm{d}u < k^p \int_0^{\infty} \frac{u^{\frac{1}{2}}}{(u+1)^2}\, \mathrm{d}u,$$

where $0 < k < 1/4$, and k depends on the choice of α. We thus obtain:

$$K_{p,1} \leqslant A_1 k^p, \tag{351}$$

where $A_1 = \pi/2$, and similarly:

$$K_{p,3} \leqslant A_1 k^p. \tag{352}$$

We have $k^p = (1/4 - \delta)^p$, where $\delta > 0$ and depends on the choice of α. It follows from what has been said that $k^p \cdot 2^{2p}\, p^{1/2} = (1-\delta)^p\, p^{1/2} \to 0$ as $p \to \infty$, and on taking (350), (351) and (352) into account, we get inequalities (349) for $K_{p,1}$ and $K_{p,3}$. We have further:

$$K_{p,2} = K_p - K_{p,1} - K_{p,3} \geqslant K_p - (\delta_p' + \delta_p''')\, K_p,$$

whence inequality (349) follows for $K_{p,2}$ and the lemma is proved.

We turn to the proof of the theorem stated in [236]. By the hypothesis of the theorem:

$$s(\lambda) = \int_0^{\infty} \frac{\varphi(x)}{(x+\lambda)^2}\, \mathrm{d}x \sim H\lambda^{-\frac{1}{2}}. \tag{353}$$

We consider the function $\lambda^2 s(\lambda)$ and show that its derivative is positive and does not decrease when λ increases:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^2 s(\lambda)\right] = 2\int_0^{\infty} \frac{\lambda x \varphi(x)}{(x+\lambda)^3}\, \mathrm{d}x = 2\int_0^{\infty} \frac{u\varphi(\lambda u)}{(u+1)^3}\, \mathrm{d}u.$$

It follows at once from the last expression and the fact that $\varphi(x)$ is non-decreasing that the derivative on the left-hand side is positive and non-decreasing. We can therefore apply Lemma I to the function $\lambda^2 s(\lambda)$ and obtain, on taking (353) into account:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^2 s(\lambda)\right] \sim \frac{3}{2} H\lambda^{\frac{1}{2}},$$

whence

$$-s'(\lambda) = 2\int_0^{\infty} \frac{\varphi(x)}{(x+\lambda)^3}\, \mathrm{d}x \sim \frac{1}{2} H\lambda^{-\frac{3}{2}}. \tag{354}$$

We obtain further:

$$-\lambda^3 s'(\lambda) \sim \frac{1}{2} H\lambda^{3/2}, \qquad -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^3 s'(\lambda)\right] = 2\cdot 3\int_0^\infty \frac{u\varphi(\lambda u)}{(u+1)^4}\,\mathrm{d}u\,,$$

and can again apply Lemma I to the function $-\lambda^3 s'(\lambda)$:

$$-\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^3 s'(\lambda)\right] \sim \frac{1}{2}\cdot\frac{3}{2} H\lambda^{\frac{1}{2}},$$

whence, on performing the differentiation and using (354), we obtain

$$s''(\lambda) = 3!\int_0^\infty \frac{\varphi(x)}{(x+\lambda)^4}\,\mathrm{d}x \sim \frac{1}{2}\cdot\frac{3}{2} H\lambda^{-\frac{5}{2}}. \tag{355}$$

On proceeding further in this way, we arrive at the formula

$$(-1)^m s^m(\lambda) = (m+1)!\int_0^\infty \frac{\varphi(x)}{(x+\lambda)^{m+2}}\,\mathrm{d}x \sim \frac{1\cdot 3\ldots(2m-1)}{2^m} H\lambda^{-\frac{2m+1}{2}}. \tag{356}$$

We now consider the asymptotic behaviour of the integral

$$J_p(\lambda) = \int_0^\infty \frac{x^p\varphi(x)}{(x+\lambda)^{2p+2}}\,\mathrm{d}x \qquad (p \geqslant 1) \tag{357}$$

for large λ. We have:

$$\frac{x^p}{(x+\lambda)^{2p+2}} = \frac{\left(1-\dfrac{\lambda}{x+\lambda}\right)^p}{(x+\lambda)^{p+2}} = \sum_{s=0}^{p}\binom{s}{p}\frac{(-\lambda)^s}{(x+\lambda)^{p+2+s}}\,,$$

where

$$\left(\frac{s}{p}\right) = \frac{p(p-1)\ldots(p-s+1)}{s!}\,; \qquad \binom{0}{p} = 1\,,$$

so that

$$J_p(\lambda) = \sum_{s=0}^{p}\binom{s}{p}(-\lambda)^s\int_0^\infty \frac{\varphi(x)}{(x+\lambda)^{p+2+s}}\,\mathrm{d}x.$$

On taking (356) into account, we get:

$$J_p(\lambda) \sim H\lambda^{-p-\frac{1}{2}}\sum_{s=0}^{p}(-1)^s\binom{s}{p}\frac{1\cdot 3\ldots(2p+2s+1)}{(p+s+1)!\,2^{p+s}}. \tag{358}$$

$$J_0(\lambda) = s(\lambda) \sim H\lambda^{-\frac{1}{2}}.$$

The first of these formulae can be written as

$$J_p(\lambda) \sim H\lambda^{-p-\frac{1}{2}} \frac{1}{\sqrt{\pi}} \sum_{s=0}^{p} (-1)^s \binom{s}{p} \frac{\Gamma\left(p+s+\frac{1}{2}\right)}{\Gamma(p+s+2)} . \tag{359}$$

We now prove the formula:

$$\frac{1}{\sqrt{\pi}} \sum_{s=0}^{p} (-1)^s \binom{s}{p} \frac{\Gamma\left(p+s+\frac{1}{2}\right)}{\Gamma(p+s+2)} = \frac{2}{\pi} K_p . \tag{360}$$

This is done by considering the integral

$$L_p = \int_0^\infty \frac{x^{p+\frac{1}{2}}}{(x+\lambda)^{2p+2}} \, dx , \tag{361}$$

which, with the aid of the substitution $x = \lambda u$, reduces to the form

$$L_p = \frac{1}{\lambda^{p+\frac{1}{2}}} \int_0^\infty \frac{u^{p+\frac{1}{2}}}{(u+1)^{2p+2}} \, du = \frac{1}{\lambda^{p+\frac{1}{2}}} K_p . \tag{362}$$

We have:

$$\left(\frac{x}{x+\lambda}\right)^p = \sum_{s=0}^{p} (-1)^s \binom{s}{p} \frac{\lambda^s}{(x+\lambda)^s} ,$$

and integral (361) can be rewritten as

$$L_p = \sum_{s=0}^{p} (-1)^s \binom{s}{p} \lambda^s \int_0^\infty \frac{x^{\frac{1}{2}}}{(x+\lambda)^{p+s+2}} \, dx$$

$$= \frac{1}{\lambda^{p+\frac{1}{2}}} \sum_{s=0}^{p} (-1)^s \binom{s}{p} \int_0^\infty \frac{u^{\frac{1}{2}}}{(u+1)^{p+s+2}} \, du \tag{363}$$

On using the substitution $u = x : (1 - x)$, we obtain

$$\int_0^\infty \frac{u^{\frac{1}{2}}}{(u+1)^{p+s+2}} \, du = \int_0^1 x^{\frac{1}{2}} (1-x)^{p+s-\frac{1}{2}} \, dx = \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(p+s+\frac{1}{2}\right)}{\Gamma(p+s+2)}$$

and we substitute this in (363):

$$L_p = \frac{\sqrt{\pi}}{2\lambda^{p+\frac{1}{2}}} \sum_{s=0}^{p} (-1)^s \binom{s}{p} \frac{\Gamma\left(p+s+\frac{1}{2}\right)}{\Gamma(p+s+2)} .$$

On comparing this with (362) we get (360), and (359) takes the form:

$$J_p(\lambda) \sim \frac{2H}{\pi} \lambda^{-p-\frac{1}{2}} K_p \tag{364}$$

or

$$J_p(\lambda) = \frac{2H}{\pi} \lambda^{-p-\frac{1}{2}} K_p (1 + \eta_\lambda), \tag{365}$$

where η_λ depends on p and λ, and $\eta_\lambda \to 0$ for fixed p as $\lambda \to \infty$.

The integral $J_p(\lambda)$ can be written as the sum of four terms:

$$J_p(\lambda) = \int\limits_0^1 \frac{x^p \varphi(x)}{(x+\lambda)^{2p+2}} \, dx + \int\limits_1^{(1-\alpha)\lambda} + \int\limits_{(1-\alpha)\lambda}^{(1+\alpha)\lambda} + \int\limits_{(1+\alpha)\lambda}^{\infty}$$
$$= J_{p,0} + J_{p,1} + J_{p,2} + J_{p,3}, \tag{366}$$

where $0 < \alpha < 1$. It follows from (345) that

$$0 \leqslant \varphi(\lambda) \leqslant A\sqrt{\lambda},$$

where A is a constant, and consequently

$$J_{p,1} \leqslant A \int\limits_0^{(1-\alpha)\lambda} \frac{x^{p+\frac{1}{2}}}{(x+\lambda)^{2p+2}} \, dx = A\lambda^{-p-\frac{1}{2}} K_{p,1},$$

whence, by Lemma II:

$$J_{p,1} = \frac{2H}{\pi} \lambda^{-p-\frac{1}{2}} K_p \eta_p,$$

where η_p depends on p and α and $\eta_p \to 0$ as $p \to \infty$ with fixed α. We obtain in the same way:

$$J_{p,3} = \frac{2H}{\pi} \lambda^{-p-\frac{1}{2}} K_p \eta_p',$$

where η_p' is analogous to η_p. We have for $J_{p,0}$:

$$J_{p,0} \leqslant B \int\limits_0^{\infty} \frac{dx}{(x+\lambda)^{2p+2}} = \frac{B}{2p+1} \left[\frac{1}{\lambda^{2p+1}} - \frac{1}{(1+\lambda)^{2p+1}} \right] \leqslant \frac{B_1}{p\lambda^{2p+1}},$$

where B and B_1 are constants (they do not depend on p and λ). It follows from this that

$$J_{p,0} = \frac{2H}{\pi} \lambda^{-p-\frac{1}{2}} K_p \eta_\lambda'$$

and we have:

$$0 < \eta_\lambda' \leqslant \frac{C}{pK_p} \lambda^{-p-\frac{1}{2}},$$

where C is constant. It follows from the previous formulae that

$$J_{p,2} = \frac{2H}{\pi} \lambda^{-p-\frac{1}{2}} K_p (1 + \eta_\lambda - \eta'_\lambda - \eta_p - \eta'_p). \tag{367}$$

On taking into account the definition of $J_{p,2}$ and the fact that $\varphi(x)$ does not decrease when x increases, we obtain:

$$J_{p,2} \leqslant \frac{\varphi(\lambda + a\lambda)}{(\lambda - a\lambda)^{1/2}} \int\limits_{\lambda - a\lambda}^{\lambda + a\lambda} \frac{x^{p+\frac{1}{2}}}{(x+\lambda)^{2p+2}} \, dx =$$

$$= \frac{\varphi(\lambda + a\lambda)}{(\lambda + a\lambda)^{1/2}} \cdot \left(\frac{1+a}{1-a}\right)^{\frac{1}{2}} \lambda^{-p-\frac{1}{2}} K_{p,2}, \tag{368}$$

whence

$$\frac{\varphi(\lambda + a\lambda)}{(\lambda + a\lambda)^{1/2}} \geqslant \lambda^{p+\frac{1}{2}} \frac{J_{p,2}}{K_{p,2}} \left(\frac{1-a}{1+a}\right)^{\frac{1}{2}} \geqslant \lambda^{p+\frac{1}{2}} \frac{J_{p,2}}{K_p} \left(\frac{1-a}{1+a}\right)^{\frac{1}{2}}.$$

We obtain on taking (367) into account:

$$\frac{\varphi(\lambda + a\lambda)}{(\lambda + a\lambda)^{1/2}} \geqslant \frac{2H}{\pi} (1 + \eta_\lambda - \eta'_\lambda - \eta_p - \eta'_p) \left(\frac{1-a}{1+a}\right)^{1/2}. \tag{369}$$

Similarly to (368), we have

$$J_{p,2} \geqslant \frac{\varphi(\lambda - a\lambda)}{(\lambda + a\lambda)^{1/2}} \int\limits_{\lambda - a\lambda}^{\lambda + a\lambda} \frac{x^{p+\frac{1}{2}}}{(x+\lambda)^{2p+2}} \, dx = \frac{\varphi(\lambda - a\lambda)}{(\lambda - a\lambda)^{1/2}} \left(\frac{1-a}{1+a}\right)^{1/2} \lambda^{-p-\frac{1}{2}} K_{p,2},$$

whence

$$\frac{\varphi(\lambda - a\lambda)}{(\lambda - a\lambda)^{1/2}} \leqslant \lambda^{p+\frac{1}{2}} \frac{J_{p,2}}{K_{p,2}} \left(\frac{1+a}{1-a}\right)^{1/2},$$

and we obtain, on taking Lemma II and (367) into account:

$$\frac{\varphi(\lambda - a\lambda)}{(\lambda - a\lambda)^{1/2}} \leqslant \frac{2H}{\pi} (1 + \eta_\lambda - \eta'_\lambda - \eta_p - \eta'_p) \left(\frac{1+a}{1-a}\right)^{1/2} (1 - \delta''_p)^{-1}. \tag{370}$$

We now show that the ratio $\varphi(\lambda) : \lambda^{1/2}$ tends to $2H/\pi$ as $\lambda \to \infty$:

$$\lim_{\lambda \to \infty} \frac{\varphi(\lambda)}{\lambda^{1/2}} = \frac{2H}{\pi}. \tag{371}$$

In general, the number A is a possible limiting value of $\varphi(\lambda) : \sqrt{\lambda}$ as $\lambda \to \infty$ if, given any positive ε and M, λ' can be found such that $| A - \varphi(\lambda') : \sqrt{\lambda'} | \leqslant \varepsilon$ and $\lambda' \geqslant M$. Similarly, $A = +\infty$ is a possible limit of $\varphi(\lambda) : \sqrt{\lambda}$ if, given any positive M and N, λ' can be found such that $\varphi(\lambda') : \sqrt{\lambda'} \geqslant N$ and $\lambda' \geqslant M$. Here, we understand by a possible limiting value an A such that there exists an indefinitely

increasing sequence λ_n of values of λ such that $\varphi(\lambda_n) : \sqrt{\lambda_n} \to A$. We have to show that only one possible limiting value exists and that it is equal to $2H/\pi$.

We return to inequalities (369) and (370) and observe that their left-hand sides do not depend on p, which appears in the right-hand sides. We first fix p and a in some way and let λ tend to infinity so that the left-hand sides of (369) and (370) tend to one of the possible limiting values A. We now obtain, in view of the fact that η_p and η_p' are independent of λ:

$$A \geqslant \frac{2H}{\pi}(1 - \eta_p - \eta_p')\left(\frac{1-a}{1+a}\right)^{1/2},$$

$$A \leqslant \frac{2H}{\pi}(1 - \eta_p - \eta_p')\left(\frac{1+a}{1-a}\right)^{1/2}.$$

The left-hand sides (i.e. A) depend neither on p nor on a, and on the assumption that a sufficiently large p is fixed and that a is sufficiently close to zero, we find that the only possible value of A is $2H/\pi$, i.e. (371) holds. We have thus proved assertion (331) of the theorem of [236]. The above proof is due to Hardy and Littlewood. These authors established a rather more general proposition, of which the present theorem is a particular case.

238. Linear equations of a more general type. Let us take an equation of the form:

$$L(u) = \sum_{i=1}^{3} u_{x_i x_i} + b(x_1, x_2, x_3)\, u = -f(x_1, x_2, x_3)\,. \tag{372}$$

In future, we shall simply write (x) or (ξ) for the coordinates (x_1, x_2, x_3) or (ξ_1, ξ_2, ξ_3) of a point of space.

Suppose we seek the solution of (372) satisfying the homogeneous boundary condition:

$$u\,|_S = 0\,. \tag{373}$$

The given functions $b(x)$ and $f(x)$ are assumed to be continuous in the closed domain $\overline{D}_i$ and to have continuous first order derivatives inside D_i.

We use the same method as in [235], and seek the solution of our problem in the form:

$$u(x) = \iiint\limits_{D_i} G(x;\ \xi)\,\mu(\xi)\,d\tau_\xi, \tag{374}$$

where $G(x;\ \xi)$ is Green's function of the Laplace operator with boundary condition (373). The latter condition, imposed on the $u(x)$ given by (374), is fulfilled with any choice of continuous function $\mu(\xi)$, and we have to choose this last function so that equation (372) is fulfilled inside D_i. Assuming that $\mu(\xi)$ has continuous derivatives, we obtain the integral equation for $\mu(\xi)$:

$$\mu(x) = f(x) + \iiint_{D_i} K(x;\ \xi)\,\mu(\xi)\,d\tau_\xi \tag{375}$$

with the kernel

$$K(x;\ \xi) = b(x)\,G(x;\ \xi). \tag{376}$$

We take the corresponding homogeneous equation:

$$\mu(x) = \iiint_{D_i} K(x;\ \xi)\,\mu(\xi)\,d\tau_\xi. \tag{377}$$

We have to ask ourselves whether it has non-zero solutions. Suppose that $b(x) \leqslant 0$ in D_i, and let $\mu_0(x)$ be a solution of (377). When $\mu(\xi) = \mu_0(\xi)$, (374) gives the solution of $L(u) = 0$ that satisfies condition (373). But such a solution is identically zero [234], i.e.

$$\iiint_{D_i} G(x;\ \xi)\,\mu_0(\xi)\,d\tau_\xi = 0. \tag{378}$$

It follows at once from (377) with $\mu(x) = \mu_0(x)$ that $\mu_0(\xi)$ must have continuous derivatives inside D_i [224], and that if we apply Laplace's operator to both sides of (378) we obtain $\mu_0(x) \equiv 0$, i.e. equation (377) with $b(x) \leqslant 0$ only has a zero solution, so that equation (375) is soluble with any function $f(x)$. Since $f(x)$ has continuous first order derivatives inside D_i by hypothesis, we can say that $\mu(x)$ also has these derivatives, whence it follows that (374) gives the solution of our original problem. It can be shown that homogeneous equation (377) has only a zero solution no matter what the sign of $b(x)$ in the case when the domain D_i is sufficiently small. Everything that has been said also holds for the plane case. If we employ the above method for the equation

$$\sum_{i=1}^{3} u_{x_i x_i} + \sum_{i=1}^{3} a_i(x)\,u_{x_i} + b(x)\,u = -f(x), \tag{379}$$

we arrive at an integral equation with the kernel:

$$K(x;\ \xi) = \sum_{i=1}^{3} a_i(x)\,G_{x_i}(x;\ \xi) + b(x)\,G(x;\ \xi). \tag{380}$$

If the inequality

$$|G_{x_i}(x;\ \xi)| \leqslant \frac{C}{r^2} \tag{381}$$

holds for the derivatives of Green's function (see D. M. Eidus, *Dokl. Akad. Nauk SSSR*, t. 106, no. 2, 1956), the usual theorems hold for the integral equation in question. But it still remains to show that the solution $\mu(x)$ of equation (375) has continuous derivatives inside D_i.

239. Linear elliptic equations of the second order. We next consider elliptic equations of the second order, of the form:

$$L(u) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) = 0 \qquad (a_{ik} = a_{ki}), \tag{382}$$

where a_{ik} are thrice continuously differentiable functions of (x) in the closed domain $\bar{D}_i$ of three-dimensional space bounded by a surface S.

By virtue of the ellipticity condition, the quadratic form

$$\sum_{i,k=1}^{3} a_{ik}\eta_i\eta_k$$

in the variables η_s is positive definite at all points of $\bar{D}_i$. Hence the determinant Δ, formed from the elements a_{ik}, is also positive. Let B_{ik} denote the cofactors A_{ik} of elements a_{ik} divided by Δ. It may easily be seen that the quadratic form

$$\sum_{i,k=i}^{3} B_{ik}\eta_i\eta_k \tag{383}$$

is also positive definite. To prove this, we only need to introduce into (383) the new variables η_s' in accordance with

$$\eta_i = a_{i1}\eta_1' + a_{i2}\eta_2' + \cdots + a_{in}\eta_n',$$

where the determinant Δ of this transformation is positive. The matrix C of the coefficients of the transformed form in variables η_s' is $C = ABA$ [III$_1$, 32], where A is the matrix of elements a_{ik} and B the matrix of elements B_{ik}. But BA is the unit matrix, so that quadratic form (383) becomes in the variables η_s':

$$\sum_{i,k=1}^{3} a_{ik}\eta_i'\eta_k',$$

whence it follows that this form is positive definite in $\bar{D}_i$. Let us define the function of a pair of points (x) and (ξ):

$$\sigma(x;\ \xi) = \sum_{i,k=1}^{3} B_{ik}(x)(x_i - \xi_i)(x_k - \xi_k). \tag{384}$$

By what has been said, $\sigma(x;\ \xi) \geqslant 0$, and the sign of equality only holds when the points (x) and (ξ) coincide. Moreover, we have

$$ar \leqslant \sigma^{\frac{1}{2}} \leqslant br, \tag{385}$$

where a and b are positive constants, and $r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}$. The numbers a and b are the least and greatest of the eigenvalues of the matrix B in $\overline{D}_l$. Let

$$\psi(x;\ \xi) = \frac{1}{[\sigma(x;\ \xi)]^{3/2}}. \tag{386}$$

We construct the function:

$$\Gamma(x, \xi) = \psi(x;\ \xi) + \iiint\limits_{D_l} \psi(x, t)\, f(t;\ \xi)\, d\tau_t, \tag{387}$$

the function $f(t;\ \xi) = f(t_1, t_2, t_3;\ \xi_1, \xi_2, \xi_3)$ being defined from the condition that $\Gamma(x;\ \xi)$, regarded as a function of (x), is a solution of equation (382). It can be shown that $f(x;\ \xi)$ is here found from a Fredholm integral equation of the second kind and that this function is a continuous function of points (x) and (ξ) provided these do not coincide, whilst it has polarity not exceeding $1/r$ when they do coincide.

The second term on the right-hand side of (387) is also a continuous function if (x) and (ξ) are distinct, and has polarity not exceeding $\log(1/r)$ when the points coincide. Hence $\psi(x;\ \xi)$ is the principal polar part of the solution $\Gamma(x;\ \xi)$. A detailed construction of this singular solution of equation (I) may be found in E. E. Levi's article *Linear elliptic partial differential equations* (O lineinykh ellipticheskikh uravneniyakh v chastnykh proizvodnykh) (*Uspekhi matematicheskikh nauk*, t. VIII, 1941). If we multiply $\Gamma(x;\ \xi)$ by a function of (ξ) and add the solution of equation (382) without a singularity, we again obtain a singular solution of this equation. Let us now write Green's formula [147] for equation (382):

$$\iiint\limits_{D_l} [uL(v) - vL(u)]\, d\tau = \iint\limits_{S} [uP(v) - vP(u)]\, dS, \tag{388}$$

where

$$P(u) = \sum_{i, k=1}^{3} a_{ik} u_{x_k} \cos(n, x_i)$$

and n is the direction of the outward normal at a point of S. It is assumed here that u and v possess the relevant continuous derivatives. We apply (388) to the solution $u(x)$ of equation (382) and to the singular solution $\Gamma(x;\ \xi)$. Here we must isolate the point ξ by a sphere C_ε with centre ξ and small radius ε. This gives us, after passing to the limit as $\varepsilon \to 0$:

$$E(\xi)\, u(\xi) = \frac{1}{4\pi} \iint\limits_{S} \{\Gamma(x;\ \xi)\, P(u) - uP[\Gamma(x;\ \xi)]\}\, dS_x, \tag{389}$$

where

$$E(\xi) = E(\xi_1, \xi_2, \xi_3) = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \iint\limits_{C_\varepsilon} P[\Gamma(x;\ \xi)]\, dS_x. \tag{390}$$

The function $E(\xi)$ is positive and has continuous derivatives up to the second order. The new singular solution is introduced:

$$K(x;\xi) = \frac{\Gamma(x;\xi)}{E(\xi)} \tag{391}$$

and the following formulae will be proved [cf. 193]:

$$\int_S\int P[K(x;\xi)]\,\mathrm{d}S_x = \begin{cases} 4\pi \text{ if } (\xi) \text{ is inside } S \\ 0 \text{ if } (\xi) \text{ is outside } S \\ 2\pi \text{ if } (\xi) \text{ is on } S \end{cases} \tag{392}$$

where the operation P is taken with respect to the point (x) in the integrand.

We shall assume that the coefficients a_{ik} of equation (382) are continued so as to retain the continuity of the derivatives up to the third order throughout space, i.e. (382) is Laplace's equation outside some sphere D_1, i.e. $L(u) = \Delta u$ outside D_1. If (x) lies outside D_1, then $\sigma(x;\xi) = r^2$ for any position of (ξ) and $E(x) = 1$. In addition, $f(x;\xi)$ appearing in (387) vanishes if (x) is outside D_1, and is determined for any positions of points (x) and (ξ) from the integral equation:

$$L[\psi(x;\xi)] - 4\pi E(x) f(x;\xi) + \int\int_D\int L[\psi(x;t)] f(t;\xi)\,\mathrm{d}\tau_t = 0.$$

The functions $\Gamma(x;\xi)$ and $K(x;\xi)$ are thereby defined throughout space, and $K(x;\xi)$ may be shown to be a symmetric function. It is analogous to the solution $1/r$ of Laplace's equation, whilst the operator $P(u)$ becomes in the case of Laplace's equation the derivative $\partial u/\partial n$ with respect to the normal.

We can form with the aid of $K(x;\xi)$ the following analogues of the potential of three-dimensional masses and the potentials of a single and double layer:

$$u(x) = \int\int_D\int \mu(\xi) K(x;\xi)\,\mathrm{d}\tau \tag{393}$$

$$v(x) = \int_S\int \mu(\xi) K(x;\xi)\,\mathrm{d}\tau \tag{394}$$

$$w(x) = \int_S\int \mu(\xi) P[K(x;\xi)]\,\mathrm{d}\tau, \tag{395}$$

the differentiation in the operator P being carried out with respect to the point (ξ) in the last formula.

The surface is assumed to be sufficiently smooth, and the density $\mu(\xi)$ to be continuous.

If, in (393), $\mu(\xi)$ has continuous derivatives inside D, the following analogue of Poisson's formula is obtained:

$$L(u) = -4\pi\mu(x) \qquad \text{(inside } D\text{)}, \tag{396}$$

and $u(x)$ satisfies equation (382) outside D. The potentials (394) and (395) satisfy (382) inside and outside S. When the point (x) approaches the point $(\xi^{(0)})$ on S from inside or outside the surface, the potential (395) has the boundary values:

$$\left.\begin{aligned} w_i(\xi^{(0)}) &= w(\xi^{(0)}) + 2\pi\mu(\xi^{(0)}) \\ w_e(\xi^{(0)}) &= w(\xi^{(0)}) - 2\pi\mu(\xi^{(0)}), \end{aligned}\right\} \tag{397}$$

where $w(\xi^{(0)})$ is the value of integral (395) at the point $(\xi^{(0)})$ on S [cf. 192]. Similarly, we have [cf. 194]:

$$\left.\begin{aligned} P_i\left[v\left(\xi^{(0)}\right)\right] &= \iint_S \mu(\xi)\, P\left[K\left(\xi^{(0)};\xi\right)\right] \mathrm{d}S + 2\pi\mu\left(\xi^{(0)}\right), \\ P_e\left[v\left(\xi^{(0)}\right)\right] &= \iint_S \mu(\xi)\, P\left[K\left(\xi^{(0)};\xi\right)\right] \mathrm{d}S - 2\pi\mu\left(\xi^{(0)}\right), \end{aligned}\right\} \tag{398}$$

and the differentiation in the operator P of $K(x;\ \xi)$ is carried out with respect to the point (x), after which we have to put $x = \xi^{(0)}$. Boundary value problems for equation (382) can be reduced with the aid of these formulae to integral equations. For instance, the interior Dirichlet problem for equation (382) with the boundary condition:

$$u\,|_S = f(\xi)$$

is sought in the form (395), and we obtain the following integral equation for the density $\mu(\xi)$ by virtue of the first of (397):

$$2\pi\mu\left(\xi^{(0)}\right) - \iint_S \mu(\xi)\, P\left[K\left(\xi^{(0)};\xi\right)\right] \mathrm{d}S = f\left(\xi^{(0)}\right), \tag{399}$$

where the differentiation in the operator P is carried out with respect to the point ξ. It may be shown that the corresponding homogeneous equation has only the zero solution, so that equation (399) is soluble with any continuous function $f(\xi)$. Similarly, the method of [220] can be used to construct Green's function $G(x;\ \xi)$ for equation (382); and we can use the function to solve the boundary value problem for the equation

$$L(u) = \sum_{i=1}^{3} a_i \frac{\partial u}{\partial x_i} + bu = -f(x) \text{ with the condition } u\,|_S = 0. \tag{400}$$

The general linear elliptic equation

$$\sum_{i,k=1}^{3} a_{ik} \frac{\partial^2 u}{\partial x_i\, \partial x_k} + \sum_{i=1}^{3} a_i \frac{\partial u}{\partial x_i} + bu = -f(x) \tag{401}$$

can be written in the form:

$$L(u) + \sum_{i=1}^{3} \left[b_i - \sum_{k=1}^{3} \frac{\partial a_{ki}}{\partial x_k}\right] \frac{\partial u}{\partial x_i} + bu = -f(x). \tag{402}$$

The extension, discussed in the present section, of potential theory to general linear elliptic equations is due to Sternberg *(Math. Zeitschr.* Bd. 21, 1924). An article by Feller gives a general treatment of the theory for elliptic equations with any number of independent variables; this is bound up with the introduction of a special metric, constructed on the basis of the coefficients a_{ik} into the space (The solutions of linear partial differential equations of the second order of the elliptic type (O resheniyakh lineinykh differentsial'nykh uravnenii v chastnykh proizvodnykh votorogo poryadka ellipticheskogo tipe) (*Uspekhi matematicheskikh nauk*, t. VIII, 1941)).

In Püschel's article, Die erste Randwertaufgabe der allgemeinen selbstadjungierten elliptischen Differentialgleichungen zweiter Ordnung im Raum für beliebige Gebiete *(Math. Zeitschr*, Bd. 32, H. 4, 1932), a generalized solution of the Dirichlet problem for equations of the form (382) is constructed

and investigated. The method is based on an approximation to the domain D with the aid of a sequence of domains D_n lying inside D, and an extension of the continuous boundary values inside D. We described this method for Laplace's equation in [217]. Püschel's article includes a detailed study of the conditions for regularity of points on the boundary.

240. Green's tensor. Let $L(\mathbf{u})$ be a linear operation on the vector $\mathbf{u}(u_1, u_2, u_3)$, which is a function of (x, y, z), and let the operation lead to another vector. We consider the equation

$$L(\mathbf{u}) = -\mathbf{f}, \tag{403}$$

where $\mathbf{f}$ is a given vector depending on (x, y, z). On taking the components of both sides, we get a system of three equations for the components (u_1, u_2, u_3) of the vector $\mathbf{u}$. Suppose we also have a homogeneous boundary condition on the surface S of the domain D, say

$$\mathbf{u}\,|_S = 0. \tag{404}$$

Green's tensor for $L(\mathbf{u})$ with boundary condition (404) is defined as the matrix

$$G(P;\, Q) = G(x, y, z;\, \xi, \eta, \zeta) = \left\| \begin{matrix} G_{11}, G_{12}, G_{13} \\ G_{21}, G_{22}, G_{23} \\ G_{31}, G_{32}, G_{33} \end{matrix} \right\|,$$

such that equation (403) with boundary condition (404) is equivalent to the equation

$$\mathbf{u}(P) = \iiint_D G(P;\, Q)\,\mathbf{f}(Q)\,\mathrm{d}v\,, \tag{405}$$

where the integrand is the result of applying matrix $G(P; Q)$ as an operator to the vector $\mathbf{f}$, i.e. the integrand is the vector with components:

$$G_{i1}f_1 + G_{i2}f_2 + G_{i3}f_3 \qquad (i = 1, 2, 3).$$

Each column of the tensor gives the components of some vector g_k $(k = 1, 2, 3)$, which, when the point Q is excluded, has continuous derivatives, satisfies the homogeneous equation (403) and boundary condition (404). The nature of the polarity at the point Q usually follows easily from the physical meaning of the problem. We can use Green's tensor to reduce the problem of the eigenvalues and eigenvectors of the equation

$$L(\mathbf{u}) + \lambda\mathbf{u} = 0$$

with boundary condition (404) to a system of integral equations, as above.

Let us write down the fundamental equation of the theory of elasticity for the displacement vector [94]:

$$\varrho \frac{\partial^2 \mathbf{u}}{\partial t^2} = G\left(\Delta \mathbf{u} + \frac{m}{m-2} \operatorname{grad} \operatorname{div} \mathbf{u}\right).$$

By using the formula [II; 112]:

$$\operatorname{curl} \operatorname{curl} \mathbf{u} = \operatorname{grad} \operatorname{div} \mathbf{u} - \Delta \mathbf{u},$$

we can write the equation for the statical case in the form

$$\Delta^* \mathbf{u} = a \operatorname{grad} \operatorname{div} \mathbf{u} - b \operatorname{rot} \operatorname{rot} \mathbf{u} = 0, \tag{406}$$

where

$$a = \frac{G(2m-2)}{m-2}; \quad b = G,$$

or, on introducing the usual Lamé constants λ and μ: $a = \lambda + 2\mu$; $b = \mu$.

A unit force acting at the point $Q(\xi, \eta, \zeta)$ in infinite space parallel to the Z axis produces a displacement with components:†

$$u_1 = A\,\frac{(x-\xi)(z-\zeta)}{r^3}; \quad u_2 = A\,\frac{(y-\eta)(z-\zeta)}{r^3};$$

$$u_3 = A\left[\frac{(z-\zeta)^2}{r^3} + \frac{\lambda+3\mu}{\lambda+\mu}\cdot\frac{1}{r}\right],$$

where

$$A = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} \quad \text{for} \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}.$$

Similar expressions are obtained for the displacements in the case of forces parallel to the X and Y axes. Green's tensor has the form in the present case:

$$G = \frac{1}{8\pi a} P_a + \frac{1}{8\pi b} P_b, \tag{407}$$

where

$$P_a = \left\| \begin{matrix} \frac{1}{r} - \frac{(x-\xi)^2}{r^3}, & -\frac{(x-\xi)(y-\eta)}{r^3}, & -\frac{(x-\xi)(z-\zeta)}{r^3} \\ -\frac{(y-\eta)(x-\xi)}{r^3}, & \frac{1}{r} - \frac{(y-\eta)^2}{r^3}, & -\frac{(y-\eta)(z-\zeta)}{r^3} \\ -\frac{(z-\zeta)(x-\xi)}{r^3}, & -\frac{(z-\zeta)(y-\eta)}{r^3}, & \frac{1}{r} - \frac{(z-\zeta)^2}{r^3} \end{matrix} \right\|$$

and

$$P_b = \left\| \begin{matrix} \frac{1}{r} + \frac{(x-\xi)^2}{r^3}, & \frac{(x-\xi)(y-\eta)}{r^3}, & \frac{(x-\xi)(z-\zeta)}{r^3} \\ \frac{(y-\eta)(x-\xi)}{r^3}, & \frac{1}{r} + \frac{(y-\eta)^2}{r^3}, & \frac{(y-\eta)(z-\zeta)}{r^3} \\ \frac{(z-\zeta)(x-\xi)}{r^3}, & \frac{(z-\zeta)(y-\eta)}{r^3}, & \frac{1}{r} + \frac{(z-\zeta)^2}{r^3} \end{matrix} \right\|.$$

† See Love, *Mathematical Theory of Elasticity.*

Here, instead of boundary condition (404), we have u vanishing at a point at infinity. The equation

$$\Delta^* \mathbf{u} = -\mathbf{f}$$

has in this case the solution (406). Tensor (407) is usually termed, in the theory of elasticity, the Somigliano displacement tensor. It can be written as

$$G = \frac{1}{8\pi\mu(\lambda + 2\mu)} \left[\frac{\lambda + 3\mu}{r} E + (\lambda + \mu) \frac{\mathbf{r} \times \mathbf{r}}{r^3} \right],$$

where E is the unit matrix and $\mathbf{r} \times \mathbf{r}$ is the tensor:

$$\mathbf{r} \times \mathbf{r} = \left\| \begin{matrix} (x-\xi)^2, & (x-\xi)(y-\eta), & (x-\xi)(z-\zeta) \\ (y-\eta)(x-\xi), & (y-\eta)^2, & (y-\eta)(z-\zeta) \\ (z-\zeta)(x-\xi), & (z-\zeta)(y-\eta), & (z-\zeta)^2 \end{matrix} \right\|.$$

Weyl† has given various analogues of Green's formula for equation (406), and also formed Green's tensor for a bounded domain and used it to investigate the eigenvalues of the equation

$$\Delta^* \mathbf{u} + \lambda \mathbf{u} = 0.$$

241. The plane statical problem of the theory of elasticity. Certain boundary value problems can be solved in the case of a plane with the aid of Cauchy's integral. This applies, for instance, to Dirichlet's problem for the harmonic and biharmonic equations, or to the problem of the conformal mapping of a singly connected domain on to a circle or of a multiply-connected domain on to a domain of a definite type (V. I. Krylov, *Mat. sbornik*, t. 4 (46): 1; 1938). By using Cauchy's integral, these problems reduce to integral equations. We shall describe the application of this method to the solution of the plane statical problem of the theory of elasticity [N. I. Muskhelishvili, Some problems of the theory of elasticity (Nekotorye zadachi teorii uprugosti)]. If our boundary condition consists in specifying the displacements on the contour of a domain B, the solution of the statical problem amounts to finding two functions $\varphi(z)$ and $\psi(z)$, regular in B and satisfying on the contour the boundary condition:

$$-\overline{k\varphi(z')} + \bar{z}'\varphi'(z') + \psi(z') = f(z') \qquad (z' \text{ on } l), \tag{408}$$

where k is a real constant and $f(z')$ is given on the contour l. On multiplying both sides of (408) by $1/2\pi i(z'-z)$, where z lies outside l, and integrating over l, we obtain:

$$-\frac{k}{2\pi i}\int\limits_l \frac{\overline{\varphi(z')}}{z'-z}\,dz' + \frac{1}{2\pi i}\int\limits_l \frac{\bar{z}'\varphi'(z')}{z'-z}\,dz' = F(z),$$

where

$$F(z) = \frac{1}{2\pi i}\int\limits_l \frac{f(z')}{z'-z}\,dz' \qquad (z \text{ outside } l)$$

† *Circolo Math. di Palermo*, 1915.

is a known function outside l. On letting z approach the contour l, we get:

$$\frac{k}{2}\overline{\varphi(t)} - \frac{k}{2\pi i}\int_l \frac{\overline{\varphi(z')}}{z'-t}\,dz' - \frac{1}{2}\bar{t}\,\varphi'(t) + \frac{1}{2\pi i}\int_l \frac{\bar{z}'\varphi'(z')}{z'-t}\,dz' = F_e(t), \tag{409}$$

where the integrals are to be understood in the sense of the principal value. In order to obtain equations containing ordinary integrals, we write:

$$\frac{k}{2}\overline{\varphi(t)} + \frac{k}{2\pi i}\int_l \frac{\overline{\varphi(z')}\,\overline{dz'}}{\bar{z}'-\bar{t}} = 0; \qquad \frac{1}{2}\varphi'(t) - \frac{1}{2\pi i}\int_l \frac{\varphi'(z')\,dz'}{z'-t} = 0.$$

On multiplying the second of these equations by $\bar{t}$ and adding both equations term by term to equation (409), we get:

$$\overline{k\varphi(t)} + \frac{k}{2\pi i}\int_l \overline{\varphi(z')}\,d\log\frac{\bar{z}'-\bar{t}}{z'-t} + \frac{1}{2\pi i}\int_l \varphi'(z')\,\frac{\bar{z}'-\bar{t}}{z'-t}\,dz' = F_e(t)$$

Finally, on integrating by parts in the integral containing $\varphi'(z')$, we get:

$$\overline{k\varphi(t)} + \frac{k}{2\pi i}\int_l \overline{\varphi(z')}\,d\left[\log\frac{\bar{z}'-\bar{t}}{z'-t}\right] - \frac{1}{2\pi i}\int_l \varphi(z')\,d\left[\frac{\bar{z}'-\bar{t}}{z'-t}\right] = F_e(t). \tag{410}$$

If we put $z'-t = re^{i\vartheta}$, the last equation can be written as

$$\overline{k\varphi(t)} + \frac{1}{\pi}\int_l \left[e^{-2i\vartheta}\varphi(z') - \overline{k\varphi(z')}\right]d\vartheta = F_e(t). \tag{411}$$

On separating real and imaginary parts, we get a system of two integral equations for the real and imaginary parts of $\varphi(z')$ on l. On solving these equations we get $\varphi(z')$ on l, and hence, by Cauchy's formula, $\varphi(z)$ inside l. To find the function $\psi(z)$, we multiply both sides of (408) by $1/2\pi i\,(z'-z)$, where z is inside l, and integrate over l:

$$\psi(z) = \frac{k}{2\pi i}\int_l \frac{\overline{\varphi(z')}}{z'-z}\,dz' - \frac{1}{2\pi i}\int_l \frac{\bar{z}'\varphi'(z')}{z'-z}\,dz' + \frac{1}{2\pi i}\int_l \frac{f(z')}{z'-z}\,dz'.$$

This method of reducing the boundary value problem (408) to an integral equation is due to N. I. Muskhelishvili *(Dokl. Akad. Nauk SSSR*, t, III, no. 1, 1934). We have assumed that the problem has a solution when forming equation (410). By using this equation, an existence theorem can be established for this plane statical problem of the theory of elasticity, not only in the case of a singlyconnected domain such as we have assumed above, but also in the case of a multiply-connected region†

A reduction of the plane statical problem of the theory of elasticity to an integral equation has been given by V. A. Fok *(Comptes Rendus*, t. 182, 1926, p. 264).

† D. I. Sherman, *Dokl. Akad. Nauk SSSR*, t. IV, no. 3, 1935.

In equation (411), ϑ is the angle formed by the radius vector from a fixed point t of contour l to a variable point z' of the contour. It may easily be seen by using this fact that homogeneous equation (411) has a non-zero solution $\varphi(z') = \text{const}$. The same can be said of equation (410). We can always assume that $z = 0$ lies inside l. It follows from the form of boundary condition (408) that we can transfer the constant term from $\varphi(z)$ to $\psi(z)$, and can take $\varphi(0) = 0$. Hence it follows that

$$\int_l \frac{\varphi(z')}{z'} \, dz' = 0 ;$$

on adding this equation to (410), we get a new equation which no longer has an eigenfunction.

Another method can be used for solving boundary value problem (408)†. We shall seek $\varphi(z)$ and $\psi(z)$ in the form:

$$\varphi(z) = \frac{1}{2\pi i} \int_l \frac{\omega(z')}{z' - z} \, dz' \qquad (z \text{ inside } l)$$

$$\psi(z) = \frac{1}{2\pi i} \int_l \frac{\overline{\omega(z')}}{z' - z} \, dz' - \frac{1}{2\pi i} \int_l \frac{\bar{z}' \, \omega(z')}{z' - z} \, dz' + \frac{1}{2\pi i} \int_l \frac{\omega(z')}{z' - z} \, \overline{dz'}.$$

where $\omega(z')$ is the required function on l. On substituting in (408) and using the properties of Cauchy type integrals, we obtain the integral equation for $\omega(z')$:

$$k\omega(t) - \frac{k}{2\pi i} \int_l \omega(z') \, d \log \frac{\bar{z}' - \bar{t}}{z' - t} - \frac{1}{2\pi i} \int_l \omega(z') \, d \frac{\bar{z}' - \bar{t}}{z' - t} = -f(t).$$

In the article by Sherman quoted, the case of a multiply-connected domain is considered, and an analysis carried out of the resulting integral equation.

§ 3. Equations of the parabolic and hyperbolic type

242. The dependence of the solutions of the heat conduction equation on the initial and boundary conditions and the function f. We have already established a uniqueness theorem for the heat conduction equation. The proof was based on a theorem which stated that *the greatest and least values of the solution of the homogeneous heat conduction equation are attained either for $t = 0$ or on the boundary of the domain.*

This latter theorem was proved for the one-dimensional case [II, 209]. The proof is essentially the same for space of any dimensions.

† Ibid., t. XXVII, no. 9, 1940.

We now consider the non-homogeneous heat conduction equation in the domain B on the (x, y) plane:

$$u_t = u_{xx} + u_{yy} + f(x, y, t) \tag{1}$$

with the initial and boundary conditions:

$$u|_{t=0} = \varphi(x, y) \text{ (in domain } \bar{B}); \quad u|_l = \psi(x, y, t), \tag{2}$$

where l is the contour of B. The function f is assumed continuous in the closed domain $\overline{B}$ for $t \geqslant 0$. Similarly, we assume that φ is continuous in $\bar{B}$ and ψ on l for $t \geqslant 0$. We imagine a cylinder D in space (x, y, t), the base of which is the domain B on the (x, y) plane and the generators of which are parallel to the t axis. Let D_1 be the part of this cylinder bounded from below by the plane $t = 0$ and from above by the plane $t = T$ $(T > 0)$. Let S' denote the lower base $t = 0$ and the lateral surface of D_1. By using arguments precisely similar to those employed in [II, 209] when proving the above-mentioned theorem, it is easily shown that:

THEOREM 1. *If u satisfies equation* (1) *inside D and is continuous as far as S', and if $f \geqslant 0$ in D_1, the least value of u in D_1 is attained on S'*, i.e. *either with $t = 0$ or on the lateral surface of D_1*, i.e. *on the boundary of domain B. If $f \leqslant 0$ in D_1, the greatest value of u is attained on S'.*

The proof of this theorem, which is very similar to the proof of [II, 209], will be indicated briefly. We shall only take the case $f \leqslant 0$, and shall use *reductio ad absurdum*. Let the greatest value of u be attained at some point (x', y', t') not on S', and be equal to M. We introduce the new function:

$$v = u - k(t - T), \tag{3}$$

where k is a positive number which will be fixed shortly.

We have in $\overline{D}_1$:

$$u \leqslant v \leqslant u + kT,$$

and we can fix k so close to zero that the greatest value of v on S' is, like that of u, less than the value of u at the point (x', y', t'). With this choice of k, the function v will attain its greatest value either inside D_1 or inside on the upper boundary $t = T$. Both these cases will be shown to lead to a contradiction.

Let v attain its greatest value at some interior point $C(x, y, t)$ of D_1. At this point v has a maximum, so that

$$v_t = 0; \quad v_{xx} \leqslant 0; \quad v_{yy} \leqslant 0 \qquad \text{at the point } C,$$

whence it follows that $v_t - v_{xx} - v_{yy} \geqslant 0$, or, by (3), $u_t - u_{xx} - u_{yy} - k \geqslant 0$ at the point C, which contradicts the fact that the equation $u_t - u_{xx} - u_{yy} - f = 0$ and $f \leqslant 0$ must be satisfied at C. Now suppose that v attains its greatest value at the point C, lying inside the base $t = T$. At this point we must have $v_t \geqslant 0$ and we find by considering the variation of v along the upper boundary: $v_{xx} \leqslant 0$ and $v_{yy} \leqslant 0$ at C. This leads us to the same contradiction as above, and the theorem is proved. By using this theorem, the following can easily be proved:

THEOREM 2. *If φ, ψ, and f satisfy the conditions $|\varphi| \leqslant a$ on the lower base of D_1, $|\psi| \leqslant a$ on the lateral surface of D_1 and $|f| \leqslant a/T$ in $\overline{D}_1$, then $|u| \leqslant 2a$ in D_1.*

We consider the function

$$v = u + \frac{a(T-t)}{T}, \tag{4}$$

which satisfies the equation

$$v_t = v_{xx} + v_{yy} + \left(f - \frac{a}{T}\right)$$

and the following conditions:

$$v|_{t=0} = \varphi + a; \quad v|_l = \psi + \frac{a(T-t)}{T}.$$

On taking into account the conditions of the theorem and the fact that $0 \leqslant t \leqslant T$ on the lateral surface of D_1, we can assert that

$$f - \frac{a}{T} \leqslant 0 \quad \text{in } D_1; \quad |\varphi + a| \leqslant 2a \text{ on the base } t = 0;$$

$$\left|\psi + \frac{a(T-t)}{S}\right| \leqslant 2a \text{ on the lateral surface of } D_1.$$

It now follows from Theorem 1 that the greatest value of v is attained on S', so that $v \leqslant 2a$ in $\overline{D}_1$. On observing that the second term on the right-hand side of (4) is non-negative, we can say that $u \leqslant 2a$. Similarly, by introducing the function

$$v = u - \frac{a(T-t)}{T},$$

we can prove that $u \geqslant -2a$, whence it follows that $|u| \leqslant 2a$. Theorem 2 gives an inequality (bound) for solutions of equation (1) via inequalities for the function f and the functions appearing in the initial and boundary conditions.

The proof of this theorem is essentially the same in three-dimensional space.

243. Potentials for the heat conduction equation in the one-dimensional case. We next show that a theory can be constructed for the heat conduction equation, analogous to the potential theory for Laplace's equation; hence we can reduce boundary value problems for the heat conduction equation to integral equations.

We take the one-dimensional heat conduction equation

$$u_t = a^2 u_{xx} \tag{5}$$

and a boundary value problem for the interval $0 \leqslant x \leqslant l$ with the boundary conditions:

$$u\,|_{x=0} = \omega_1(t)\,; \quad u\,|_{x=l} = \omega_2(t) \tag{6}$$

and the initial conditions:

$$u\,|_{t=0} = f(x) \qquad (0 \leqslant x \leqslant l). \tag{7}$$

We continue the function $f(x)$, given in the interval $[0, l]$, throughout the x axis, in such a way that it is continuous and vanishes outside some finite interval, and we form the solution of equation (5) [II, 204]:

$$u_0(x, t) = \frac{1}{2a\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(\xi)\, e^{-\frac{(\xi-x)^2}{4a^2 t}}\, d\xi \qquad (t > 0), \tag{8}$$

which satisfies the condition:

$$u_0\,|_{t=0} = f(x) \qquad (-\infty < x < +\infty). \tag{9}$$

On replacing $u(x, t)$ by the new function $w(x, t) = u(x, t) - u_0(x, t)$, we obtain equation (5) for w with the homogeneous initial condition:

$$w\,|_{t=0} = 0 \qquad (0 \leqslant x \leqslant l)$$

and with certain conditions at $x = 0$ and $x = l$, the right-hand sides of which are equal to the differences $\omega_1(t) - w(0, t)$ and $\omega_2(t) - w(l, t)$. In future, therefore, we shall seek the solution of equation (5) with boundary conditions (6) and the homogeneous initial condition:

$$u\,|_{t=0} = 0 \qquad (0 \leqslant x \leqslant l). \tag{10}$$

The fundamental singular solution, corresponding to a source located at the point $x = \xi$ at the instant $t = \tau$ is [II, 204]:

$$u = \frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-\frac{(\xi-x)^2}{4a^2(t-\tau)}}. \tag{11}$$

On differentiating with respect to ξ and adding the constant factor $2a^2$, we obtain the singular solution corresponding to a dipole:

$$u = \frac{1}{2a\sqrt{\pi}\,(t-\tau)^{3/2}}\,(x-\xi)\,e^{-\frac{(\xi-x)^2}{4a^2(t-\tau)}}. \tag{12}$$

On multiplying the last solution by some function $\varphi(\tau)$ and integrating with respect to τ from $\tau = 0$ to $\tau = t$, we get the solution:

$$u(x, t) = \int_0^t \frac{\varphi(\tau)}{2a\sqrt{\pi}\,(t-\tau)^{3/2}}\,(x-\xi)\,e^{-\frac{(\xi-x)^2}{4a^2(t-\tau)}}\,d\tau, \tag{13}$$

corresponding to a dipole at the point $x = \xi$ acting from the instant $\tau = 0$, with intensity $\varphi(\tau)$. The fact that function (13) with $x \neq \xi$ satisfies equation (5) follows immediately by simple differentiation; differentiation with respect to the upper limit gives zero, since the integrand tends to zero for $x \neq \xi$ as $\tau \to t$. We show that function (13) satisfies the following boundary relationships if x tends to ξ from the left or right:

$$u(\xi+0, t) = \varphi(t); \quad u(\xi-0, t) = -\varphi(t). \tag{14}$$

Assuming $x \neq \xi$, we introduce instead of τ the new variable of integration:

$$\alpha = \frac{x-\xi}{2a\sqrt{t-\tau}}.$$

If $x > \xi$, then $\alpha \to +\infty$ as $\tau \to t$, and if $x < \xi$, $\alpha \to -\infty$ as $\tau \to t$. We obtain in the new variable:

$$u(x, t) = \frac{1}{2\sqrt{\pi}} \int_{\frac{x-\xi}{2a\sqrt{t}}}^{+\infty} \varphi\left[t - \frac{(\xi-x)^2}{4a^2\alpha^2}\right] e^{-\alpha^2}\,d\alpha \qquad (x > \xi), \tag{15}$$

and we obtain in the limit as $x \to \xi + 0$:

$$u(\xi+0, t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \varphi(t)\,e^{-\alpha^2}\,d\alpha = \varphi(t)\,\frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\alpha^2}\,d\alpha = \varphi(t).$$

The proof of the second of equations (14) is similar. Moreover, solution (13) obviously satisfies the homogeneous initial condition:

$$u|_{t=0} = 0. \tag{16}$$

We shall not dwell on a more detailed proof of the passage to the limit in formula (15). The proof is easy if we assume the continuity of $\varphi(\tau)$.

Suppose we have a statement of the above problem with boundary conditions (6) and initial condition (10). Let us seek the solution as the sum of two dipoles — one located at the point $x = 0$, and the other at $x = l$. Let $\varphi(\tau)$ be the unknown intensity of the first, and $\psi(\tau)$ of the second:

$$u(x, t) = \int_0^t \frac{\varphi(\tau)}{2a\sqrt{\pi}(t-\tau)^{3/2}} x e^{-\frac{x^2}{4a^2(t-\tau)}} d\tau + \\ + \int_0^t \frac{\psi(\tau)}{2a\sqrt{\pi}(t-\tau)^{3/2}} (x-l) e^{-\frac{(l-x)^2}{4a^2(t-\tau)}} d\tau \qquad (17)$$

By (14), boundary conditions (6) can be written as

$$\left.\begin{aligned} \varphi(t) - l\int_0^t \frac{\psi(\tau)}{2a\sqrt{\pi}(t-\tau)^{3/2}} e^{-\frac{l^2}{4a^2(t-\tau)}} d\tau = \omega_1(t), \\ -\psi(t) + l\int_0^t \frac{\varphi(\tau)}{2a\sqrt{\pi}(t-\tau)^{3/2}} e^{-\frac{l^2}{4a^2(t-\tau)}} d\tau = \omega_2(t). \end{aligned}\right\} \qquad (18)$$

These equations represent a system of Volterra integral equations for $\varphi(\tau)$ and $\psi(\tau)$, and the kernel of the equations depends only on the difference $(t-\tau)$, so that the Laplace transformation, described in [46], can be applied to our system. If, for instance, the derivative $\partial u/\partial x$ is given instead of u itself at one of the ends, we have to locate a simple source instead of a dipole at this end, the action of the source being given by (11). Suppose, for instance, that the boundary conditions have the form:

$$u\big|_{x=0} = \omega_1(t); \quad \frac{\partial u}{\partial x}\bigg|_{x=l} = \omega_2(t), \qquad (19)$$

and the initial conditions have the form (16) as above.

In order to simplify later formulae, we multiply (11) by $2a^2$ and therefore seek the solution in the form:

$$u(x, t) = \int_0^t \frac{\varphi(\tau)}{2a\sqrt{\pi}(t-\tau)^{3/2}} x e^{-\frac{x^2}{4a^2(t-\tau)}} d\tau + \\ + \int_0^t \frac{a\psi(\tau)}{\sqrt{\pi}\sqrt{t-\tau}} e^{-\frac{(l-x)^2}{4a^2(t-\tau)}} d\tau. \qquad (20)$$

The first of conditions (19) gives:

$$\varphi(t) + \int_0^t \frac{a e^{-\frac{l^2}{2a^2(t-\tau)}}}{\sqrt{\pi}\sqrt{t-\tau}} \psi(\tau)\, d\tau = \omega_1(t).$$

On differentiating (20) with respect to x and letting x tend to l, we obtain, by (14) and the second of conditions (19):

$$\psi(t) + \int_0^t \frac{e^{-\frac{l^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi}\,(t-\tau)^{3/2}} \varphi(\tau)\, d\tau - l^2 \int_0^t \frac{e^{-\frac{l^2}{4a^2(t-\tau)}}}{4a^3\,(t-\tau)^{5/2}} \varphi(\tau)\, d\tau = \omega_2(t),$$

so that we again have a system of integral equations for $\varphi(\tau)$ and $\psi(\tau)$ with kernels depending on the difference $(t - \tau)$.

244. Heat sources in the multi-dimensional case. The concept of potential can also be applied to multi-dimensional problems of heat conduction. We shall confine ourselves to indicating the results, which are similar to the above. The proof of the properties of potentials presents much greater difficulties in the multi-dimensional than in the one-dimensional case. We shall consider the plane case, i.e. the equation

$$u_t = a^2\,(u_{xx} + u_{yy}). \tag{21}$$

Let B be a domain on the (x, y) plane with contour l. The basic singular solution corresponding to a source at the point (ξ, η) acting at the instant τ has the form:

$$u = \frac{1}{2\pi\,(t-\tau)} e^{-\frac{r^2}{4a^2(t-\tau)}} \qquad [r^2 = (\xi - x)^2 + (\eta - y)^2].$$

The analogue of the potential of a simple layer is given by

$$u(x, y, t) = \frac{1}{2\pi}\int_0^t d\tau \int_l \frac{a(\sigma, \tau)}{t-\tau} e^{-\frac{r^2}{4a^2(t-\tau)}}\, d\sigma, \tag{22}$$

where σ is the length of arc of the contour l measured from some fixed point, and $a(\sigma, \tau)$ is a function of the variable point σ of the contour and the parameter τ; r denotes the distance from the point (x, y) to the variable point σ of l. The heat potential of a double layer is given by

$$v(x, y, t) = \frac{1}{2\pi}\int_0^t d\tau \int_l \frac{b(\sigma, \tau)}{t-\tau} \frac{\partial}{\partial n} e^{-\frac{r^2}{4a^2(t-\tau)}}\, d\sigma, \tag{23}$$

where n is the direction of the outward normal at the variable point of integration, or

$$v(x, y, t) = \int_0^t d\tau \int_l \frac{b(\sigma, \tau)}{4\pi a^2\,(t-\tau)^2} e^{-\frac{r^2}{4a^2(t-\tau)}}\, r\cos(r, n)\, d\sigma, \tag{23_1}$$

where the direction r is reckoned from the point σ to the point (x, y). If we bring in the angle $d\varphi$ subtended by an element of length $d\sigma$ at the point (x, y), the above formula can be written as

$$v(x, y, t) = -\int_l d\varphi \int_0^t \frac{b(\sigma, \tau)}{4\pi a^2 (t-\tau)^2} e^{-\frac{r^2}{4a^2(t-\tau)}} r^2 \, d\tau. \tag{23_2}$$

The boundary values of the double layer potential at the point $\sigma_0(x_0, y_0)$ of the contour are given by

$$v_i(x_0, y_0, t) = -b(\sigma_0, t) + \int_0^t d\tau \int_l \frac{b(\sigma, \tau)}{4\pi a_2 (t-\tau)^2} e^{-\frac{r_0^2}{4a^2(t-\tau)}} r_0 \cos(r_0, n) \, d\sigma$$

$$v_e(x_0, y_0, t) = b(\sigma_0, t) + \ldots, \tag{24}$$

where r_0 is the distance from the variable point of integration to the point $\sigma_0(x_0, y_0)$. The simple layer potential (22) is continuous on passing through the contour l, whilst its derivative with respect to the normal n at the point σ_0 of the contour has boundary values at this point given by

$$\left(\frac{\partial u(x_0, y_0, t)}{\partial n_0}\right)_i = a(\sigma_0, t) - \int_0^t d\tau \int_l \frac{a(\tau, \sigma)}{4\pi a^2 (t-\tau)^2} e^{-\frac{r_0^2}{4a^2(t-\tau)}} r_0 \cos(r_0, n_0) \, d\sigma$$

$$\left(\frac{\partial u(x_0, y_0, t)}{\partial n_0}\right)_e = -a(\sigma_0, t) - \ldots \tag{25}$$

These formulae can be used to reduce the solution of boundary value problems to integral equations. For instance, suppose we want the function $v(x, y, t)$, satisfying equation (21) inside B, and having on the contour l given boundary values:

$$v|_l = \omega(s, t), \tag{26}$$

where s is the coordinate of a point of the contour, defined by the arc length s measured from a given point. The initial data are taken to be zero. On seeking the solution as the double layer potential (23), we obtain, with the aid of the first of equations (24), an integral equation for the function $b(\sigma, \tau)$:

$$-b(s, t) + \int_0^t d\tau \int_l \frac{b(\sigma, t)}{4\pi a^2 (t-\tau)^2} e^{-\frac{r^2}{4a^2(t-\tau)}} r \cos(r, n) \, d\sigma = \omega(s, t), \tag{27}$$

where r is the distance between the points s and σ of contour l, and the direction of r is reckoned from σ to s. The integration with respect to σ is carried out over the fixed interval $(0, L)$, where L is the length of contour l, whilst the upper limit is variable when integrating with respect to τ. In other words, this last integral equation is a Fredholm equation with respect to the variable σ and a Volterra equation with respect to the variable τ. In spite of the mixed nature of equation (27), the usual method of successive approximations, such as we described for Volterra equations, is convergent for (27). The method is also applicable for a domain bounded by several contours. It is also easily generalized

for the three-dimensional case and can be used for exterior problems. The reduction of the initial condition to zero is accomplished as in the one-dimensional case, with the aid of a solution of the problem for the entire plane or the whole of space. The formula was given by us in [II,204] for the three-dimensional case. In the two-dimensional case the formula is

$$u(x, y, t) = \frac{1}{4\pi a^2 t} \int_{-\infty}^{+\infty}\int e^{-\frac{r^2}{4a^2 t}} f(\xi, \eta)\, d\xi\, d\eta.$$

An investigation of heat potentials and their application to boundary value problems may be found in the following: (1) E. Levi, *Annali di Matematica*, 1908; (2) Gevrey, *J. de Mathem. Pure et Appl.*, t. 9, 1913; (3) Muntz, *Math. Zeitschr.* Bd. 38, H. 3, 1934; (4) Muntz, *Integral Equations* (Integral'nye uravneniya), Leningrad, 1934; (5) A. N. Tikhonov, Byull. Moskovskogo Universiteta, 1938.

245. Green's function for the heat conduction equation. A Green's function can be formed for the heat conduction equation in precisely the same way as for Laplace's equation. For convenience in writing later formulae, we shall write $u_0(x - \xi, t - \tau)$ for the basic singular solution (11). The Green's function for the segment $0 \leqslant x \leqslant l$ with the homogeneous boundary conditions:

$$u|_{x=0} = u|_{x=l} = 0 \tag{28}$$

is defined as:

$$G(x, t;\ \xi, \tau) = \begin{cases} u_0(x - \xi, t - \tau) - u(x, t; \xi, \tau) & t \geqslant \tau \\ 0 & t \leqslant \tau, \end{cases} \tag{29}$$

where $u(x, t;\ \xi, \tau)$ satisfies the heat conduction equation with respect to (x, t) for $0 < x < l$ and $t > \tau$, the homogeneous initial condition for $t = \tau$:

$$u(x, \tau; \xi, \tau) = 0 \tag{30}$$

and the boundary conditions

$$u(x, t;\ \xi, \tau) = u_0(x, t) \text{ for } x = 0 \text{ and } x = l \text{ and } t \geqslant \tau. \tag{30$_1$}$$

In the above formulae ξ and τ are fixed, whilst $0 < \xi < l$. It follows at once from our definition that $u(x, t;\ \xi, \tau)$ and Green's function depend only on the difference $a = t - \tau$; thus we can write $u(x, \xi, a)$ instead of $u(x, t;\ \xi, \tau)$, and $G(x, \xi, a)$ instead of $G(x, t;\ \xi, \tau)$. Conditions (30) and (30_1) give the boundary values of $u(x, \xi, a)$ on the contour of the half-strip formed by the straight lines $x = 0$ and $x = l$ $(t \geqslant \tau)$, and by the segment $0 \leqslant x \leqslant l$ of the straight line $t = \tau$. These boundary values are continuous at the corners of the half-strip. This follows at once from the fact that solution (11) tends to zero as t tends to $(\tau + 0)$ with fixed x not equal to ξ. On observing that the boundary values indicated are non-negative, we can say that $u(x, t, a) \geqslant 0$, so that, by (29): $G(x, \xi, a) \leqslant u_0$. Green's function has a singularity at $t = \tau + 0$ and $x = \xi$, characterized by the singularity of u_0. We have $u_0 \geqslant 0$ and, by (30), $u(x, \xi, a) \to 0$ as $a \to +0$, and hence it follows directly that the second inequality holds for Green's function, i.e. $G(x, \xi, a) \geqslant 0$. It can be shown that this Green's function is symmetrical with respect to x and ξ.

We can use Green's function to form the solution of the non-homogeneous heat conduction equation satisfying a homogeneous initial condition and homogeneous boundary conditions. In other words, if $\pi(x, t)$ is continuous and has continuous first order derivatives in the interval $(0, l)$ when $t > 0$, the function

$$w(x, t) = \int_0^t d\tau \int_0^l G(x, \xi, a)\, \pi(\xi, \tau)\, d\xi \tag{31}$$

satisfies the equation:

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + \pi(x, t)$$

and the zero initial and boundary conditions.

Everything that has been said can be carried over to the multi-dimensional case. A proof of the above assertions can be found in the work quoted, by Tikhonov.

246. Application of Laplace transforms. As already mentioned, Laplace transforms can be used when solving the system of integral equations (18). The transformation can be applied directly to differential equation (5) itself. We shall employ here the one-sided transformation

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt = L_1(F). \tag{32}$$

Suppose we have boundary conditions (6) and the homogeneous initial condition (10). We replace $u(x, t)$ by its Laplace transform as the required function:

$$\varphi(x, s) = \int_0^\infty e^{-st} u(x, t)\, dt. \tag{33}$$

We use integration by parts, on the assumption that the product $e^{-st} u(x, t)$ vanishes for $t = \infty$. We obtain, on using homogeneous initial condition (16):

$$\varphi(x, s) = -\frac{1}{s} \int_0^\infty u(x, t)\, de^{-st} = \frac{1}{s} \int_0^\infty \frac{\partial u}{\partial t} e^{-st}\, dt.$$

We can change the scale of t or x so as to assume that $a = 1$ in equation (5). On applying the Laplace transformation to both sides of the equation and assuming that we can differentiate with respect to x under the integral sign in (33), we obtain an equation for $\varphi(x, s)$ which contains the derivative with respect to x only:

$$\frac{\partial^2 \varphi}{\partial x^2} = s\varphi. \tag{34}$$

On applying the Laplace transformation to equations (6) also, we obtain the boundary conditions for φ:

$$\varphi|_{x=0} = a_1(s); \quad \varphi|_{x=l} = a_2(s), \tag{35}$$

where

$$a_k(s) = \int_0^\infty e^{-st}\,\omega_k(t)\,dt \quad (k = 1, 2). \tag{36}$$

The solution of (34) with boundary conditions (35) is readily found in the explicit form:

$$\varphi(x, s) = a_1(s)\,\varphi_1(x, s) + a_2(s)\,\varphi_2(x, s), \tag{37}$$

where

$$\varphi_1(x, s) = \frac{\sin(l-x)\sqrt{-s}}{\sin l\sqrt{-s}}; \quad \varphi_2(x, s) = \frac{\sin x\sqrt{-s}}{\sin l\sqrt{-s}}. \tag{38}$$

On applying to function (37) the inverse transformation to (32), we obtain the required function $u(x, t)$. As a matter of fact, this function is simply expressible in terms of the functions $\omega_1(t)$ and $\omega_2(t)$ appearing in the boundary conditions, and in terms of the Jacobi function $\theta_3(v)$ [III$_2$, 176]; in forming this last function, we take $h = e^{-\pi t}$. Let us write $\theta_3(v, t)$ for this Jacobi function:

$$\vartheta_3(v, t) = \sum_{n=-\infty}^{+\infty} e^{2ni\pi v - n^2\pi^2 t}. \tag{39}$$

The following formula will lie at the basis of our future working:

$$L_1[\vartheta_3(v, t)] = -\frac{\cos(2v-1)\sqrt{-s}}{\sqrt{-s}\sin\sqrt{-s}} = \psi(v, s) \quad (0 \leqslant v \leqslant 1), \tag{40}$$

where, for brevity, we have written $\psi(v, s)$ for the fraction indicated. Formula (38) can be rewritten as

$$\left.\begin{aligned} \varphi_1(x, s) &= -\frac{1}{2}\left[\frac{\partial\psi(v, l^2 s)}{\partial v}\right]_{v=\frac{x}{2l}} & (0 \leqslant x \leqslant 2l)\\ \varphi_2(x, s) &= -\frac{1}{2}\left[\frac{\partial\psi(v, l^2 s)}{\partial v}\right]_{v=\frac{l-x}{2l}} & (-l \leqslant x \leqslant l). \end{aligned}\right\} \tag{41}$$

We obviously have, in addition:

$$f(l^2 s) = \int_0^\infty e^{-l^2 st} F(t)\,dt = \frac{1}{l^2}\int_0^\infty e^{-st} F\left(\frac{t}{l^2}\right)dt,$$

i.e. the passage from $f(s)$ to $f(l^2 s)$ in transformation (32) is equivalent to a passage from $F(t)$ to $1/l^2\, F(t/l^2)$. On taking this fact into account, together with (40)

and (41), and carrying out the differentiation with respect to v under the integral sign, we obtain

$$L_1^{-1}\{\varphi_1(x, s)\} = -\frac{1}{2l^2}\left[\frac{\partial\vartheta_3\left(v, \frac{t}{l^2}\right)}{\partial v}\right]_{v=\frac{x}{2l}} =$$

$$= -\frac{1}{l}\,\frac{\partial\vartheta_3\left(\frac{x}{2l}, \frac{t}{l^2}\right)}{\partial x} \qquad (0 \leqslant x \leqslant 2l)$$

$$L_1^{-1}\{\varphi_2(x, s)\} = -\frac{1}{2l^2}\left[\frac{\partial\vartheta_3\left(v, \frac{t}{l^2}\right)}{\partial v}\right]_{v=\frac{l-x}{2l}} =$$

$$= \frac{1}{l}\,\frac{\partial\vartheta_3\left(\frac{l-x}{2l}, \frac{t}{l^2}\right)}{\partial x} \qquad (-l \leqslant x \leqslant l).$$

On now applying the transformation L_1^{-1} to function (37) and using (36) and the convolution theorem, we finally obtain:

$$u(x, t) = -\frac{1}{l}\,\omega_1(t)*\frac{\partial\vartheta_3\left(\frac{x}{2l}, \frac{t}{l^2}\right)}{\partial x} +$$

$$+\frac{1}{l}\,\omega_2(t)*\frac{\partial\vartheta_3\left(\frac{l-x}{2l}, \frac{t}{l^2}\right)}{\partial x}, \tag{42}$$

$$(0 < x < l)$$

where we have introduced the notation:

$$F_1(t)*F_2(t) = \int_0^t F_1(\tau)\,F_2(t-\tau)\,d\tau.$$

We can easily express Green's function, described in the previous section, in terms of the function $\theta_3(v, t)$. We remark first of all that (40) holds only for the interval $0 \leqslant v \leqslant 1$. If $-1 \leqslant v \leqslant 0$, then $0 \leqslant v + 1 \leqslant 1$ and we can write, on taking the periodicity of the function $\theta_3(v, t)$ into account:

$$L_1[\vartheta_3(v, t)] = L_1[\vartheta_3(v+1, t)] = -\frac{\cos[2(v+1)-1]\sqrt{-s}}{\sqrt{-s}\sin\sqrt{-s}} \quad (0 \leqslant v+1 \leqslant 1),$$

i.e.

$$L_1[\vartheta_3(v, t)] = -\frac{\cos(2v+1)\sqrt{-s}}{\sqrt{-s}\sin\sqrt{-s}} \qquad (-1 \leqslant v \leqslant 0). \tag{43}$$

We now take the non-homogeneous equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \pi(x, t) \tag{44}$$

with homogeneous initial and boundary conditions. On introducing the function

$$\sigma(x, s) = L_1[\pi(x, t)] = \int_0^\infty e^{-st}\,\pi(x, t)\,dt \tag{45}$$

and applying the Laplace transformation to equation (44), we obtain

$$\frac{\partial^2 \varphi(x, s)}{\partial x^2} - s\varphi(x, s) = -\sigma(x, s) \tag{46}$$

and the boundary conditions:

$$\varphi(0, s) = \varphi(l, s) = 0. \tag{47}$$

Green's function for the operator on the left-hand side of (46) is easily seen to be, with these boundary conditions:

$$\gamma(x, \xi; s) = \begin{cases} \dfrac{\sin(l-\xi)\sqrt{-s}\cdot\sin x\sqrt{-s}}{\sqrt{-s}\sin l\sqrt{-s}} & (x \leqslant \xi) \\[2ex] \dfrac{\sin(l-x)\sqrt{-s}\cdot\sin\xi\sqrt{-s}}{\sqrt{-s}\sin l\sqrt{-s}} & (x \geqslant \xi), \end{cases} \tag{48}$$

and the solution of equation (46) satisfying boundary conditions (47) can be expressed in terms of this Green's function as

$$\varphi(x, s) = \int_0^l \gamma(x, \xi;\ s)\,\sigma(\xi; s)\,d\xi. \tag{49}$$

In order to perform the transformation L_1^{-1}, we write function (48) in the form:

$$\gamma(x,\xi;\ s) = \begin{cases} -\dfrac{\cos(x-\xi+l)\sqrt{-s}}{2\sqrt{-s}\sin l\sqrt{-s}} + \dfrac{\cos(x+\xi-l)\sqrt{-s}}{2\sqrt{-s}\sin l\sqrt{-s}} & (x \leqslant \xi) \\[2ex] -\dfrac{\cos(x-\xi-l)\sqrt{-s}}{2\sqrt{-s}\sin l\sqrt{-s}} + \dfrac{\cos(x-\xi-l)\sqrt{-s}}{2\sqrt{-s}\sin l\sqrt{-s}} & (x \geqslant \xi). \end{cases} \tag{50}$$

On using the fact that, if $0 \leqslant x \leqslant \xi \leqslant l$, then $-1/2 \leqslant (x-\xi)/2l \leqslant 0$ and $0 \leqslant (x+\xi)/2l \leqslant 1$, whilst if $0 \leqslant \xi \leqslant x \leqslant l$, then $0 \leqslant (x-\xi)/2l \leqslant 1/2$ and $0 \leqslant (x+\xi)/2l \leqslant 1$, together with (40) and (43), we obtain:

$$L_1^{-1}[\gamma(x, \xi; s)] = G(x, \xi; t) = \frac{1}{2l}\left[\vartheta_3\left(\frac{x-\xi}{2l}, \frac{t}{l^2}\right) - \vartheta_3\left(\frac{x+\xi}{2l}, \frac{t}{l^2}\right)\right]. \tag{51}$$

It follows from the convolution theorem that

$$L_1^{-1}[\gamma(x, \xi; s)\,\sigma(\xi; s)] = \int_0^t \pi(\xi, \tau)\,G(x, \xi; t-\tau)\,d\tau\,,$$

so that, by (49):

$$u(x, t) = \int_0^l d\xi \int_0^t \pi(\xi, \tau)\,G(x, \xi; t-\tau)\,d\tau\,. \tag{52}$$

On comparing this formula with (31), we see that the function $G(x, \xi; t - \tau)$, given in terms of $\theta_3(v, t)$ by (51), is the Green's function of the heat conduction equation which we discussed in the previous section.

We must mention the proof of (40), on which all the above working is based. We have had the expression:

$$\cos zx = \frac{2z \sin \pi z}{\pi} \left(\frac{1}{2z^2} + \frac{\cos x}{1^2 - z^2} - \frac{\cos 2x}{2^2 - z^2} + \cdots \right),$$

which holds in the interval $-\pi \leqslant x \leqslant \pi$ [II, 145]. On setting in this $x = 2\pi v - \pi$ and $z = \sqrt{-s}: \pi$, we get:

$$-\frac{\cos (2v-1)\sqrt{-s}}{\sqrt{-s}\sin\sqrt{-s}} = \frac{1}{s} + 2 \sum_{n=1}^{\infty} \frac{\cos 2n\pi v}{s + n^2 \pi^2},$$

and the above inequality for x gives $0 \leqslant v \leqslant 1$. On the other hand, we have the Fourier expansion of $\theta_3(v, t)$ [III$_2$, 176]:

$$\vartheta_3(v, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \cos 2n\pi v .$$

The above series is uniformly convergent with respect to t in any finite interval $0 < \varepsilon \leqslant t \leqslant T$ lying to the right of zero. Assuming that the real part of s is positive and integrating by parts, we get:

$$\int_{\varepsilon}^{T} e^{-st} \vartheta_3(v, t)\, dt = \frac{e^{-s\varepsilon} - e^{-sT}}{s} + 2 \sum_{n=1}^{\infty} \frac{\cos 2n\pi v}{s + n^2 \pi^2} [e^{-(s+n^2\pi^2)\varepsilon} - e^{-(s+n^2\pi^2)T}]$$

In view of the presence of n^2 in the denominator, this series is uniformly convergent with respect to ε and T, and passage to the limit as $\varepsilon \to 0$ and $T \to \infty$ gives us

$$\int_{0}^{\infty} e^{-st} \vartheta_3(v, t)\, dt = \frac{1}{s} + 2 \sum_{n=1}^{\infty} \frac{\cos 2n\pi v}{s + n^2 \pi^2},$$

which yields (40).

A detailed account of the application of Laplace transforms to problems of heat conduction can be found in articles by Doetsch *(Mathem. Zeitschr.* Bd. 22, 25, 26, 28) and in his book *Theorie und Anwendung der Laplace-Transformation.*

247. Application of finite differences. We take the non-homogeneous heat conduction equation:

$$u_t = a^2 u_{xx} + \pi(x, t) \tag{53}$$

with the initial condition:

$$u|_{t=0} = f(x) \qquad (0 \leqslant x \leqslant l) \tag{54}$$

and the homogeneous boundary conditions:

$$u\big|_{x=0} = 0; \quad u\big|_{x=l} = 0\,, \tag{55}$$

we shall assume in future formulae that $a = 1$ and $l = 1$, this being always possible by changing the scales of t and x. We take the interval $[0, T]$ of variation of t and divide it into n equal parts by the points $t_k = kh$ ($k = 0, 1, \ldots, n$), where $h = T : n$. We put $t = t_{k+1}$ in equation (53) and replace the derivative with respect to t by the ratio of the increment of the function to the increment h of the independent variable. We obtain as a result of this replacement a system of ordinary differential equations for the functions $u_k(x)$, which are approximate values for $u(x, t_{k+1})$, since the derivative with respect to t has been replaced by the above-mentioned ratio. The system of differential equations for the $u_k(x)$ obviously has the form:

$$\frac{d^2 u_{k+1}(x)}{dx^2} = \frac{u_{k+1}(x) - u_k(x)}{h} - \pi(x, t_{k+1}) \quad (k = 0, 1, \ldots, n-1). \tag{56}$$

Taking (54) into account, we put $u_0(x) = f(x)$, whilst we subject all the remaining functions $u_{k+1}(x)$ to the boundary conditions (55):

$$u_{k+1}(0) = u_{k+1}(1) = 0 \quad (k = 0, 1, \ldots, n-1). \tag{57}$$

The process of evaluation amounts to the following. On putting $k = 0$ in equation (56) and substituting $u_0(x) = f(x)$, we get a second order equation for $u_1(x)$, which has to be integrated with boundary conditions (57). Having thus found $u_1(x)$, and after putting $k = 1$ in equation (56), we get an equation for $u_2(x)$, which has to be integrated with boundary conditions (57), and so on. It becomes a question of investigating in future an equation of the form

$$\frac{d^2 y}{dx^2} - m^2 y = -\pi(x) \tag{58}$$

with the boundary conditions:

$$y(0) = y(1) = 0\,, \tag{59}$$

where we have written $m^2 = 1/h$. We introduce Green's function for the operator on the left-hand side of (58), with boundary conditions (59). It may easily be seen to have the form [172]:

$$G(x, \xi) = \begin{matrix} -(e^{mx} - e^{-mx})\,[e^{m(\xi-1)} - e^{-m(\xi-1)}] : 2m\,(e^m - e^{-m}) & (x \leqslant \xi) \\ -[e^{m(x-1)} - e^{-m(x-1)}]\,(e^{m\xi} - e^{-m\xi}) : 2m\,(e^m - e^{-m}) & (x \geqslant \xi), \end{matrix} \tag{60}$$

and the solution of (58), satisfying boundary conditions (59), is given by

$$y(x) = \int_0^1 G(x, \xi)\, \pi(\xi)\, d\xi. \tag{61}$$

Let us prove a lemma: *Solutions of* (58), *satisfying boundary conditions* (59), *satisfy the inequality*

$$|y(x)| \leqslant \frac{1}{m^2} \max_{0 \leqslant x \leqslant 1} |\pi(x)|. \tag{62}$$

We first take the case when $\pi(x) \geqslant 0$ in the interval $[0, 1]$. We show that now $y(x) \geqslant 0$. For, if this were not the case, $y(x)$ would have to have a negative minimum inside the interval, and we should have $y'' \geqslant 0$ and $m^2 y < 0$ at the corresponding point, which contradicts (58) with $\pi(x) \geqslant 0$. The inequality $y(x) \geqslant 0$ also follows from (61).

Thus all the values of $y(x)$ are non-negative, and the function takes its greatest positive value at a point inside the interval $[0, 1]$. We must have $y''(x) \leqslant 0$ at this point, and it follows at once from (58) that $-m^2 y(x) \geqslant -\pi(x)$, whence inequality (62) follows. If $\pi(x)$ takes negative values, by using (61) and the fact that Green's function (60) does not take negative values, we obtain the inequality

$$|y(x)| \leqslant \int_0^1 G(x, \xi)\, |\pi(\xi)|\, d\xi. \tag{63}$$

The right-hand side of this inequality is the solution of the equation

$$\frac{d^2 z}{dx^2} - m^2 z = -|\pi(x)|,$$

satisfying boundary conditions (59). As we have just proved, the inequality holds for this solution:

$$z(x) \leqslant \frac{1}{m^2} \max_{0 \leqslant x \leqslant 1} |\pi(x)|.$$

By (63), (62) holds all the more.

Let us introduce the error $\gamma_{k+1}(x)$, due to replacing $u(x, l_{k+1})$ by $u_{k+1}(x)$, and the error $\eta_{k+1}(x)$ due to replacing the derivative by the ratio of the increment of the function to the increment of the independent variable:

$$\left.\begin{aligned} \gamma_{k+1}(x) &= u(x, t_{k+1}) - u_{k+1}(x); \\ \eta_{k+1}(x) &= \left.\frac{\partial u(x, t)}{\partial t}\right|_{t=t_{k+1}} - \frac{u(x, t_{k+1}) - u(x, t_k)}{h}, \end{aligned}\right\} \tag{64}$$

where, obviously, $\gamma_0(x) = 0$. On putting $t = t_{k+1}$ in equation (53) and adding the equation obtained to (56), we obtain

$$\frac{d^2\gamma_{k+1}(x)}{dx^2} = \frac{\gamma_{k+1}(x) - \gamma_k(x)}{h} + \eta_{k+1}(x)$$

or

$$\frac{d^2\gamma_{k+1}(x)}{dx^2} - m^2\gamma_{k+1}(x) = -m^2\gamma_k(x) + \eta_{k+1}(x). \tag{65}$$

If $u(x, t)$ is assumed to have a derivative with respect to t, continuous as far as $t = 0$, application of the finite increments formula to expression (64) for $\eta_{k+1}(x)$ shows us that $|\eta_{k+1}(x)| \leqslant \tau$, where τ does not depend on k and x, and tends to zero along with h. Let δ_k denote the maximum of $|\gamma_k(x)|$ for $0 \leqslant x \leqslant 1$. Application of our lemma to equation (65) gives us: $\delta_{k+1} \leqslant \delta_k + h\tau$. On summing this inequality from $k = 0$ to $k = n - 1$, and using the fact that $\delta_0 = 0$, we get $\delta_n \leqslant nh\tau = T\tau$. This inequality will hold all the more if the summation is taken from $k = 0$ to some $k = m \leqslant n - 1$, i.e.

$$|u(x, t_m) - u_m(x)| \leqslant T\tau \quad (m = 1, 2, \ldots, n-1). \tag{66}$$

The error $\gamma_m(x)$ is thus seen to tend to zero along with h. We have assumed in the proof of this that the solution $u(x, t)$ of the problem exists and that this function has a derivative with respect to t which is continuous up to $t = 0$.

The application of the method of finite differences as described above is due to Rothe (see Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben, *Math. Annal.* Bd. 102, Heft 4/5, 1929). This work considers the more general equation

$$\frac{\partial^2 u}{\partial x^2} = a(x, t)\frac{\partial u}{\partial t} + \pi(x, t, u),$$

and the method described is used for proving the existence of the solution.

If we have the non-homogeneous boundary conditions:

$$u|_{x=0} = \omega_1(t); \quad u|_{x=l} = \omega_2(t),$$

we replace u by the new required function v given by

$$v = u - (1 - x)\omega_1(t) - x\omega_2(t),$$

and reduce the boundary conditions to the homogeneous type. This replacement of the required function changes the function $\pi(x, t)$, but this is of no real significance.

248. Fourier's method. Fourier's method has often been employed above for solving boundary value problems. We shall provide a basis for this method by making use of the theory of integral equations. In the case of three independent variables, we take the homogeneous equation

$$u_t = u_{xx} + u_{yy} \tag{67}$$

in the domain B with contour l, with the conditions:

$$u\,|_{t=0} = f(P) \quad (P \text{ of } \bar{B})\,; \tag{68}$$

$$u\,|_l = 0\,. \tag{69}$$

Fourier's method gives the solution of this problem formally as

$$u(P;\, t) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} v_k(P)\,, \tag{70}$$

where λ_k, $v_k(P)$ are the eigenvalues and eigenfunctions of the equation

$$\Delta v + \lambda v = 0$$

with the boundary condition

$$v\,|_l = 0 \tag{71}$$

and a_k are the Fourier coefficients of the function $f(P)$:

$$a_k = \iint_B f(P)\, v_k(P)\, \mathrm{d}S\,. \tag{72}$$

Suppose that $f(P)$ is itself continuous, has continuous derivatives up to the second order in the closed domain $\bar{B}$ and vanishes on l. Now [22]:

$$f(P) = \sum_{k=1}^{\infty} a_k v_k(P) \tag{73}$$

and this series is regularly convergent in $\bar{B}$, i.e. the series

$$\sum_{k=1}^{\infty} |\, a_k v_k(P)\,| \tag{74}$$

is uniformly convergent in $\bar{B}$.

Since $0 \leqslant e^{-\lambda_k t} \leqslant 1$ for $t \geqslant 0$, we can say that series (70) is also regularly convergent if P belongs to $\bar{B}$ and $t \geqslant 0$. Hence its sum $u(P, t)$ is a continuous function of P and t if P belongs to $\bar{B}$ and $t \geqslant 0$. It follows from this that

$$\lim_{t \to +0} u(P;\, t) = u(P;\, 0) = \sum_{k=1}^{\infty} a_k v_k(P) = f(P)\,,$$

i.e. the function $u(P, t)$ defined by (70) satisfies the initial condition (68). Further, each of the functions $v_k(P)$ satisfies the boundary condition (69), so that $u(P, t)$ also satisfies this condition for $t \geqslant 0$. It remains to show that $u(P, t)$ has a continuous derivative with respect to t and continuous derivatives u_{xx}, u_{yy}, and satisfies equation (67) inside B and with $t > 0$.

We differentiate series (70) term by term with respect to t:

$$-\sum_{k=1}^{\infty} a_k \lambda_k e^{-\lambda_k t} v_k(P), \tag{75}$$

and let a be an arbitrarily chosen positive number. On taking into account the uniform convergence of series (74) and the fact that $0 < \lambda_k e^{-\lambda_k a} < 1$ for all sufficiently large k, we can assert that series (75) is regularly convergent if P belongs to $\overline{B}$ and $t \geqslant a$. It can similarly be shown that the series

$$\sum_{k=1}^{\infty} a_k \lambda_k^2 e^{-\lambda_k t} v_k(P),$$

obtained by term by term differentiation of series (75) with respect to t, is also regularly convergent under the above conditions. Hence it follows that $u(P, t)$ has continuous derivatives of the first and second orders with respect to t for $t > 0$ and P belonging to $\overline{B}$, and we have for these derivatives:

$$u_t(P; t) = -\sum_{k=1}^{\infty} a_k \lambda_k e^{-\lambda_k t} v_k(P); \tag{76$_1$}$$

$$u_{tt}(P; t) = \sum_{k=1}^{\infty} a_k \lambda_k^2 e^{-\lambda_k t} v_k(P). \tag{76$_2$}$$

A similar argument can be employed for the derivatives of any order with respect to t.

But we have:

$$v_k(P) = \lambda_k \iint_B G(P; Q)\, v_k(Q)\, dS,$$

where $G(P; Q)$ is Green's function for Laplace's operator, subject to boundary condition (71), and expression (76_1) can be rewritten as

$$u_t(P; t) = -\sum_{k=1}^{\infty} \iint a_k \lambda_k^2 e^{-\lambda_k t} G(P; Q)\, v_k(Q)\, dS.$$

On taking into account the uniform convergence of series (76_2) in $\bar{B}$ for $t > 0$, we can interchange the summation and integration, and obtain:

$$u_t(P;t) = -\iint_B G(P;Q)\, u_{tt}(Q;t)\, dS\,, \tag{77}$$

and similarly:

$$u(P;t) = -\iint_B G(P;Q)\, u_t(Q;t)\, dS\,. \tag{78}$$

The function $u_{tt}(Q;t)$ is continuous in $\bar{B}$ for $t > 0$, and it follows from (77) that $u_t(P;t)$ has continuous first order derivatives with respect to the coordinates (x, y) of the point P inside B for $t > 0$. After this, (78) shows that $u(P;t)$ has continuous derivatives up to the second order inside B for $t > 0$ and satisfies the equation:

$$\Delta u(P;t) = u_t(P;t)\,,$$

which is what we wanted to show.

249. Non-homogeneous equations. We now consider the non homogeneous equation:

$$u_t = u_{xx} + u_{yy} + \pi(x, y, t) \tag{79}$$

with homogeneous initial and boundary conditions:

$$\lim_{t\to+0} u = 0; \tag{80}$$

$$u\,|_l = 0\,. \tag{81}$$

We introduce the Fourier coefficients of the function π:

$$b_k(t) = \iint_B \pi(P;t)\, v_k(P)\, dS \tag{82}$$

and seek the solution of the problem as

$$u(P;t) = \sum_{k=1}^{\infty} c_k(t)\, v_k(P)\,. \tag{83}$$

On substituting in equation (79) and taking into account the fact that $\Delta v_k = -\lambda_k v_k$, we obtain the differential equation for coefficients $c_k(t)$:

$$c_k'(t) = -\lambda_k\, c_k(t) + b_k(t)\,,$$

whence, on taking (80) into account, i.e. $c_k(0) = 0$, we obtain

$$c_k(t) = \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' ,$$

and we substitute this in (83):

$$u(P; t) = \sum_{k=0}^{\infty} v_k(P) \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' . \tag{84}$$

This solution is justified with the following assumptions regarding the function $\pi(P, t)$: it has continuous first order derivatives with respect to the coordinates of the point P for every $t \geqslant 0$ inside B, and the series

$$\sum_{k=1}^{\infty} b_k(t)\, v_k(P); \quad \sum_{k=1}^{\infty} b_k(t)\, \lambda_k\, v_k(P); \quad \sum_{k=1}^{\infty} b_k(t)\, \lambda_k^2\, v_k(P) \tag{85}$$

are regularly convergent if P belongs to the closed domain $\overline{B}$ and t to any finite interval $[0, T]$. On taking into account the regular convergence of the first of series (85) and the fact that $0 \leqslant e^{\lambda_k(t'-t)} \leqslant 1$ for $0 \leqslant t' \leqslant t$, we can assert that the series on the right-hand side of (84) is uniformly convergent with the conditions indicated for P and t. Its sum $u(P, t)$ is a continuous function of P and t, given these conditions for P and t. It follows at once from the form of the right-hand side of (84) that $u(P, t)$ satisfies conditions (80) and (81).

It remains to verify that the function $u(P, t)$ defined by (84) has the corresponding continuous derivatives inside B for $t > 0$, and satisfies equation (79). On differentiating the series appearing in (84) term by term with respect to t, we obtain

$$\sum_{k=1}^{\infty} b_k(t)\, v_k(P) - \sum_{k=1}^{\infty} \lambda_k\, v_k(P) \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt'.$$

On taking into account the regular convergence of the second of series (85), we can say that the series forming the subtrahend in the above difference is uniformly convergent under the above conditions for P and t. The sum of the series forming the first term of the difference is equal to $\pi(P, t)$, since this series is regularly convergent by hypothesis [22]. We therefore have:

$$u_t(P; t) = \pi(P; t) - \sum_{k=1}^{\infty} v_k(P)\, \lambda_k \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' , \tag{86}$$

$u_t(P, t)$ being continuous with the indicated conditions for P and t.

On replacing P by Q in this formula, multiplying both sides by $G(P, Q)$ and integrating over B, we obtain, on taking into account the integral equation for $v_k(P)$:

$$\int_B\int G(P; Q)\, u_t(Q; t)\, dS =$$
$$= \int_B\int G(P; Q)\, \pi(Q; t)\, dS - \sum_{k=1}^{\infty} v_k(P) \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' ,$$

the sum of the last series being equal to $u(P, t)$. Thus

$$u(P; t) = -\int_B\int G(P; Q)\, u_t(Q; t)\, dS + \int_B\int G(P; Q)\, \pi(Q; t)\, dS . \quad (87)$$

Since $\pi(Q, t)$ has continuous derivatives inside B, we can say that the last integral has continuous derivatives inside B up to the second order with respect to the coordinates of the point P, and Laplace's operator of this integral is equal to $[-\pi(Q; t)]$.

We now make use of the regular convergence of the third of series (85) and show that $u(P, t)$ has continuous derivatives up to the second order inside B and satisfies equation (79).

We use the notation:

$$w(Q; t) = -u_t(P; t) + \pi(P; t) = \sum_{k=1}^{\infty} v_k(P)\, \lambda_k \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' .$$

On taking into account the regular convergence of the third of series (85), we can differentiate the uniformly convergent series written above term by term with respect to t, which gives us:

$$w_t(P; t) = \sum_{k=1}^{\infty} \lambda_k\, b_k(t)\, v_k(P) - \sum_{k=1}^{\infty} v_k(P)\, \lambda_k^2 \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' ,$$

these last series being uniformly convergent. On replacing P by Q in the last expression, multiplying both sides by $G(P; Q)$, integrating and taking into account the integral equation for $v_k(P)$, we get:

$$\int_B\int G(P; Q)\, w_t(Q; t)\, dS =$$
$$= \sum_{k=1}^{\infty} b_k(t)\, v_k(P) - \sum_{k=1}^{\infty} v_k(P)\, \lambda_k \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt' ,$$

and hence we have, on taking (86) into account:

$$u_t(P; t) = \int_B\int G(P; Q)\, w_t(Q; t)\, dS ,$$

whence it follows that $u_t(P, Q)$ has continuous first order derivatives inside B. After this, (87) shows that $u(P, t)$ has continuous derivatives up to the second order inside B and satisfies the equation

$$\Delta u(P; t) = u_t(P; t) - \pi(P; t),$$

and hence (84) is fully proved.

If we are talking of only the generalized solution of equation (79), the formula can be justified with fewer assumptions regarding the function π. We recall the definition of the generalized solution of (79). Let D be the cylinder which we referred to in [242], and D_1 the part of it bounded from above by the plane $t = T$. The function $u(P, t)$ is called a generalized solution of the equation if the following formula holds for any function $\sigma(P, t)$, having continuous derivatives up to the second order inside D_1 and vanishing at all points sufficiently close to the boundary of D_1:

$$\iiint_{D_1} u(\sigma_{xx} + \sigma_{yy} + \sigma_t)\, dx\, dy\, dt = - \iiint_{D_1} \pi\sigma\, dx\, dy\, dt. \qquad (88)$$

We now assume that the first of series (85) is regularly convergent if P belongs to B and t lies in the finite interval $[0, T]$. The sum of the series is now equal to $\pi(P, t)$, and, as we saw above, series (84) is uniformly convergent.

Let $\pi_n(P, t)$ denote the segment of the first of series (85):

$$\pi_n(P; t) = \sum_{k=1}^{n} b_k(t)\, v_k(P),$$

and $u_n(P, t)$ the segment of series (84):

$$u_n(P; t) = \sum_{k=1}^{n} v_k(P) \int_0^t e^{\lambda_k(t'-t)}\, b_k(t')\, dt',$$

The function $u_n(P, t)$ satisfies equation (79) with π replaced by $\pi_n(P, t)$. We can therefore write:

$$\iiint_{D_1} u_n(\sigma_{xx} + \sigma_{yy} + \sigma_t)\, dx\, dy\, dt = - \iiint_{D_1} \pi_n\, \sigma\, dx\, dy\, dt.$$

On passing to the limit as $n \to \infty$ and using the fact that $\pi_n(P, t) \to \to \pi(P, t)$ and $u_n(P, t) \to u(P, t)$ uniformly in D_1, we obtain (88), i.e. the function $u(P, t)$ defined by (84) is the generalized solution of equation (79). It is immediately evident, in addition, that this sum satisfies conditions (80) and (81).

If we make use of the fact that a generalized solution of the homogeneous heat conduction equation is an actual solution of this equation [160], together with the uniqueness theorem for the solution of a boundary value problem of the heat conduction equation, it can be shown, precisely as for Poisson's equation, that the generalized solution of non-homogeneous equation (79) with given initial and boundary conditions is unique [224].

250. Properties of the solutions of the heat conduction equation. We take the equation

$$u_t - u_{xx} = 0. \tag{89}$$

Let $u(x, t)$ be a solution of this equation, having continuous derivatives u_x and u_t at and in the neighbourhood of some point M. It follows from (89) that the derivative u_{xx} is now continuous.

We surround the point M with a sufficiently small rectangle $ABCD$, with sides parallel to the axes (Fig. 19), such that the above solution $u(x, t)$ exists in this rectangle. We take the origin at the point A, and let l be the length of AB. We write $\omega_1(t)$, $\omega_2(t)$ for the values of our solution on the sides AD and BC, and $f(x)$ for its value on the side AB. We first take the case when $f(x) \equiv 0$. We can write the solution $u(x, t)$ in accordance with (17) as

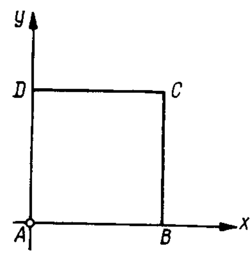

FIG. 19

$$u(x, t) = \int_0^t \frac{\varphi(\tau)}{2a\sqrt{\pi}\,(t-\tau)^{3/2}}\, x e^{-\frac{x^2}{4a^2(t-\tau)}}\, d\tau +$$

$$+ \int_0^t \frac{\psi(\tau)}{2a\sqrt{\pi}\,(t-\tau)^{3/2}}\,(x - l)\, e^{-\frac{(x-l)^2}{4a^2(t-\tau)}}\, d\tau, \tag{90}$$

where the continuous functions $\varphi(\tau)$ and $\psi(\tau)$ are found from integral equations (18). The uniqueness theorem for equation (89) must be borne in mind here.

Let the point (x_0, t_0) lie inside $ABCD$. Let us take say the first of the integrals appearing on the right-hand side of (90). If we replace x_0 in it by $x' + x''i$, where x' is sufficiently close to x_0 and x'' is

sufficiently close to zero, the real part of $(x' + x''i)^2$ will be positive, whence it is clear that the integral in question is uniformly convergent at the limit $\tau = t$ with respect to the parameter $x = x' + x''i$ for all complex x sufficiently close to x_0; on the other hand, the integrand is here an entire function of x for $0 \leqslant \tau < t$. Hence it follows that the magnitude of the integral is a holomorphic function of x in the neighbourhood of every point (x, t) lying inside $ABCD$ [III_2, 70], and in particular, at the point M. The same can be said as regards the second of the integrals appearing on the right-hand side of (90).

Therefore, *the solutions of equation* (89) *are analytic functions of the variable* x.

This assertion is not valid as regards the variable t. For, if every solution of (89) were an analytic function of t, the values of the function on any straight line parallel to the t axis and belonging to the half-strip illustrated in Fig. 19 would be fully defined, by virtue of the principle of analytic continuation, by the values which this function has on the segment of the straight line belonging to $ABCD$. But this is not the case, since the values of u obviously depend on the actual method by which the functions $\omega_1(t)$ and $\omega_2(t)$ are continued, these being given initially only on the segments AD and BC of $x = 0$ and $x = 1$.

We have so far assumed that $f(x_0) \equiv 0$ in the interval $(0, l)$. If this is not the case, we can continue this function on the wider interval $[a, b]$ such that it is equal to zero at the ends of the interval, and then continue it as zero outside the interval. We form the difference:

$$u - \frac{1}{2\sqrt{\pi t}} \int_a^b f(\xi)\, e^{-\frac{(\xi - x)^2}{4t}}\, d\xi.$$

This difference has zero values on the segment AB, and the above arguments are applicable to it. It remains to consider the solution:

$$u_0(x, t) = \frac{1}{2\sqrt{\pi t}} \int_0^b f(\xi)\, e^{-\frac{(\xi - x)^2}{4t}}\, d\xi.$$

On applying the theorem on an integral depending on a parameter [III_2, 70], we see that $u_0(x, t)$ is a regular function of (x, t) in the neighbourhood of any point lying on the $t = 0$ axis, i.e. for $t > 0$. We remark further that it follows at once from (90) that the function u has derivatives of all orders with respect to t for $0 < x < l$.

An upper bound can be found for the derivatives of the solution of equation (89) with respect to t. We take the solution u of (89) having continuous derivatives u_x and u_t at the origin and in its neighbourhood, and assume that u is an odd function of x. We have the Maclaurin expansion:

$$u = u_1(t)\, x + \frac{u_3(t)}{3!} x^3 + \ldots + \frac{u_{2n+1}(t)}{(2n+1)!} x^{2n+1} + \ldots, \tag{91}$$

where

$$u_{2n+1}(t) = \left.\frac{\partial^{2n+1} u}{\partial x^{2n+1}}\right|_{x=0} \qquad (n = 0, 1, 2, \ldots)$$

On using (89), we can write:

$$u_{2n+1}(t) = \frac{\partial^n}{\partial t^n}\left(\frac{\partial u}{\partial x}\right)_{x=0} = \frac{d^n u_1(t)}{dt^n}\,. \tag{92}$$

If ϱ is a positive number less than the radius of convergence of series (91), we have the inequality [III_2, 83]:

$$\left|\frac{u^{2n+1}(t)}{(2n+1)!}\right| \leqslant \frac{M}{\varrho^n}\,,$$

where M is a positive number. The following inequality for the derivatives of function $u_1(t)$ is a consequence of (92):

$$\left|\frac{d^n u_1(t)}{dt^n}\right| \leqslant \frac{M(2n+1)!}{\varrho^n}\,.$$

This inequality does not guarantee that $u_1(t)$ is analytic. If we had the stronger inequality:

$$\left|\frac{d^n u_1(t)}{dt^n}\right| \leqslant \frac{M \cdot n!}{\varrho^n}\,,$$

the Maclaurin series of $u_1(t)$ would be convergent, and this function would be regular in the neighbourhood of the origin.

251. The generalized potentials of a simple and double layer in the one-dimensional case. We discussed in [243] the solution of the boundary value problem in the half-strip bounded from below by the characteristic $t = 0$ of equation (5), and at the sides by $x = 0$ and $x = l$. We now consider the domain on the (x, t) plane which is

bounded from below by the characteristic $t = b$, and at the sides by the two curves l_i with explicit equations (Fig. 20):

$$x = \sigma_1(t); \quad x = \sigma_2(t) \qquad [\sigma_1(t) < \sigma_2(t)], \tag{93}$$

where $\sigma_i(t)$ have continuous derivatives for $t \geqslant b$. To solve the problem in this domain, we have to construct the generalized potentials of a

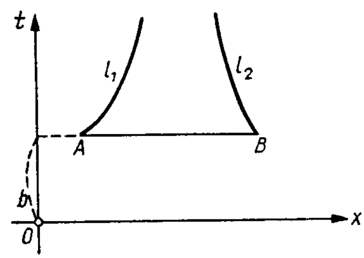

FIG. 20

simple and double layer, which, with $\sigma_i(t) = \text{const}$, become the potentials indicated in [243]. These generalized potentials have the form:

$$u_i(x, t) = \frac{1}{2a\sqrt{\pi}} \int\limits_b^t \frac{\varphi_i(t')}{\sqrt{t-t'}} \, e^{-\frac{[\sigma_i(t')-x]^2}{4a^2(t-t')}} \, dt', \tag{94}$$

$$v_i(x, t) = \frac{1}{2a\sqrt{\pi}} \int\limits_b^t \frac{\psi_i(t')}{(t-t')^{3/2}} [x - \sigma_i(t')] \, e^{-\frac{[\sigma_i(t')-x]^2}{4a^2(t-t')}} \, dt', \tag{95}$$

where $\varphi_i(t')$ and $\psi_i(t')$ are continuous functions.

The functions $u_i(x, t)$ and $v_i(x, t)$ have continuous derivatives and satisfy equation (5) everywhere outside the curves l_i. Both potentials also have a meaning when the point (x, t) lies on the curves l_i. This is immediately obvious for the potential $u_i(x, t)$, since the integrand is bounded by $C(t - t')^{-1/2}$, where C is a constant. If the point (x, t) lies on l_i, we can write for the potential $v_i(x, t)$:

$$\frac{[x - \sigma_i(t')]}{(t-t')^{3/2}} = \frac{[\sigma_i(t) - \sigma_i(t')]}{(t-t')^{3/2}} = \frac{\sigma_i'(t_0)}{(t-t')^{1/2}} \quad (t' < t_0 < t),$$

whence the convergence of integral (95) follows.

The magnitude of integral (94) over the small section: $t - \delta \leqslant t' \leqslant t$ tends to zero as $\delta \to 0$ for any position of the point (x, t), and

hence it follows immediately that $u_i(x, t)$ is continuous as far as l_i. Integral (95) has different limits as (x, t) tends to the point (x_0, t_0) on l_i:

$$\lim_{(x,t)\to(x_0,t_0)} v_i(x, t) = \pm \psi_i(t_0) + v_i(x_0, t_0), \tag{96}$$

where $v_i(x_0, t_0)$ is the value of integral (95) at (x_0, t_0) itself; the $(+)$ sign has to be taken if $(x, t) \to (x_0, t_0)$ from the right of l_i, and the $(-)$ sign if $(x, t) \to (x_0, t_0)$ from the left of l_i. If $\sigma_i(t) =$ const., obviously $v_i(x_0, t_0) = 0$, and we get the result of [243]. We omit the suffix i when proving (96).

We consider integral (95) with $\varphi(t') = 1$:

$$v_0(x, t) = \frac{1}{2a\sqrt{\pi}} \int_b^t \frac{1}{(t-t')^{3/2}} [x - \sigma(t')] \, e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}} \, dt' \tag{97}$$

and the function

$$w_0(x, t) = \frac{1}{2a\sqrt{\pi}} \int_b^t \frac{-2\sigma'(t')}{\sqrt{t-t'}} \, e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}} \, dt'. \tag{98}$$

On replacing t' by the new variable of integration

$$z = \frac{x - \sigma(t')}{2a\sqrt{t-t'}},$$

we obtain

$$v_0(x, t) + w_0(x, t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x-\sigma(b)}{2a\sqrt{t-b}}}^{\pm\infty} e^{-z^2} \, dz, \tag{99}$$

where the $(+)$ sign must be taken in the upper limit if $x - \sigma(t) > 0$, and the $(-)$ sign if $x - \sigma(t) < 0$. If the point (x_0, t_0) lies on l, i.e. $x_0 - \sigma(t_0) = 0$, then

$$v_0(x_0, t_0) + w_0(x_0, t_0) = \frac{2}{\pi} \int_{\frac{x_0-\sigma(b)}{2a\sqrt{t_0-b}}}^{0} e^{-z^2} \, dz. \tag{100}$$

It follows at once from the definition of $w_0(x, t)$, as above, that $w_0(x, t)$ is continuous as far as l. It follows from (99) that

$$\lim_{(x,t)\to(x_0,t_0)} [v_0(x, t) + w_0(x, t)] = \frac{2}{\sqrt{\pi}} \int_{\frac{x_0-\sigma(b)}{2a\sqrt{t_0-b}}}^{\pm\infty} e^{-z^2} \, dz,$$

and, on subtracting (100) term by term from the last formula, we get

$$\lim_{(x,\,t)\to(x_0,\,t_0)} v_0(x,t) = v_0(x_0,t_0) + \frac{2}{\sqrt{\pi}} \int_0^{\pm\infty} e^{-z^2}\, dz,$$

i.e.

$$\lim_{(x,\,t)\to(x_0,\,t_0)} v_0(x,t) = v_0(x_0,t_0) \pm 1,$$

and we have obtained (96) with $\psi(t') \equiv 1$.

We turn to the general case. We rewrite the expression for $v(x, t)$ as

$$v(x,t) = \frac{1}{2a\sqrt{\pi}} \int_b^t [\psi(t') - \psi(t_0)] \frac{x-\sigma(t')}{(t-t')^{3/2}}\, e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}}\, dt' +$$
$$+ \frac{\psi(t_0)}{2a\sqrt{\pi}} \int_b^t \frac{1}{(t-t')^{3/2}} [x-\sigma(t')]\, e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}}\, dt'. \qquad (101)$$

Precisely as in [193], it is sufficient to show that the first term retains its continuity when the point (x, t) cuts l at the point (x_0, t_0). Let ε be a given positive number. We choose a positive δ so small that

$$|\psi(t') - \psi(t_0)| \leqslant \varepsilon \text{ for } |t' - t_0| \leqslant \delta,$$

and divide the interval of integration $b \leqslant t' \leqslant t$ into parts $b \leqslant t' \leqslant t_0 - \delta$ and $t_0 - \delta \leqslant t' \leqslant t$. The function which is expressed by the integral over the first of these intervals is continuous at the point (x_0, t_0), and it is sufficient to show that

$$\frac{1}{2a\sqrt{\pi}} \int_{t_0-\delta}^t [\psi(t') - \psi(t_0)] \frac{x-\sigma(t')}{(t-t')^{3/2}}\, e^{-\frac{[\sigma(t')-x]^2}{4a^2(-)}}\, dt$$

is sufficiently small for all positions of the point (x, t) provided the latter is sufficiently close to the point (x_0, t_0) or coincides with it. The absolute value of this last integral does not exceed

$$\frac{\varepsilon}{2a\sqrt{\pi}} \int_{t_0-\delta}^t \frac{|x-\sigma(t')|}{(t-t')^{3/2}}\, e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}}\, dt'.$$

We assume that the difference $x - \sigma(t')$ changes sign not more than k times, where k is a definite positive integer, for $t_0 - \delta \leqslant t' \leqslant t$

and an arbitrary position of (x, t) in some neighbourhood of (x_0, t_0). The integral

$$\int_{t_0-\delta}^{t} \frac{|x-\sigma(t')|}{(t-t')^{3/2}} e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}} dt'$$

is now the sum of not more than k integrals, of the form:

$$\pm \int_{t_i}^{t_{i+1}} \frac{x-\sigma(t')}{(t-t')^{3/2}} e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}} dt' \quad (t_0-\delta \leqslant t_i \leqslant t_{i+1} \leqslant t),$$

which differ from the integrals

$$\pm 4a \int_{\frac{x-\sigma(t_i)}{2a\sqrt{t-t_i}}}^{\frac{x-\sigma(t_{i+1})}{2a\sqrt{t-t_{i+1}}}} e^{-z^2} dz \text{ by the magnitude } \pm \int_{t_i}^{t_{i+1}} \frac{-2\sigma(t')}{\sqrt{t-t'}} e^{-\frac{[\sigma(t')-x]^2}{4a^2(t-t')}} dt',$$

the absolute value of which does not exceed some constant. The above-mentioned integral therefore remains bounded when (x, t) is situated in some neighbourhood of (x_0, t_0). This integral is multiplied by ε, and hence we can use the same method as in [193] to complete the proof that the first term on the right-hand side of (101) is continuous when (x, t) cuts l at the point (x_0, t_0). We can make use of the above potentials to reduce a boundary value problem for the domain indicated in Fig. 20 to an integral equation, as we did in [243]. Suppose that the conditions are: $u = 0$ on the characteristic $t = b$ and $u = \omega_i(t)$ on l_i. We shall seek the solution in the form

$$u(x,t) = \frac{1}{2a\sqrt{\pi}} \sum_{i=1}^{2} \int_b^t \frac{\psi_i(t)}{(t-t')^{3/2}} [x-\sigma_i(t')] e^{-\frac{[\sigma_i(t')-x]^2}{4a^2(t-t')}} dt'. \quad (102)$$

Now, equation (5) and the boundary condition on the characteristic $t = b$ are satisfied for every $\psi_i(t')$, whilst we obtain from the boundary conditions on l_i the system of Volterra integral equations for $\psi_i(t)$:

$$\left.\begin{aligned} \omega_1(t) &= \psi_1(t) + \frac{1}{2a\sqrt{\pi}} \sum_{i=1}^{2} \int_b^t \frac{\psi_i(t')}{(t-t')^{3/2}} [\sigma_1(t) - \sigma_i(t')] e^{-\frac{[\sigma_i(t')-\sigma_1(t)]^2}{4a^2(t-t')}} dt'; \\ \omega_2(t) &= -\psi_2(t) + \frac{1}{2a\sqrt{\pi}} \sum_{i=1}^{2} \int_b^t \frac{\psi_i(t')}{(t-t')^{3/2}} [\sigma_2(t) - \sigma_i(t')] e^{-\frac{[\sigma_i(t')-\sigma_2(t)]^2}{4a^2(t-t')}} dt'. \end{aligned}\right\} \quad (103)$$

We consider the integrals appearing in the first equation:

$$\int_b^t \frac{\varphi_1(t')}{(t-t')^{3/2}} [\sigma_1(t) - \sigma_1(t')] \, e^{-\frac{[\sigma_1(t')-\sigma_1(t)]^2}{4a^2(t-t')}} \, dt', \tag{104}$$

$$\int_b^t \frac{\varphi_2(t')}{(t-t')^{3/2}} [\sigma_1(t) - \sigma_2(t')] \, e^{-\frac{[\sigma_2(t')-\sigma_1(t)]}{4a^2(t-t')}} \, dt'. \tag{105}$$

The polarity in integral (104) for $t' = t$ is lowered on account of the numerator of the fraction

$$\frac{\sigma_1(t) - \sigma_1(t')}{(t-t')^{3/2}},$$

in the same way as remarked above. In the second integral, the index of e tends to $(-\infty)$ as $t' \to t$, and this completely removes the polarity. The integrals for the second of equations (103) may be similarly considered. Hence system (103) has a unique solution and can be solved by the method of successive approximations.

The solution of the above boundary value problem can also be sought as the sum of two simple layer potentials:

$$u(x, t) = \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^t \frac{\varphi_j(t')}{\sqrt{t-t'}} \, e^{-\frac{[\sigma_j(t')-x]^2}{4a^2(t-t')}} \, dt'. \tag{106}$$

We now arrive at a system of integral equations of the first kind:

$$\begin{aligned} \omega_1(t) &= \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^t \frac{\varphi_j(t')}{\sqrt{t-t'}} \, e^{-\frac{[\sigma_j(t)-\sigma_1(t)]^2}{4a^2(t-t')}} \, dt'. \\ \omega_2(t) &= \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^t \frac{\varphi_j(t')}{\sqrt{t-t'}} \, e^{-\frac{[\sigma_j(t)-\sigma_2(t)]^2}{4a^2(t-t')}} \, dt'. \end{aligned} \tag{107}$$

We multiply both sides by $(y-t)^{-1/2}$ and integrate with respect to t from $t = b$ to $t = y$:

$$\left.\begin{aligned} f_1(y) &= \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^y \varphi_j(t') K_{1j}(t', y) \, dt'; \\ f_2(y) &= \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^y \varphi_j(t') K_{2j}(t', y) \, dt', \end{aligned}\right\} \tag{108}$$

where

$$\left.\begin{aligned} f_i(y) &= \int_b^y \frac{\omega_i(t)}{\sqrt{y-t}}\,\mathrm{d}t; \\ K_{ij}(t', y) &= \int_{t'}^{y} \frac{1}{\sqrt{(y-t)(t-t')}}\, e^{-\frac{[\sigma_j(t')-\sigma_i(t)]^2}{4a^2(t-t')}}\,\mathrm{d}t; \end{aligned}\right\} \tag{109}$$

here, we have changed the order of integration on the right-hand side and applied Dirichlet's formula [II, 79]. System (108) is equivalent to (107) [cf. II, 79]. We obviously have:

$$\lim_{t\to t'+0} e^{-\frac{[\sigma_j(t')-\sigma_i(t)]^2}{4a^2(t-t')}} = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j, \end{cases}$$

and, on taking into account the formula [II, 79]:

$$\int_{t'}^{y} \frac{\mathrm{d}t}{\sqrt{(y-t)(t-t')}} = \pi,$$

we obtain

$$K_{11}(y, y) = K_{22}(y, y) = \pi; \qquad K_{12}(y, y) = K_{21}(y, y) = 0.$$

We differentiate system (108) with respect to y, on the assumption that the $\omega_i(t)$ have continuous derivatives, in which case the functions $f_i(y)$ also have continuous derivatives [II, 85]:

$$\left.\begin{aligned} f_1'(y) &= \frac{\sqrt{\pi}}{2a}\varphi_1(y) + \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^y \varphi_j(t') \frac{\partial K_{1j}(t', y)}{\partial y}\,\mathrm{d}t'; \\ f_2'(y) &= \frac{\sqrt{\pi}}{2a}\varphi_2(y) + \frac{1}{2a\sqrt{\pi}} \sum_{j=1}^{2} \int_b^y \varphi_j(t') \frac{\partial K_{2j}(t', y)}{\partial y}\,\mathrm{d}t'. \end{aligned}\right\} \tag{110}$$

To evaluate the derivatives, we write $K_{ij}(t', y)$ in the form:

$$K_{ij}(t', y) = \int_{t'}^{y} e^{-\frac{[\sigma_j(t')-\sigma_i(t)]^2}{4a^2(t-t')}}\,\mathrm{d}\left[\arcsin\left(2\frac{t-t'}{y-t'} - 1\right)\right].$$

On integrating by parts and differentiating with respect to y, we obtain

$$\frac{\partial K_{ij}(t', y)}{\partial y} = -\frac{2}{(y-t')}\int_{t'}^{y} \frac{1}{\sqrt{(y-t)(t-t')}}\, e^{-\frac{[\sigma_j(t')-\sigma_i(t)]^2}{4a^2(t-t')}} \times$$

$$\times \left\{[\sigma_i(t) - \sigma_j(t')]\,\sigma_i'(t) - \frac{[\sigma_j(t') - \sigma_i(t)]^2}{2(t-t')}\right\}\mathrm{d}t.$$

When $i \neq j$, the convergence of the integral at the limit $t = t'$ is guaranteed by the exponential function, whilst when $i = j$ the fraction in the braces has no polarity, and the entire expression in the braces tends to zero with $(t - t')$. On taking all this into account and using elementary inequalities for the integrand with $i \neq j$ and $i = j$, we arrive in both cases at an integral of the form

$$C \int_{t'}^{y} \frac{dt}{\sqrt{y-t}},$$

where C is a constant. It is evident from this that

$$\frac{\partial K_{ij}(t', y)}{\partial y} = \frac{L_{ij}(t', y)}{\sqrt{y-t'}},$$

where $L_{ij}(t', y)$ is a continuous function of (t', y), and the method of successive approximations [43] can be applied to system (110).

252. Sub- and superparabolic functions. When solving a boundary value problem for the heat conduction equation, a method can be used analogous to the method of upper and lower functions which we described in [217]. We take a domain B on the (x, t) plane, bounded from above and below by the characteristics $t = 0$ and $t = b$, and from the left and right by curves having the equations (93); no assumptions will be made as yet regarding the properties of the functions $\sigma_i(t)$, except that they are single-valued continuous functions and $\sigma_1(t) < \sigma_2(t)$. In order to define sub- and superparabolic functions, we have to choose some basic domain, for which we are able to solve the boundary value problem for the equation

$$u_t - u_{xx} = 0 \tag{111}$$

with any continuous boundary conditions. This domain was a circle for Laplace's equation. In the case of equation (111) we choose say an isosceles triangle β, the base of which is parallel to the $t = 0$ axis, whilst the sides are directed towards increasing t. The solution $u(x, t)$ of the boundary value problem for such a triangle can be obtained by the method indicated in [251] and is unique. This solution, attains its maximum and minimum value on the sides of the triangle [II, 209]. The function $\varphi(M) = \varphi(x, t)$, continuous in the closed domain $\overline{B}$, is said to be subparabolic if its value $\varphi(M_0)$ at any interior point M_0 of B is not greater than the value at this point of the solution of equation (111) in any sufficiently small triangle β containing M_0 as an interior point, the value of the solution on the sides of β being equal to $\varphi(M)$. A superparabolic function $\psi(M) = \psi(x, t)$ is defined in a similar way, except that $\psi(M_0)$ must be not less than the values of the solutions of (111) in the triangles β. The least value of a superparabolic function and the greatest value of a subparabolic function are attained on the boundary of B.

It may easily be seen that, *if $\psi(x, t)$ has continuous derivatives $\psi_t, \psi_x, \psi_{xx}$ inside B, and $\psi_t - \psi_{xx} \geqslant 0$ inside B, then $\psi(x, t)$ is a superparabolic function.* For, let u be a function satisfying (111) and coinciding with ψ on the sides of β. The difference $w = \psi - u$ now vanishes on the sides of β and $w_t - w_{xx} \geqslant 0$

inside β. But the function w must attain its least value on the boundary of β [242], where it vanishes, i.e. $w \geqslant 0$ throughout the triangle β, i.e. $\psi \geqslant u$ in β, which is what we wanted to prove.

Similarly, if $\varphi_t - \varphi_{xx} \leqslant 0$ inside B, then φ is a subparabolic function. Every solution of equation (111) is simultaneously both a sub- and superparabolic function. It can be shown, precisely as in [216], that, if $f_1(M), \ldots, f_m(M)$ are superparabolic functions, $\psi(M) = \min [f_1(M), \ldots, f_m(M)]$ is also a superparabolic function. Let $f_\beta(M)$ denote the function which coincides with $f(M)$ outside the triangle β and on its sides and is equal, inside β, to the solution of equation (111) with values on the contour of β equal to $f(M)$. As in [216], it can be shown that, if $f(M)$ is a superparabolic function, the same can be said of $f_\beta(M)$, where $f_\beta(M) \leqslant f(M)$ in B.

The boundary values in B are specified on the lower base $t = 0$ and on the sides l_i. Let l' denote this part of the contour of B. The definition of upper and lower functions is the same as for Laplace's equation. In particular, an upper function is defined as any superparabolic function which has values on l' greater than or equal to the specified boundary values.

The function $u(x, t)$ can then be defined as the strict lower bound of values of all the upper functions. It can be shown that this function satisfies equation (111) [cf. 217]. It is a generalized solution of the above-mentioned boundary value problem for equation (111). An investigation of the behaviour of this function $u(x,t)$ on approaching l' can be found in I. G. Petrovskii's article First boundary value problem for the heat conduction equation (O pervoi predel'noi zadache dlya uravneniya teplorovodnosti) (*Compositio Mathematica*, t. 1, fasc. 3, 1935).

253. Fundamental inequalities for solutions of the wave equation. We were concerned in the previous chapter with the question of finding inequalities for the solutions of a linear hyperbolic equation in terms of the initial data and coefficients. It was assumed either that the process occurred in unbounded space, or else that we were entirely concerned with the part of space and the instants such that no perturbation had yet had time to arrive on the boundary. We shall next establish similar inequalities for a boundary value problem, our consideration being here confined to the wave equation. For the sake of clarity, we shall carry out all our arguments for the wave equation with two spatial coordinates. We thus have three coordinates (x, y, t), and we can make use of three-dimensional spatial forms. The entire discussion can be extended to the case of three spatial coordinates.

Let $u(P; t) = u(x, y, t)$ be a given solution of the wave equation

$$u_{tt} = u_{xx} + u_{yy} \tag{112}$$

in the domain B with contour l, satisfying certain initial conditions:

$$u|_{t=0} = f_0(x, y); \quad u_t|_{t=0} = f_1(x, y) \quad [(x, y) \text{ in } B] \tag{113}$$

and the homogeneous boundary condition

$$u|_l = 0. \tag{114}$$

Our aim is to explain the nature of the dependence of the solution on the initial conditions (113). Let D be the three-dimensional domain in space (x, y, t) bounded by the planes $t = 0$, $t = T$ and by the lateral surface S_1 of a cylinder with directrix l and generators parallel to the t axis. We assume that the above-mentioned solution is itself continuous, has continuous first order derivatives in the closed domain $\bar{D}$ and has continuous second order derivatives inside D. We shall start out from the following elementary identity, which we made use of when proving the uniqueness theorem:

$$2u_t(u_{tt} - u_{xx} - u_{yy}) = \frac{\partial}{\partial t}(u_x^2 + u_y^2 + u_t^2) - 2(u_t u_x)_x - 2(u_t u_y)_y. \tag{115}$$

On integrating both sides with respect to the domain D and using equation (112), and transforming the integral on the right-hand side in accordance with Ostrogradskii's formula, we get:

$$\iint_S [(u_x^2 + u_y^2 + u_t^2)\cos(n, t) - 2u_t u_x \cos(n, x) - \\ - 2u_t u_y \cos(n, y)]\, dS = 0, \tag{116}$$

where S is the total surface of D and n is the direction of the outward normal to S. Strictly speaking, given our assumptions regarding u, we should apply Ostrogradskii's formula first to the cylinder contained inside D and formed by parallel surfaces, then pass to the limit and use the continuity of the first order derivatives right up to S. On the lateral surface of the body D, i.e. on S_1, we have $\cos(n, t) = 0$ and $u_t = 0$. This last follows from the fact that points of S_1 are points of the contour l at different instants, whilst we have condition (114) on l, i.e. $u = 0$ on S_1. The lemma of [162] should also be borne in mind here. On the upper and lower bases of the cylinder we have $\cos(n, x) = \cos(n, y) = 0$. In addition, $\cos(n, t) = 1$ on the upper, and $\cos(n, t) = -1$ on the lower base. Formula (116) thus leads to the following fundamental equation:

$$\iint_B (u_x^2 + u_y^2 + u_t^2)\, dS\,\Big|_{t=T} = \iint_B (u_x^2 + u_y^2 + u_t^2)\, dS\,\Big|_{t=0}$$

or, on taking (113) into account:

$$\iint_B (u_x^2 + u_y^2 + u_t^2)\, dS = \iint_B \left[\left(\frac{\partial f_0}{\partial x}\right)^2 + \left(\frac{\partial f_0}{\partial y}\right)^2 + f_1^2\right] dS, \tag{117}$$

and the integral on the left-hand side can be taken with any value of t in $[0, T]$. It is assumed that the solution u with the above-mentioned properties exists in the cylinder D defined above. It follows at once from (117) that the left-hand side of (117) does not depend on t. We shall now give an inequality for the square of the function u over the domain B. We suppose that straight lines parallel to an axis cut the contour l in not more than two points; also, let $y = \varphi_1(x)$ and $y = \varphi_2(x)$ be the equations of the lower and upper parts of the contour, where obviously $0 \leqslant \varphi_2(x) - \varphi_1(x) \leqslant M$, where M is a constant. We can write:

$$u^2(x, y, t) = \int_{\varphi_1(x)}^{y} \frac{\partial}{\partial y_1} u^2(x, y_1, t)\, \mathrm{d}y_1,$$

or

$$u^2(x, y, t) = \int_{\varphi_1(x)}^{y} 2u(x, y_1 t) \frac{\partial u(x, y_1 t)}{\partial y_1}\, \mathrm{d}y_1,$$

whence

$$\iint_B u^2(x, y, t)\, \mathrm{d}x\, \mathrm{d}y = \iint_B \Big[\int_{\varphi_1(x)}^{y} 2u(x, y_1, t) \frac{\partial u(x, y_1, t)}{\partial y_1}\, \mathrm{d}y_1 \Big] \mathrm{d}x\, \mathrm{d}y.$$

On writing a and b for the abscissae of the extreme left- and right-hand points of the contour l, we can write:

$$\iint_B u^2(x, y, t)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \Big\{ \int_{\varphi_1(x)}^{\varphi_2(x)} \Big[\int_{\varphi_1(x)}^{y} 2u(x, y_1, t) \frac{\partial u(x, y_1, t)}{\partial y_1}\, \mathrm{d}y_1 \Big] \mathrm{d}y \Big\} \mathrm{d}x.$$

We change the order of the integrations with respect to y_1 and y by applying Dirichlet's formula [II, 79]:

$$\iint_B u^2(x, y, t)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \Big\{ \int_{\varphi_1(x)}^{\varphi_2(x)} \Big[\int_{y}^{\varphi_2(x)} 2u(x, y_1, t) \frac{\partial u(x, y_1, t)}{\partial y_1}\, \mathrm{d}y \Big] \mathrm{d}y_1 \Big\} \mathrm{d}x,$$

or, on recalling that the integrand does not depend on y:

$$\iint_B u^2(x, y, t)\, \mathrm{d}x\, \mathrm{d}y = \iint_B 2u(x, y_1, t) \frac{\partial u(x, y_1, t)}{\partial y_1} [\varphi_2(x) - y_1]\, \mathrm{d}x\, \mathrm{d}y_1.$$

We have $\varphi_1(x) \leqslant y_1 \leqslant \varphi_2(x)$, so that $0 \leqslant \varphi_2(x) - y_1 \leqslant M$. On putting y for y_1, we can write:

$$\iint_B u^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y \leqslant 2M \iint_B |u(x, y, t)| \left| \frac{\partial u(x, y, t)}{\partial y} \right| \mathrm{d}x\,\mathrm{d}y.$$

We apply Buniakowski's inequality to the last integral:

$$\iint_B u^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y \leqslant$$

$$\leqslant 2M \left[\iint_B u^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y \right]^{\frac{1}{2}} \left[\iint_B \left(\frac{\partial u(x, y, t)}{\partial y} \right)^2 \mathrm{d}x\,\mathrm{d}y \right]^{\frac{1}{2}}.$$

On squaring both sides and dividing through by the integral of $u^2(x, y, t)$, we get:

$$\iint_B u^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y \leqslant 4M^2 \iint_B \left(\frac{\partial u(x, y, t)}{\partial y} \right)^2 \mathrm{d}x\,\mathrm{d}y, \tag{118}$$

and, on taking (117) into account, we obtain as our final inequality for the integral of $u^2(x, y, t)$ in terms of the initial data:

$$\iint_B u^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y \leqslant 4M^2 \iint_B \left[\left(\frac{\partial f_0}{\partial x} \right)^2 + \left(\frac{\partial f_0}{\partial y} \right)^2 + f_1^2 \right] \mathrm{d}x\,\mathrm{d}y. \tag{119}$$

By using this formula, we can estimate the mean square error in the solution from the mean square error in the initial data. Suppose we have found a solution $u_1(x, y, t)$ of equation (112), satisfying, instead of initial data (113), the different initial data:

$$u_1 \Big|_{t=0} = \varphi_0(x, y); \quad \frac{\partial u_1}{\partial t} \Big|_{t=0} = \varphi_1(x, y)$$

and the homogeneous boundary condition (114). The difference $(u - u_1)$ satisfies (112), the homogeneous boundary condition (114) and the initial data:

$$(u - u_1)|_{t=0} = f_0(x, y) - \varphi_0(x, y);$$

$$\frac{\partial (u - u_1)}{\partial t} \Big|_{t=0} = f_1(x, y) - \varphi_1(x, y).$$

On applying inequality (119) to this difference, we obtain an inequality for the mean square maximum error in the solution in terms of the mean square error in the initial data:

$$\iint\limits_B [u(x, y, t) - u_1(x, y, t)]^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant$$

$$\leqslant 4M^2 \iint\limits_B \Big[\Big(\frac{\partial f_0}{\partial x} - \frac{\partial \varphi_0}{\partial x}\Big)^2 + \Big(\frac{\partial f_0}{\partial y} - \frac{\partial \varphi_0}{\partial y}\Big)^2 + (f_1 - \varphi_1)^2\Big] \mathrm{d}\sigma. \quad (120)$$

It follows from this that, if the right-hand side tends to zero, the same can be said of the mean square error in the solution.

254. The case of the non-homogeneous equation. Let us now take the non-homogeneous equation

$$u_{tt} = u_{xx} + u_{yy} + \pi(x, y, t) \quad (121)$$

with homogeneous boundary condition (117) and initial conditions (113). If the boundary condition is non-homogeneous:

$$u|_l = \omega(x, y, t), \quad (122)$$

we can replace u by a new function $v = u - u_0$, where u_0 is any function satisfying boundary condition (122), and thus obtain homogeneous boundary conditions for v. We remark that, if we had the homogeneous equation (112) for u, we should have a non-homogeneous equation for v. Thus an inequality for the solution of equation (121) in terms of initial data (113) and the function $\pi(x, y, t)$ enables us to find an inequality as above for the mean square error when the initial and boundary conditions are modified.

By (121), the triple integral of the left-hand side of (115) over D reduces to an integral of the product $2u_t\,\pi$, and we obtain instead of (117):

$$\iint\limits_B (u_x^2 + u_y^2 + u_z^2)\,\mathrm{d}S|_{t=T} =$$

$$= \iint\limits_B \Big[\Big(\frac{\partial f_0}{\partial x}\Big)^2 + \Big(\frac{\partial f_0}{\partial y}\Big)^2 + f_1^2\Big] \mathrm{d}S + 2\iiint\limits_D u_t\,\pi\,\mathrm{d}\tau. \quad (123)$$

We put:

$$K(t) = \iint\limits_B (u_x^2 + u_y^2 + u_t^2)\,\mathrm{d}S; \quad A(t) = \iint\limits_B \pi^2\,\mathrm{d}S. \quad (124)$$

The previous formula can be written as

$$K(T) - K(0) = 2 \iiint_D u_t \pi \, \mathrm{d}\tau = 2 \int_0^T \Big[\iint_B u_t \pi \, \mathrm{d}S \Big] \mathrm{d}t. \qquad (125)$$

We differentiate this equation with respect to T and replace T by t:

$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} = 2 \iint_B u_t \pi \, \mathrm{d}S,$$

whence, on using the inequality $2 \mid ab \mid \leqslant a^2 + b^2$, we obtain

$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} \leqslant A(t) + \iint_B u_t^2 \mathrm{d}S,$$

or, if we take into account the fact that

$$\iint_B u_t^2 \, \mathrm{d}S \leqslant K(t), \qquad (126)$$

we can write:

$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} \leqslant A\,(t) + K(t), \qquad \frac{\mathrm{d}[\mathrm{e}^{-t} K(t)]}{\mathrm{d}t} \leqslant \mathrm{e}^{-t} A(t),$$

and integration of this inequality gives us

$$K(t) \leqslant \mathrm{e}^t K(0) + \int_0^t \mathrm{e}^{t-t'} A(t') \, \mathrm{d}t'. \qquad (127)$$

If we put

$$L(t) = \iint_B u^2 \, \mathrm{d}S,$$

we have:

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = 2 \iint_B u u_t \, \mathrm{d}S \leqslant \iint_B u^2 \, \mathrm{d}S + \iint_B u_t^2 \mathrm{d}S \leqslant L(t) + K(t). \qquad (128)$$

We can put the absolute value sign on the left-hand side of this inequality. It follows from (128) that

$$\frac{\mathrm{d}[\mathrm{e}^{-t} L(t)]}{\mathrm{d}t} \leqslant \mathrm{e}^{-t} K(t),$$

and we arrive, as above, at the inequality

$$L(t) \leqslant \mathrm{e}^t L(0) + \int_0^t \mathrm{e}^{\,-t'} K(t') \, \mathrm{d}t'. \qquad (129)$$

In addition to these inequalities, we can obtain simple inequalities for $K(t)$ and $L(t)$ with homogeneous initial as well as boundary con-

ditions. In this case $K(0) = 0$, and application of Buniakowski's inequality to the right-hand side of (125) yields

$$K(T) \leqslant 2\int_0^T \Big[\iint_B u_t^2 \, \mathrm{d}S\Big]^{\frac{1}{2}} \Big[\iint_B \pi^2 \, \mathrm{d}S\Big]^{\frac{1}{2}} \mathrm{d}t,$$

or, by (126):

$$K(T) \leqslant \int_0^T [K(t)]^{\frac{1}{2}} [4A(t)]^{\frac{1}{2}} \mathrm{d}t. \tag{130}$$

Let $K = \max K(t)$ and $A = \max 4A(t)$ in the interval $0 \leqslant t \leqslant T$. The previous inequality gives

$$K(T) \leqslant K^{\frac{1}{2}} A^{\frac{1}{2}} T.$$

We put t in place of T in this formula, where $0 \leqslant t \leqslant T$:

$$K(t) \leqslant K^{\frac{1}{2}} A^{\frac{1}{2}} t \qquad (0 \leqslant t \leqslant T). \tag{131}$$

On substituting this inequality for $K(t)$ in integral (130), we obtain

$$K(t) \leqslant K^{\frac{1}{4}} A^{\frac{3}{4}} t^{\frac{3}{2}} \cdot \frac{2}{3} \qquad (0 \leqslant t \leqslant T).$$

On again substituting this inequality in integral (130), we obtain

$$K(t) \leqslant K^{\frac{1}{8}} A^{\frac{7}{8}} t^{\frac{7}{4}} \cdot \frac{4}{7} \left(\frac{2}{3}\right)^{\frac{1}{2}}.$$

On proceeding in this way, we obtain after n substitutions:

$$K(t) \leqslant K^{\frac{1}{2^n}} A^{\frac{2^n-1}{2^n}} t^{\frac{2^n-1}{2^{n-1}}} \frac{2^{n-1}}{2^n-1} \left(\frac{2^{n-2}}{2^{n-1}-1}\right)^{\frac{1}{2}} \cdots \left(\frac{2}{3}\right)^{\frac{1}{2^{n-2}}}. \tag{132}$$

We consider the sequence of positive numbers:

$$\left\{\frac{2^{n-1}}{2^n-1} \cdot \left(\frac{2^{n-2}}{2^{n-1}-1}\right)^{\frac{1}{2}} \cdots \left(\frac{2}{3}\right)^{\frac{1}{2^{n-2}}}\right\}. \tag{133}$$

Each of these fractions decreases as n increases, since

$$\frac{2^n}{2^{n+1}-1} : \frac{2^{n-1}}{2^n-1} = \frac{2^{n+1}-2}{2^{n+1}-1} < 1.$$

Hence each factor of product (133) decreases as n increases by unity, and furthermore, a fresh factor $(2/3)^{1/(2n-1)}$, less than unity, is added,

i.e. sequence (133) is decreasing and therefore has a limit, which we denote by C. On passing to the limit in (132), we obtain

$$K(t) \leqslant CAt^2 \quad (0 \leqslant t \leqslant T).$$

On substituting this inequality in (130), we obtain

$$K(t) \leqslant \frac{1}{2} \sqrt{C}\, At^2.$$

On substituting once more, we obtain:

$$K(t) \leqslant \frac{1}{2} \sqrt{\frac{\sqrt{C}}{2}}\, At^2 = \frac{C^{\frac{1}{4}}}{2^{1+\frac{1}{2}}} At^2;$$

on proceeding with these substitutions, we find that

$$K(t) \leqslant \frac{C^{\frac{1}{2^n}}}{2^{1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}}} At^2,$$

and, on passing to the limit, we get an inequality which does not contain C:

$$K(t) \leqslant \frac{1}{4} At^2 \qquad (0 \leqslant t \leqslant T). \tag{134}$$

We recall that the A in this inequality is max $4A(t)$ in the interval $0 \leqslant t \leqslant T$. Having obtained inequality (134) for $K(t)$, we can write for the integral of the squared function $u^2(x, y, t)$, in accordance with (118):

$$L(t) \leqslant AM^2 t^2 \qquad (0 \leqslant t \leqslant T). \tag{135}$$

Suppose we have homogeneous equation (112) with initial and boundary conditions. By using the solution of (112) for the case of an unbounded plane [II, 172], we can reduce the initial conditions to the homogeneous form. Suppose that, after this reduction, the function $\omega(x, y, t)$ appearing in the boundary conditions is such that we can find an auxiliary function u_0, as mentioned at the beginning of the present section, satisfying the homogeneous initial conditions. The substitution $v = u - u_0$ now makes the boundary condition homogeneous and does not destroy the homogeneity of the initial conditions, but reduces the equation to the form (121). We in fact have inequality (135) for this case. This inequality can be used in practice for estimating the difference between two solutions having different boundary conditions. The above inequalities were obtained by Sobolev.

By using these inequalities, we can prove the continuity, in a certain sense, of the dependence of the solutions of the wave equation on the initial and boundary conditions, as also on the function π.

Let us first investigate the dependence on the function π. Let u_1 and u_2 be two solutions of the non-homogeneous equation with different functions $\pi_1(x, y, t)$ and $\pi_2(x, y, t)$, where the solutions satisfy homogeneous initial and boundary conditions. The difference $(u_2 - u_1)$ satisfies the non-homogeneous equation with function $(\pi_2 - \pi_1)$ and homogeneous initial and boundary conditions. If, with this,

$$\iint\limits_B (\pi_2 - \pi_1)^2 \, \mathrm{d}S \leqslant \frac{\varepsilon}{4} \qquad (0 \leqslant t \leqslant T),$$

we obtain on applying inequalities (134) and (135):

$$\iint (u_2 - u_1)^2 \, \mathrm{d}S \leqslant \varepsilon M^2 t^2 \qquad (0 \leqslant t \leqslant T);$$

$$\iint\limits_B \left[\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial t}\right)^2\right] \mathrm{d}S \leqslant \frac{\varepsilon}{4} t^2$$
$$(0 \leqslant t \leqslant T). \qquad (136)$$

This shows that the solution is continuously dependent on the function π, the continuity being in the sense of the mean square deviation.

Now let v_1 and v_2 be two solutions of the homogeneous wave equation with homogeneous initial conditions and different boundary conditions:

$$v_1 |_l = \psi_1 (x, y, t); \quad v_2 |_l = \psi_2 (x, y, t),$$

the functions ψ_i being defined in the closed domain $\bar{B}$, where they have continuous derivatives up to the second order for $t \geqslant 0$, whilst they satisfy

$$\psi_1 (x, y, 0) = \psi_2 (x, y, 0) = \left.\frac{\partial \psi_1 (x, y, t)}{\partial t}\right|_{t=0} = \left.\frac{\partial \psi_2 (x, y, t)}{\partial t}\right|_{t=0} = 0.$$

On introducing the new required functions $u_1 = v_1 - \psi_1$, $u_2 = v_2 - \psi_2$, we obtain for these the non-homogeneous equation

$$\frac{\partial^2 u_i}{\partial t^2} = \Delta u_i + \left(\frac{\partial^2 \psi_i}{\partial t^2} - \Delta \psi_i\right)$$

with homogeneous initial and boundary conditions.

We can now apply (134) and (135) to $(u_2 - u_1)$, where

$$A = \max_{0 \leqslant t \leqslant T} \iint_B \left[\left(\frac{\partial^2 \psi_2}{\partial t^2} - \Delta\psi_2\right) - \left(\frac{\partial^2 \psi_1}{\partial t^2} - \Delta\psi_1\right)\right]^2 \mathrm{d}S,$$

On taking into account the obvious inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, we obtain

$$A \leqslant \max_{0 \leqslant t \leqslant T} 3 \iint_B \left[\left(\frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \psi_2}{\partial y^2} - \frac{\partial^2 \psi_1}{\partial y^2}\right)^2 + \left(\frac{\partial^2 \psi_2}{\partial t^2} - \frac{\partial^2 \psi_1}{\partial t^2}\right)^2\right] \mathrm{d}S,$$

and if the integral on the right is not greater than $\varepsilon/4$ for $0 \leqslant t \leqslant T$, we can write inequality (136).

We recall that formula (120) gave the inequality for the mean square deviation of the solutions of the homogeneous wave equation with homogeneous boundary condition as a function of the mean square deviation in the initial conditions. By further using (117), we obtain here, in addition to (120):

$$\iint_B \left[\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial t}\right)^2\right] \mathrm{d}S =$$

$$= \iint_B \left[\left(\frac{\partial f_0}{\partial x} - \frac{\partial \varphi_0}{\partial x}\right)^2 + \left(\frac{\partial f_0}{\partial y} - \frac{\partial \varphi_0}{\partial y}\right)^2 + (f_1 - \varphi_1)^2\right] \mathrm{d}S. \tag{137}$$

255. Fourier's method and generalized solutions. We take the boundary value problem for the wave equation on a plane:

$$\Box\, u = u_{x_1x_1} + u_{x_2x_2} - u_{tt} = 0. \tag{138}$$

Suppose that a solution of (138), satisfying the homogeneous boundary condition:

$$u\,|_l = 0 \tag{139}$$

and the initial conditions:

$$u\,|_{t=0} = \varphi_0(x_1, x_2); \quad u_t\,|_{t=0} = \varphi_1(x_1, x_2). \tag{140}$$

is sought inside the domain B with contour l. By applying Fourier's method formally, we obtain the solution of the problem in the form

$$u(P; t) = \sum_{m=1}^{\infty} \left(a_m \cos\sqrt{\lambda_m}\,t + b_m \sin\sqrt{\lambda_m}\,t\right) v_m(P), \tag{141}$$

where λ_m and $v_m(P)$ are the eigenvalues and eigenfunctions of the equation

$$\Delta v + \lambda v = 0 \tag{142}$$

with boundary condition (139) for v, and

$$\sum_{m=1}^{\infty} a_m v_m(P); \tag{143}$$

$$\sum_{m=1}^{\infty} b_m \sqrt{\lambda_m} v_m(P) \tag{144}$$

are the Fourier series of the functions $\varphi_0(P)$ and $\varphi_1(P)$, i.e.

$$a_m = \iint_B \varphi_0(P)\, v_m(P)\, \mathrm{d}S; \quad b_m \sqrt{\lambda_m} = \iint_B \varphi_1(P)\, v_m(P)\, \mathrm{d}S. \tag{145}$$

The justification of Fourier's method involves considerable difficulties as compared with its justification for the heat conduction equation. In Sobolev's *Partial Differential Equations of Mathematical Physics*, Pergamon Press, 1964, p. 311 the justification is carried out from the point of view of generalized solutions of the wave equation. The author makes use in essence of the familiar Riesz–Fischer theorem and the convergence in the mean. We shall justify Fourier's method here from the point of view of generalized solutions with more restrictive assumptions regarding $\varphi_0(P)$ and $\varphi_1(P)$, but with the aid of more elementary methods.

We shall assume that the series

$$\sum_{m=1}^{\infty} a_m \sqrt{\lambda_m}\, v_m(P); \tag{146}$$

$$\sum_{m=1}^{\infty} b_m \sqrt{\lambda_m} v_m(P) \tag{147}$$

are regularly convergent in the closed domain $\overline{B}$. Now, the series

$$\sum_{m=1}^{\infty} a_m v_m(P); \qquad \sum_{m=1}^{\infty} b_m v_m(P)$$

are all the more regularly convergent in $\overline{B}$. Hence, given any real t, (141) defines a function $u(P, t)$ continuous in $\overline{B}$, whilst it follows from the regular convergence of series (146) and (147) that this function has a partial derivative with respect to t:

$$u_t(P; t) = \sum_{m=1}^{\infty} \left(- a_m \sqrt{\lambda_m} \sin \sqrt{\lambda_m} t + b_m \sqrt{\lambda_m} \cos \sqrt{\lambda_m} t \right) v_m(P), \tag{148}$$

continuous in $\overline{B}$, the series written being regularly convergent in $\overline{B}$ for all t. We can conclude from (141) and (148) that the function $u(P, t)$ satisfies the initial conditions (140). The boundary condition

(139) immediately follows from the fact that all the functions $v_m(P)$ satisfy this condition.

We now define a generalized solution of equation (138) in the presence of the boundary condition (139) and initial conditions (140). We show further on that, given our assumptions, (141) gives the generalized solution, which is unique.

Let D' be a cylindrical domain in space (x, y, t) with base B and generators parallel to the t axis, where $-\infty < t < +\infty$, and let S' be the boundary of this cylindrical domain.

Suppose the function $\sigma(P; t)$ is continuous in D' as far as S', vanishes on S' and has continuous derivatives up to the second order in D'. Suppose, in addition, that $\sigma(P; t)$ vanishes for all sufficiently large absolute values of t.

Let $u(P; t)$ be any twice continuously differentiable solution of equation (138) in the part of D' where $t > 0$, satisfying on S' boundary condition (139) and having a regular normal derivative. Furthermore, let $u(P; t)$ satisfy initial conditions (140). We apply to $u(P; t)$ and $\sigma(P; t)$ Green's formula for the part of D' where $t > 0$. On recalling that $\Box u = 0$ and that the integral over S' falls out, since $u = \sigma = 0$ on S', we obtain:

$$\iiint\limits_{D';\, t \geqslant 0} u \Box \sigma \,\mathrm{d}\tau = \iint\limits_{B} \left[\varphi_0 \left(\frac{\partial \sigma}{\partial t}\right)_{t=0} - \varphi_1 (\sigma)_{t=0}\right] \mathrm{d}S, \tag{149}$$

the integral on the left-hand side being taken in fact only over a bounded domain, since σ, and therefore $\Box\sigma$, vanishes for sufficiently large t.

Equation (149) leads naturally to the following definition of a generalized solution: the function $u(P; t)$, defined in the part of D' where $t > 0$, continuous along with $u_t(P; t)$ up to S' and $t = 0$ and satisfying boundary condition (139) is described as a generalized solution of the problem (138)—(140) if equation (149) holds for any choice of $\sigma(P; t)$ with the properties indicated above. It can be shown that conditions (139) and (140) follow from (149) and the continuity of u and u_t.

Formula (141) gives us the solution of the problem in the sense just indicated. For, on observing that the functions $v_m(P)$ have regular normal derivatives on the contour l of the domain, we see that (149) holds if we take as $u(P; t)$ the segment $s_n(P; t)$ of series (141), i.e.

$$\iiint\limits_{D';\, t \geqslant 0} s_n (P; t) \Box \sigma (P; t) \,\mathrm{d}\tau = \iint\limits_{B} \left[s_n^{(0)} (P) \left(\frac{\partial \sigma}{\partial t}\right)_{t=0} - s_n^{(1)} (P) (\sigma)_{t=0}\right] \mathrm{d}S,$$

where $s_n^{(0)}(P)$ and $s_n^{(1)}(P)$ are segments of the Fourier series of $\varphi_0(P)$ and $\varphi_1(P)$. Since $s_n(P; t) \to u(P; t)$, $s_n^{(0)}(P) \to \varphi_0(P)$, $s_n^{(1)} \to \varphi_1(P)$ as $n \to \infty$ uniformly in the domain D' as far as S', it can be shown by passing to the limit in the last equation as $n \to \infty$ that $u(P; t)$, defined by (141), is a generalized solution of the problem with initial conditions (140).

We show that the solution of the boundary value problem i n the new statement is uniquely defined by the initial conditions. Forthis, we only need to show that the generalized solution of the boundary value problem $u^{(0)}(P; t)$, satisfying the homogeneous initial conditions

$$u^{(0)}(P; 0) = u_t^{(0)}(P; 0) = 0,$$

is identically zero in D' $(t \geqslant 0)$.

In accordance with the definition of a generalized solution, we have for any choice of $\sigma(P; t)$:

$$\int\int_{D';\ t\geqslant 0}\int u^{(0)}(P; t) \, \Box \, \sigma(P; t) \, d\tau = 0.$$

We put $u^{(0)}(P; t) = 0$ for $t < 0$. Now, $u^{(0)}(P; t)$ and $u_t^{(0)}(P; t)$ are continuous throughout D', and

$$\int\int_{D'}\int u^{(0)}(P; t) \, \Box \, \sigma(P; t) \, d\tau = 0. \tag{150}$$

We put $\sigma(P; t) = v_m(P) f(t + \xi)$, where $f(t)$ is an arbitrary twice continuously differentiable function for all t, vanishing outside some finite interval of values of t, whilst ξ is some fixed number.

The function $\sigma(P; t)$ thus chosen has the above-mentioned properties. We have:

$$\begin{aligned} \Box \, \sigma(P; t) &= f(t + \xi) \, \Delta v_m(P) - v_m(P) f''(t + \xi) = \\ &= -[f''(t + \xi) + \lambda_m f(t + \xi)] \, v_m(P), \end{aligned}$$

so that equation (150) becomes:

$$\int\int_{D'}\int u^{(0)}(P; t) [f''(t + \xi) + \lambda_m f(t + \xi)] \, v_m(P) \, d\tau = 0 \tag{151}$$

We put:

$$\psi_m(\xi) = \int\int_{D'}\int u^{(0)}(P; t) f(t + \xi) \, v_m(P) \, d\tau. \tag{152}$$

We have:

$$\psi_m''(\xi) = \int\int_{D'}\int u^{(0)}(P; t) f''(t + \xi) \, v_m(P) \, d\tau,$$

and (151) gives:

$$\psi_m''(\xi) + \lambda_m \psi_m(\xi) = 0,$$

i.e.

$$\psi_m(\xi) = C_1 \cos\sqrt{\lambda_m}\xi + C_2 \sin\sqrt{\lambda_m}\xi. \tag{153}$$

We show that $\psi_m(\xi) \equiv 0$. Let $[t_1, t_2]$ be the interval outside which $f(t)$ vanishes. Let $\xi > t_2$. We now have $f(t+\xi) = 0$ for $t \geqslant 0$ in the integrand of (152), whilst $u^{(0)}(P;t) = 0$ for $t < 0$, whence it follows that $\psi_m(\xi) = 0$ for $\xi > t_2$, and hence $C_1 = C_2 = 0$ in (153), i.e. $\psi_m(\xi) \equiv 0$. It is now quite easy to show that $u^{(0)}(P;t) \equiv 0$. We form the function:

$$\omega(P;\xi) = \int_{-\infty}^{+\infty} u^{(0)}(P;t) f(t+\xi)\,\mathrm{d}t.$$

It is a continuous function of P in $\overline{B}$ for any fixed ξ, and by (152) and $\psi_m(\xi) = 0$, it is orthogonal to all the functions $v_m(P)$.

The functions $v_m(P)$ form a closed system, and it follows at once from what has been said above that $\omega(P;\xi) \equiv 0$, i.e.

$$\int_{-\infty}^{+\infty} u^{(0)}(P;t) f(t+\xi)\,\mathrm{d}t = 0$$

for any choice of function $f(t)$ with the above-mentioned properties. On applying the fundamental lemma of the calculus of variations [62], we get $u^{(0)}(P;t) \equiv 0$, which proves the uniqueness of the generalized solution of the boundary value problem with the given conditions.

This proof of uniqueness is taken from Kh. L. Smolitskii's dissertation *The Boundary Value Problem for the Wave Equation* (Predelnaya zadacha dlya volnovogo uravneniya). The whole of the treatment of the present section is due to this author. It is easy to indicate the conditions to be fulfilled by $\varphi_0(P)$ and $\varphi_1(P)$ in order for series (146) and (147) to be regularly convergent in $\overline{B}$. This will be the case, for instance, if $\varphi_0(P)$ has continuous derivatives up to the fourth order in $\overline{B}$ and satisfies the conditions:

$$\varphi_0|_l = \Delta\varphi_0|_l = 0,$$

whilst $\varphi_1(P)$ has continuous derivatives up to the second order and satisfies the condition:

$$\varphi_1|_l = 0.$$

The regular convergence of series (147) here follows directly from the fact that this is the Fourier series of $\varphi_1(P)$, whilst the regular convergence of series (146) follows from the formulae:

$$a_m = \iint\limits_B \varphi_0(P)\, v_m(P)\, dS = -\frac{1}{\lambda_m}\iint\limits_B \varphi_0(P)\, \Delta v_m\ (P)\, dS$$
$$= \frac{1}{\lambda_m}\iint\limits_B v_m(P)\, \Delta\varphi_0(P)\, dS.$$

256. Investigation of Fourier series. We shall be concerned next with the possibility of term by term differentiation of series (141) given certain assumptions regarding the functions $\varphi_0(P)$ and $\varphi_1(P)$ appearing in the initial conditions (140) [cf. 184]. Our investigation will be based on an inequality that will be given shortly, and which will be proved in later sections. As a preliminary, we introduce the following new notations:

$$I_k(u, v) = I_k(v, u) = \iint\limits_B \sum_{i_1,\ldots,i_k=1}^{2} \frac{\partial^k u}{\partial x_{i_1}\ldots\partial x_{i_k}} \cdot \frac{\partial^k v}{\partial x_{i_1}\ldots\partial x_{i_k}}\, dS,$$

$$H_k(u) = \sum_{j=1}^{k} I_j(u, u), \quad I_0(u, v) = \iint\limits_B uv\, dS. \tag{154}$$

It is assumed that $u(P)$ and $v(P)$ are continuous in $\overline{B}$ and have continuous derivatives in B as far as l up to the order appearing in these last formulae. We now state the fundamental inequality:
if a function $u(P)$ is continuous in $\overline{B}$, has continuous derivatives up to the fifth order as far as l and satisfies the conditions:

$$u|_l = \Delta u|_l = 0, \tag{155}$$

the inequality holds:

$$H_4(u) \leqslant A\,[I_0(\Delta^2 u, \Delta^2 u) + I_1(\Delta u, \Delta u) + I_0(\Delta u, \Delta u) + I_1(u, u)], \tag{156}$$

where A is a constant depending only on the domain B, Δu is Laplace's operator and $\Delta^2 u = \Delta(\Delta u)$. The assumptions regarding the contour l will be formulated when the inequality (156) is proved.

A remark must be made in regard to this inequality: only the derivatives of u up to the fourth order play a part in it, yet in order to deduce it we assume the existence of the fifth derivatives as far as l, as stated in the condition above. It is not difficult to free ourselves of this condition. In fact, if $u(P)$ is four times continuously differentiable in $\overline{B}$, given a sufficiently smooth contour l the function $u(P)$ can be

continued throughout the plane in such a way that the continued function is four times continuously differentiable throughout the plane. The mean functions for the continued $u(P)$ now have continuous derivatives of any order in $\overline{B}$, so that inequality (156) holds for the mean functions. But the mean functions and their derivatives up to the fourth order tend uniformly in $\overline{B}$ to the functions $u(P)$ and the corresponding derivatives of $u(P)$. On passing to the limit in inequality (156) for the mean functions, we can conclude that (156) holds for the function $u(P)$ itself. This remark could be utilized to reduce the requirements that we impose on the contour l.

We shall assume that $\varphi_0(P)$ and $\varphi_1(P)$ are continuous in $\overline{B}$, that $\varphi_0(P)$ has continuous derivatives up to the fourth order as far as l and $\varphi_1(P)$ continuous derivatives up to the third order as far as l, and that the functions satisfy the conditions:

$$\varphi_0\,|_l = \Delta\varphi_0\,|_l = \varphi_1\,|_l = \Delta\varphi_1\,|_l = 0. \tag{157}$$

We write down series (141):

$$u(P;t) = \sum_{m=1}^{\infty} \left(a_m \cos\sqrt{\lambda_m}t + b_m \sin\sqrt{\lambda_m}t\right) v_m(P) =$$

$$= \sum_{m=1}^{\infty} c_m(t)\, v_m(P) \tag{158}$$

and use (156) to write an inequality for $H_4(u_{p,q})$, where

$$u_{p,q} = \sum_{m=p+1}^{p+q} c_m(t)\, v_m(P).$$

We have shown that the functions $v_m(P)$ have regular normal derivatives on l. If we postulate the usual smoothness of contour l, the $v_m(P)$ will have derivatives up to a definite order as far as l [cf. 226]. We shall assume in future that the $v_m(P)$ have derivatives up to the fifth order, continuous right up to l. In this case we can put $u = u_{p,q}$ in inequality (156). The derivatives of $v_m(P)$ are continuous as far as l if e.g. l is a circle. In this case the $v_m(P)$ are expressible in terms of Bessel functions [II, 178] and have derivatives of all orders, continuous as far as l. We shall later mention some conditions for the contour l which are sufficient for the $v_m(P)$ to have continuous derivatives up to the fifth order as far as l. Let us pass on to the evaluation

of the integrals on the right-hand side of (156) with $u = u_{p,q}$. We have:

$$\Delta^2 u_{p,q} = \sum_{m=p+1}^{p+q} c_m(t)\, \lambda_m^2 v_m(P),$$

whence

$$I_0(\Delta^2 u_{p,q}, \Delta^2 u_{p,q}) = \iint_B \left[\sum_{m=p+1}^{p+q} c_m(t)\, \lambda_m^2 v_m(P)\right]^2 dS =$$
$$= \sum_{m=p+1}^{p+q} c_m^2(t)\, \lambda_m^4.$$

and similarly:

$$I_0(\Delta u_{p,q}, \Delta u_{p,q}) = \sum_{m=p+1}^{p+q} c_m^2(t)\, \lambda_m^2.$$

To evaluate $I_3(\Delta u_{p,q}, \Delta u_{p,q})$ and $I_1(u_{p,q}, u_{p,q})$, it may be observed that the application of Green's formula [II, 193] gives us:

$$\iint_B \left(\frac{\partial v_i}{\partial x_1} \cdot \frac{\partial v_j}{\partial x_1} + \frac{\partial v_i}{\partial x_2} \cdot \frac{\partial v_j}{\partial x_2}\right) dS = \begin{matrix} 0 \ \text{ for } i \neq j \\ \lambda_m \text{ for } i = j, \end{matrix} \tag{159}$$

whence it follows immediately that

$$I_1(\Delta u_{p,q}, \Delta u_{p,q}) = \iint_B \left(\sum_{m=p+1}^{p+q} c_m(t)\, \lambda_m \sum_{i=1}^{2} \frac{\partial v_m}{\partial x_i}\right)^2 dS = \sum_{m=p+1}^{p+q} c_m^2(t)\, \lambda_m^3$$

$$I_1(u_{p,q}, u_{p,q}) = \iint_B \left(\sum_{m=p+1}^{p+q} c_m(t) \sum_{i=1}^{2} \frac{\partial v_m}{\partial x_i}\right)^2 dS = \sum_{m=p+1}^{p+q} c_m^2(t)\, \lambda_m.$$

On substituting all this in the right-hand side of (156), we obtain

$$H_4(u_{p,q}) \leqslant A \sum_{m=p+1}^{p+q} c_m^2(t)\, (\lambda_m^4 + \lambda_m^3 + \lambda_m^2 + \lambda_m).$$

Suppose that p is so large that $\lambda_m > 1$ for $m > p$. Now, $\lambda_m < \lambda_m^2 <$ $< \lambda_m^3 < \lambda_m^4$. We have in addition:

$$c_m^2(t) = (a_m \cos \sqrt{\lambda_m} t + b_m \sin \sqrt{\lambda_m} t)^2 \leqslant$$
$$\leqslant (a_m + b_m^2)\left(\cos^2 \sqrt{\lambda_m} t + \sin^2 \sqrt{\lambda_m} t\right) = a_m^2 + b_m^2,$$

and hence

$$H_4(u_{p.q}) \leqslant 4A \sum_{m=p+1}^{p+q} (a_m^2 + b_m^2)\, \lambda_m^4. \tag{160}$$

We now show that, given our assumptions regarding $\varphi_0(P)$ and $\varphi_1(P)$, the series

$$\sum_{m=1}^{\infty} (a_m^2 + b_m^2)\,\lambda_m^4. \tag{161}$$

is convergent. We have, using the equation for $v_m(P)$:

$$a_m = \iint_B \varphi_0 v_m \,\mathrm{d}S = \frac{1}{\lambda_m^2} \iint_B \varphi_0 \Delta^2 v_m \,\mathrm{d}S.$$

On taking into account the conditions imposed on φ_0, and applying Green's formula twice, we obtain:

$$a_m = \frac{1}{\lambda_m^2} \iint_B \Delta^2\varphi_0 \cdot v_m \,\mathrm{d}S = \frac{\alpha_m}{\lambda_m^2}, \text{ where } \alpha_m = \iint_B \Delta^2 \varphi_0 \cdot v_m \,\mathrm{d}S,$$

It follows from the closure equation for $\Delta^2 \varphi_0$ that the series

$$\sum_{m=1}^{\infty} \alpha_m^2 = \sum_{m=1}^{\infty} a_m^2 \lambda_m^4.$$

is convergent. We now prove that the series

$$\sum_{m=1}^{\infty} b_m^2 \lambda_m^4. \tag{162}$$

is convergent. If we use the properties of $\varphi_1(P)$, the equation for $v_m(P)$ and Green's formula, we get:

$$b_m = \frac{1}{\sqrt{\lambda_m}} \iint_B \varphi_1 v_m \,\mathrm{d}S = \frac{1}{\lambda_m^{\frac{5}{2}}} \iint_B \varphi_1 \,\Delta^2 v_m \,\mathrm{d}S =$$

$$= \frac{1}{\lambda_m^{\frac{5}{2}}} \iint_B \Delta\varphi_1 \,\Delta v_m \,\mathrm{d}S = -\frac{1}{\lambda_m^{\frac{5}{2}}} \iint_B \sum_{i=1}^{2} \frac{\partial \Delta\varphi_1}{\partial x_i} \cdot \frac{\partial v_m}{\partial x_i} \,\mathrm{d}S,$$

i.e.

$$b_m = \frac{\beta_m}{\lambda_m^2}, \tag{163}$$

where

$$\beta_m = -\frac{1}{\sqrt{\lambda_m}} \iint_B \sum_{i=1}^{2} \frac{\partial \Delta\varphi_1}{\partial x_i} \cdot \frac{\partial v_m}{\partial x_i} \,\mathrm{d}S. \tag{164}$$

It will be seen, on making use of (163), that a proof of the convergence of series (162) requires a proof of the convergence of

$$\sum_{m=1}^{\infty} \beta_m^2. \tag{165}$$

We have the obvious inequality:

$$\iint_B \left\{\left[\frac{\partial}{\partial x_1}\left(\Delta\varphi_1 + \sum_{m=1}^{N} \frac{\beta_m}{\sqrt{\lambda_m}} v_m\right)\right]^2 + \left[\frac{\partial}{\partial x_2}\left(\Delta\varphi_1 + \sum_{m=1}^{N} \frac{\beta_m}{\sqrt{\lambda_m}} v_m\right)\right]^2\right\} \mathrm{d}S =$$

$$= \iint_B \left[\left(\frac{\partial\Delta\varphi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\Delta\varphi_1}{\partial x_2}\right)^2\right] \mathrm{d}S +$$

$$+ 2 \sum_{m=1}^{N} \frac{\beta_m}{\sqrt{\lambda_m}} \iint_B \left(\frac{\partial\Delta\varphi_1}{\partial x_1} \cdot \frac{\partial v_m}{\partial x_1} + \frac{\partial\Delta\varphi_1}{\partial x_2} \cdot \frac{\partial v_m}{\partial x_2}\right) \mathrm{d}S +$$

$$+ \sum_{m,n=1}^{N} \frac{\beta_m \beta_n}{\sqrt{\lambda_m \lambda_n}} \iint_B \left(\frac{\partial v_m}{\partial x_1} \cdot \frac{\partial v_n}{\partial x_1} + \frac{\partial v_m}{\partial x_2} \cdot \frac{\partial v_n}{\partial x_2}\right) \mathrm{d}S \geqslant 0.$$

This gives us, if we make use of (159) and (164):

$$\iint_B \left[\left(\frac{\partial\Delta\varphi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\Delta\varphi_1}{\partial x_2}\right)^2\right] \mathrm{d}S - 2 \sum_{m=1}^{N} \beta_m^2 + \sum_{m=1}^{N} \beta_m^2 \geqslant 0,$$

i.e. for any N:

$$\sum_{m=1}^{N} \beta_m^2 \leqslant \iint_B \left[\left(\frac{\partial\Delta\varphi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\Delta\varphi_1}{\partial x_2}\right)^2\right] \mathrm{d}S,$$

whence it follows that series (165) is convergent. We have thus proved that series (161) is convergent. After this, it follows from (160) that

$$H_4(u_p, u_q) \to 0 \quad \text{as} \quad p \to \infty \quad \text{and} \quad q \to \infty. \tag{166}$$

Furthermore, the obvious inequality

$$\iint_B u_{p,q}^2 \, \mathrm{d}S = \sum_{m=p+1}^{p+q} c_m^2(t) \leqslant \sum_{m=p+1}^{p+q} (a_m^2 + b_m^2)$$

has the immediate consequence that

$$\iint_B u_{p,q}^2 \, \mathrm{d}S \to 0 \quad \text{as} \quad p \to \infty. \tag{167}$$

But $H_4(u_{p,q})$ is the integral over B of the sum of the squares of all the derivatives of $u_{p,q}$ with respect to x_1 and x_2 up to the fourth order, and by taking (166) and (167) into account, we can state the following: given any positive ε, there exists a number $M(\varepsilon)$ such that

$$\iint_B \left(\frac{\partial^k u_{p,q}}{\partial x_1^{k_1} \partial x_2^{k_2}}\right)^2 \mathrm{d}S \leqslant \varepsilon \text{ for } p \geqslant M(\varepsilon) \text{ and } q > 0 \tag{168}$$

$$(k = 0, 1, 2, 3, 4).$$

This inequality will hold all the more if the integration is over a circular domain lying inside B, instead of over the whole of B.

But now, by using the theorem of [156], where $l = 4$ and $n = 2$, i.e. $l - [n/2] - 1 = 2$ in the present case, we can say that, in a circular domain D_1, concentric with D and of smaller radius, the inequalities hold:

$$\left| \frac{\partial^k u_{p,q}}{\partial x_1^{k_1} \partial x_2^{k_2}} \right| \leqslant C\varepsilon \qquad (k = 0, 1, 2), \tag{169}$$

where the constant C depends only on the choice of D_1. But we can cover any closed domain E contained in B with a finite number of circular domains D_1 lying inside B, and if we take C as the greatest of the constants for these domains D_1, we can say that inequality (169) holds throughout E, where C depends on the choice of E. It follows at once from (169) that series (158) and the series obtained from it by term by term differentiation once or twice with respect to x_1 and x_2 are uniformly convergent in E.

If we differentiate series (158) term by term twice with respect to t, we get majorant series of the form

$$\sum_{m=1}^{\infty} \lambda_m |a_m v_m(P)| \text{ and } \sum_{m=1}^{\infty} \lambda_m |b_m v_m(P)|. \tag{170}$$

Let us prove that say the first of these uniform convergences in $\overline{B}$. We can write the general term of this series in the form:

$$\lambda_m |a_m v_m(P)| = |a_m| \lambda_m^2 \frac{|v_m(P)|}{\lambda_m}.$$

If we use the inequality $\alpha_m \beta_m \leqslant (\alpha_m^2 + \beta_m^2)/2$, it becomes a question of the convergence of the series

$$\sum_{m=1}^{\infty} a_m^2 \lambda_m^4 \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{v_m^2(P)}{\lambda_m^2}.$$

The convergence of these has already been proved; we have for the second series [235]:

$$\sum_{m=1}^{\infty} \frac{v_m^2(P)}{\lambda_m^2} = \iint_B G^2(P;\ Q)\, dS. \tag{171}$$

But the right-hand side is easily shown to be a continuous function of P in B which vanishes on l [cf. 235 and 224], whence it follows, in view of Dini's theorem [23] that the series on the left-hand side of (171) is uniformly convergent in $\overline{B}$. We can assert all the more the uniform

convergence in B of series (170) and the series obtained from term by term differentiation once or twice with respect to t of series (158). This in fact shows that the function $u(P; t)$ defined by (158) satisfies equation (138) and conditions (139) and (140). We now turn to the proof of inequality (156). This proof will be developed in several stages. We shall start by explaining the conditions imposed on the contour l.

257. Assumptions regarding the contour. Let l be a simple closed rectifiable contour, the equation of which can be written as

$$x_1 = x_1(s); \quad x_2 = x_2(s), \tag{172}$$

where s is the arc length measured on l from a definite point in a direction shortly to be fixed, whilst the periodic functions $x_1(s)$ and $x_2(s)$ have continuous derivatives up to the fourth order. It is assumed further that there exists in some domain $\overline{B}$, containing the closed domain B in its interior, a continuous function $\Phi(x_1, x_2)$ with continuous derivatives up to the fourth order, such that the equation of l can be written as

$$\Phi(x_1, x_2) = 0, \tag{173}$$

where

$$(\text{grad } \Phi) = \Phi_{x_1}^2 + \Phi_{x_2}^2 > 0 \text{ on } l. \tag{174}$$

Suppose say that $\Phi(x_1, x_2) > 0$ inside l. Let N be a point on l and (y_1, y_2) Cartesian coordinates with origin at N; the y_2 axis is directed along the outward normal to l at N, and y_1 along the tangent, the orientation of the (y_1, y_2) axes being the same as (x_1, x_2). A direction is now defined on l, in which the arc length s is measured. The y_2 direction is opposite to that of grad $\Phi(x_1, x_2)$.

Let $c_i^{(j)}$ be the cosine of the angle between the x_i and x_j axes, and $(x_1^{(N)}, x_2^{(N)})$ the coordinates of the point N. We have:

$$x_i = c_i^{(1)} y_1 + c_i^{(2)} y_2 + x_i^{(N)} \quad (i = 1, 2), \tag{175}$$

and the equation of l in the (y_1, y_2) coordinates becomes

$$\Phi\left(c_1^{(1)} y_1 + c_1^{(2)} y_2 + x_1^{(N)},\ c_2^{(1)} y_1 + c_2^{(2)} y_2 + x_2^{(N)}\right) = \tilde{\Phi}(y_1, y_2) = 0, \tag{176}$$

where

$$\left(\frac{\partial \tilde{\Phi}}{\partial y_2}\right)_N = -\,|\text{grad } \Phi| \neq 0; \quad \left(\frac{\partial \tilde{\Phi}}{\partial y_1}\right)_N = 0.$$

By using the implicit function theorem [I, 159], we can write the equations of the part of contour l in the neighbourhood of the point N in the explicit form:

$$y_2 = \omega_N(y_1). \tag{177}$$

This equation holds on condition that $|y_1| < h_N$, where h_N is a positive constant depending on N, the function $\omega_N(y_1)$ has continuous derivatives up to the fourth order [cf. I, 159] and $\omega_N'(0) = 0$.

Let m denote the least value of $|\text{grad } \Phi|$ on l $(m > 0)$.

Let B_γ be the closed set of all points of $\overline{B}$ at which $|\operatorname{grad} \Phi| \geqslant \gamma m$, and B'_γ the set of points of $\overline{B}$ at which $|\operatorname{grad} \Phi| \leqslant \gamma m$. We extend in a continuous manner throughout the plane the function that is equal to unity in $B_{1/2}$ and outside $\overline{B}$ and to zero in $B'_{1/4}$, then average this extended function [157]. On taking a sufficiently small averaging radius, we can construct a function $\eta(x_1, x_2)$, defined throughout the plane, equal to unity in $B_{3/4}$ and outside $\overline{B}$, to zero in $B'_{1/5}$ and having derivatives of all orders. The functions

$$a_1(x_1, x_2) = -\frac{\Phi_{x_1}}{|\operatorname{grad} \Phi|}\eta(x_1, x_2); \quad a_2(x_1, x_2) = -\frac{\Phi_{x_2}}{|\operatorname{grad} \Phi|}\eta(x_1 x_2) \tag{178}$$

are now three times continuously differentiable in $\overline{B}$ and become equal to $\cos(n, x_1)$ and $\cos(n, x_2)$ on l, where n is the direction of the outward normal to l. If $|y_1| < h_N$, we have the identity

$$\tilde{\Phi}(y_1, \omega_N(y_1)) = 0.$$

On differentiating it four times with respect to y_1, then putting $y_1 = 0$ and recalling that $\omega'_N(0) = 0$, we obtain the equations at the point N:

$$\left.\begin{aligned}
&\tilde{\Phi}_{y_1} = 0\\
&\tilde{\Phi}_{y_1^2} + \tilde{\Phi}_{y_2}\omega''_N(0) = 0\\
&\tilde{\Phi}_{y_1^3} + 3\tilde{\Phi}_{y_1 y_2}\omega''_N(0) + \tilde{\Phi}_{y_2}\omega'''_N(0) = 0\\
&\tilde{\Phi}_{y_1^4} + 6\tilde{\Phi}_{y_1^2 y_2}\omega''_N(0) + 4\tilde{\Phi}_{y_1 y_2}\omega'''_N(0) + 3\tilde{\Phi}_{y_2^2}[\omega''_N(0)]^2 + \tilde{\Phi}_{y_2}\omega^{(4)}_N(0) = 0.
\end{aligned}\right\} \tag{179}$$

Since $|\tilde{\Phi}y_2| \geqslant m$, we can define $\omega_N^{(k)}(0)$ $(k = 2, 3, 4)$ from these equations and the inequalities hold:

$$\left|\omega_N^{(k)}(0)\right| \leqslant C \quad (k = 2, 3, 4), \tag{180}$$

where the constant C is independent of N.

258. Auxiliary propositions. Let $u(x_1, x_2)$ be a given function, continuous in B and having derivatives up to the fourth order continuous as far as l. If (x_1, x_2) is replaced by (y_1, y_2) in accordance with (175), a function is obtained which we write as $\tilde{u}(y_1, y_2)$.

If $u = 0$ on l, formulae (179) hold, in which $\tilde{\Phi}$ has to be replaced by $\tilde{u}$. For, $u[y_1, \omega_N(y_1)] = 0$, and the formulae in question are deduced exactly as above. It follows at once from these formulae that:

LEMMA 1. *If $u = 0$ on l, then $\partial\tilde{u}/\partial y_1 = 0$ and $\partial^k\tilde{u}/\partial y_1^k$ $(k = 2, 3, 4)$ are linear combinations of the derivatives up to order $(k - 1)$ of $\tilde{u}$, the coefficients of which have the form $a[\omega_N^{(k)}(0)]^m$ $(k = 2, 3, 4;\ m = 1, 2)$, where a is a number.*

Before turning to a statement of the next lemma, we introduce some symbols and conditions. Let $\varphi(x_1, x_2)$ and $\psi(x_1, x_2)$ be functions continuous in $\overline{B}$ and having derivatives, up to order $(p + 1)$ for φ, and up to order $(q + 1)$ for ψ, continuous as far as l.

We write $\tilde{\varphi}$ and $\tilde{\psi}$ as above for the functions $\varphi(x_1, x_2)$ and $\psi(x_1, x_2)$, expressed in the coordinates (y_1, y_2). We consider the expression:

$$\left\{\frac{\partial}{\partial y_1}\left[\frac{\partial^p \tilde{\Phi}}{\partial y_2^\alpha \partial y_1^\gamma} \cdot \frac{\partial^q \tilde{\psi}}{\partial y_2^\beta \partial y_1^\delta}\right]\right\}_{y_1 = y_2 = 0}. \tag{181}$$

It has a definite value at every point N of contour l and is therefore a function of the arc length s. We write $m(s)$ for this function and introduce the further notation:

$$l(s) = \left\{\frac{\partial^p \tilde{\varphi}}{\partial y_2^\alpha \partial y_1^\gamma} \cdot \frac{\partial^q \tilde{\psi}}{\partial y_2^\beta \partial y_1^\delta}\right\}_{y_1=y_2=0} . \tag{182}$$

Also, let $K(s)$ denote the curvature of contour l, equal to $\omega_N''(0)$.

LEMMA 2. *Given these assumptions, we have the equation:*

$$m(s) = \frac{\mathrm{d}l(s)}{\mathrm{d}s} + K(s)\left\{\alpha \frac{\partial^p \tilde{\varphi}}{\partial y_2^{\alpha-1} \partial y_1^{\gamma+1}} \cdot \frac{\partial^q \tilde{\psi}}{\partial y_2^\beta \partial y_1^\delta} + \right.$$

$$+ \beta \frac{\partial^p \tilde{\varphi}}{\partial y_2^\alpha \partial y_1^\gamma} \cdot \frac{\partial^q \tilde{\psi}}{\partial y_2^{\beta-1} \partial y_1^{\delta+1}} - \gamma \frac{\partial^p \tilde{\varphi}}{\partial y_2^{\alpha+1} \partial y_1^{\gamma-1}} \cdot \frac{\partial^q \tilde{\psi}}{\partial y_2^\beta \partial y_1^\delta} -$$

$$\left. - \delta \frac{\partial^p \tilde{\varphi}}{\partial y_2^\alpha \partial y_1^\gamma} \cdot \frac{\partial^q \tilde{\psi}}{\partial y_2^{\beta+1} \partial y_1^{\delta-1}}\right\} . \tag{183}$$

There will be no loss of generality if we prove (183) for the point N_0 at which $s = 0$. We shall in future write (y_1, y_2) for the local coordinates at the point N_0. Let N_1 be any other point of l, for which the corresponding value of s is s_1 and let (z_1, z_2) be the local system of coordinates at the point N_1. We shall write $\theta(s_1)$ for the angle through which the direction y_1 must be rotated in order to obtain the direction z_1, the direction of rotation being determined by the orientation of the axes. We have:

$$y_1 = z_1 \cos\theta - z_2 \sin\theta + y_1^{(0)} = c_{11} z_1 + c_{12} z_2 + y_1^{(0)}$$

$$y_2 = z_1 \sin\theta + z_2 \cos\theta + y_2^{(0)} = c_{21} z_1 + c_{22} z_2 + y_2^{(0)},$$

where $(y_1^{(0)}, y_2^{(0)})$ are the coordinates of N_1 in the (y_1, y_2) coordinates, and $\theta(s_1) \to 0$ as $s_1 \to 0$. If $\tilde{\tilde{\varphi}}$ and $\tilde{\tilde{\psi}}$ denote the functions φ and ψ expressed in terms of z_1 and z_2, we can write:

$$\frac{\partial^p \tilde{\tilde{\varphi}}}{\partial z_2^\alpha \partial z_1^\gamma} = \sum_{g_1, \ldots, g_\alpha, h_1, \ldots, h_\gamma = 1}^{2} \frac{\partial^p \tilde{\varphi}}{\partial y_{g_1} \ldots \partial y_{g_\alpha} \partial y_{h_1} \ldots \partial y_{h_\gamma}} c_{g_1 2} \ldots c_{g_\alpha 2} c_{h_1 1} \ldots c_{h_\gamma 1}$$

$$\frac{\partial^q \tilde{\tilde{\psi}}}{\partial z_2^\beta \partial z_1^\delta} = \sum_{l_1, \ldots, l_\beta, m_1, \ldots, m_\delta = 1}^{2} \frac{\partial^q \tilde{\psi}}{\partial y_{l_1} \ldots \partial y_{l_\beta} \partial y_{m_1} \ldots \partial y_{m_\delta}} c_{l_1 2} \ldots c_{l_\beta 2} c_{m_1 1} \ldots c_{m_\delta 1}$$

so that

$$l(s_1) = \sum \frac{\partial^p \tilde{\varphi}}{\partial y_{g_1} \ldots \partial y_{g_\alpha} \partial y_{h_1} \ldots \partial y_{h_\gamma}} \times$$

$$\times \frac{\partial^q \tilde{\psi}}{\partial y_{l_1} \ldots \partial y_{l_\beta} \partial y_{m_1} \ldots \partial y_{m_\delta}} c_{g_1 2} \ldots c_{g_\alpha 2} c_{l_1 2} \ldots c_{l_\beta 2} c_{h_1 1} \ldots c_{h_\gamma 1} c_{m_1 1} \ldots c_{m_\delta 1} \tag{184}$$

where the summation is with respect to $g_1, \ldots, g_\alpha, h_1, \ldots, h_\gamma, l_1, \ldots, l_\beta, m_1, \ldots, m_\delta$ from 1 to 2. We have as $s_1 \to 0$:

$$c_{g_n 2} \to \begin{cases} 1 \text{ for } g_n = 2 \\ 0 \text{ for } g_n = 1 \end{cases}; \quad c_{g_n 1} \to \begin{cases} 1 \text{ for } g_n = 1 \\ 0 \text{ for } g_n = 2 \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} c_{g_n 2} = \begin{cases} -\theta'(s_1)\cos\theta(s_1) \to -K_0 & \text{for } g_n = 1 \\ \theta'(s_1)\sin\theta(s_1) \to 0 & \text{for } g_n = 2 \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} c_{g_n 1} = \begin{cases} -\theta'(s_1)\sin\theta(s_1) \to 0 & \text{for } g_n = 1 \\ \theta'(s_1)\cos\theta(s_1) \to K_0 & \text{for } g_n = 2, \end{cases}$$

where $K_0 = \mathrm{d}\theta(s_1)/\mathrm{d}s_1$ for $s_1 = 0$. On differentiating both sides of (184) with respect to s_1 and taking into account the continuity of $\mathrm{d}l(s_1)/\mathrm{d}s_1$, which follows immediately from definition (182) and the assumptions made above, we obtain:

$$\begin{aligned} \left.\frac{\mathrm{d}l(s_1)}{\mathrm{d}s_1}\right|_{s_1=0} &= \lim_{s_1\to 0}\left[\frac{\mathrm{d}}{\mathrm{d}s_1}\left(\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta}\right)\right] - \\ &- K_0\alpha\frac{\partial^p\tilde{\varphi}}{\partial y_2^{\alpha-1}\,\partial y_1^{\gamma+1}}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta} - K_0\beta\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^{\beta-1}\,\partial y_1^{\delta+1}} + \\ &+ K_0\gamma\frac{\partial^p\tilde{\varphi}}{\partial y_2^{\alpha+1}\,\partial y_1^{\gamma-1}}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta} + K_0\delta\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^{\beta+1}\,\partial y_1^{\delta-1}}. \end{aligned} \tag{185}$$

Let $y_1 = y_1(s)$, $y_2 = y_2(s)$ be the equations of the contour l in the (y_1, y_2) coordinates. We have:

$$\frac{\mathrm{d}}{\mathrm{d}s_1} = \frac{\partial}{\partial y_1} y_1'(s_1) + \frac{\partial}{\partial y_2} y_2'(s_1),$$

where $y_1'(s_1) \to 1$ and $y_2'(s_1) \to 0$ as $s_1 \to 0$. Hence it follows that

$$\lim_{s_1\to 0}\left[\frac{\mathrm{d}}{\mathrm{d}s_1}\left(\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta}\right)\right] = \left\{\frac{\partial}{\partial y_1}\left(\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta}\right)\right\}_{s_1=0} = m(0).$$

On substituting this in (185), we get (183).

Corollary. Integration of both sides of (183) over l gives a formula which will come in useful later:

$$\begin{aligned} &\int \frac{\partial}{\partial y_1}\left\{\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta}\right\}_{y_1=y_2=0}\mathrm{d}s = \\ &= \int_l K(s)\left\{\alpha\frac{\partial^p\tilde{\varphi}}{\partial y_2^{\alpha-1}\,\partial y_1^{\gamma+1}}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta} + \beta\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^{\beta-1}\,\partial y_1^{\delta+1}} - \right. \\ &\left. - \gamma\frac{\partial^p\tilde{\varphi}}{\partial y_2^{\alpha+1}\,\partial y_1^{\gamma-1}}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^\beta\,\partial y_1^\delta} - \delta\frac{\partial^p\tilde{\varphi}}{\partial y_2^\alpha\,\partial y_1^\gamma}\cdot\frac{\partial^q\tilde{\psi}}{\partial y_2^{\beta+1}\,\partial y_1^{\delta-1}}\right\}_{y_1=y_2=0}\mathrm{d}s. \end{aligned} \tag{186}$$

259. Transformation of the contour integrals. Let $u(x_1, x_2)$ and $v(x_1, x_2)$ be functions continuous in B and having in $\overline{B}$ derivatives up to the fifth order for u and up to the fourth order for v, continuous as far as l. In addition to the notation $I_k(u, v)$ and $H_k(u)$, which we introduced in [256], we also put

$$\Phi_k(u, v) = \int_l \sum_{i_1, \ldots, i_{k-1}=1}^{2} \frac{\partial^k u}{\partial n\, \partial x_{i_1} \ldots \partial x_{i_{k-1}}} \cdot \frac{\partial^{k-1} v}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}}\, ds, \qquad (187)$$

where n is the direction of the outward normal to l. It may be remarked that the integrands in $I_k(u,v)$ and $\Phi_k(u,v)$ are invariant under orthogonal transformations of the coordinates (x_1, x_2). This is readily proved if we use formulae (175) and the conditions for orthogonality of the matrices with elements $c_i^{(k)}$.

If we take the obvious identity:

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \sum_{i_1, \ldots, i_{k-1}=1}^{2} \frac{\partial^k u}{\partial x_{i_1} \ldots \partial x_{i_{k-1}} \partial x_i} \frac{\partial^{k-1} v}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}} =$$

$$= \sum_{i_1, \ldots, i_{k-1}=1}^{2} \frac{\partial^{k-1} \Delta u}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}} \cdot \frac{\partial^{k-1} v}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}} +$$

$$+ \sum_{i_1, \ldots, i_{k-1}, i=1}^{2} \frac{\partial^k u}{\partial x_{i_1} \ldots \partial x_{i_{k-1}} \partial x_i} \cdot \frac{\partial^k v}{\partial x_{i_1} \ldots \partial x_{i_{k-1}} \partial x_i},$$

integrate with respect to x and apply Ostrogradskii's formula to the left-hand side, we obtain:

$$I_k(u, v) + I_k(\Delta u, v) = \Phi_k(u, v). \qquad (188)$$

Repeated application of this formula gives us:

$$\begin{aligned} I_2(u, u) &= \Psi_2(u) + I_0(\Delta u, \Delta u) \\ I_3(u, u) &= \Psi_3(u) + I_1(\Delta u, \Delta u) \\ I_4(u, u) &= \Psi_4(u) + I_0(\Delta^2 u, \Delta^2 u), \end{aligned} \qquad (189)$$

where $\Psi_k(u)$ are the integrals over l defined by the formulae:

$$\begin{aligned} \Psi_2(u) &= \Phi_2(u, u) - \Phi_1(u, \Delta u) \\ \Psi_3(u) &= \Phi_3(u, u) - \Phi_2(u, \Delta u) \\ \Psi_4(u) &= \Phi_4(u, u) - \Phi_3(u, \Delta u) + \Phi_2(\Delta u, \Delta u) - \Phi_1(\Delta u, \Delta^2 u). \end{aligned} \qquad (190)$$

We now state a theorem which will be needed for the proof of the fundamental inequality (156).

THEOREM. *If $u(x_1, x_2)$ is continuous in $\overline{B}$, has derivatives up to the fifth order continuous as far as l, and satisfies the conditions*

$$u|_l = \Delta u|_l = 0, \qquad (191)$$

$\Psi_2(u)$, $\Psi_3(u)$, $\Psi_4(u)$ can be transformed in such a way that derivatives of order not higher than the first, second and third respectively appear under the integral over l.

Since the integrands in the $\Phi_k(u, v)$ are invariant under orthogonal transformation, we can express them in terms of the local coordinates (y_1, y_2)

at points of the contour; this gives us, on observing that the direction of n coincides with the y_2 direction:

$$\Phi_k(u, v) = \int_l \sum_{i_1,\ldots,i_{k-1}=1}^{2} \frac{\partial^k \tilde{u}}{\partial y_2\, \partial y_{i_1} \ldots \partial y_{i_{k-1}}} \cdot \frac{\partial^{k-1} \tilde{v}}{\partial y_{i_1} \ldots \partial y_{i_{k-1}}}\, \mathrm{d}s, \tag{192}$$

Let $\overline{\Phi}_k(\tilde{u}, \tilde{v})$ denote the integrand in this integral:

$$\Phi_k(u, v) = \int_l \overline{\Phi}_k(\tilde{u}, \tilde{v})\, \mathrm{d}s, \tag{193}$$

and let us also use the following notation for brevity:

$$\frac{\partial^s \tilde{u}}{\partial y_1^s} = \tilde{u}_s; \quad \frac{\partial^s \tilde{u}}{\partial y_2^s} = \tilde{u}^{(s)}; \quad \frac{\partial^m \tilde{u}}{\partial y_1^k\, \partial y_2^l} = \tilde{u}_k^{(l)}.$$

If we collect the terms for which $i_{k-1} = 1$ and the terms for which $i_{k-1} = 2$ in the integrand of (192), we obtain the identity:

$$\overline{\Phi}_k(\tilde{u}, \tilde{v}) = \Phi_{k-1}(\tilde{u}_1, \tilde{v}_1) + \overline{\Phi}_{k-1}(\tilde{v}, \tilde{u}^{(2)}). \tag{194}$$

We have further, on observing that $\Delta\tilde{u} = \tilde{u}_2 + \tilde{u}^{(2)}$:

$$\overline{\Phi}_k(\tilde{u}, \tilde{v}) - \overline{\Phi}_{k-1}(\tilde{v}, \Delta\tilde{u}) = \overline{\Phi}_{k-1}(\tilde{u}_1, \tilde{v}_1) - \overline{\Phi}_{k-1}(\tilde{v}, \tilde{u}_2). \tag{195}$$

We obtain on making use of (194) and (195):

$$\Psi_2(u) = \int_l \overline{\Phi}_1(\tilde{u}_1, \tilde{u}_1)\, \mathrm{d}s - \int_l \overline{\Phi}_1(\tilde{u}, \tilde{u}_2)\, \mathrm{d}s, \tag{196}$$

$$\begin{aligned} \Psi_3(u) = {} & \int_l [\overline{\Phi}_1(\tilde{u}_2, \tilde{u}_2) - \overline{\Phi}_1(\tilde{u}_2, \tilde{u}^{(2)})]\, \mathrm{d}s + \\ & + \int_l [\overline{\Phi}_1(\tilde{u}_1, \tilde{u}_1^{(2)}) - \overline{\Phi}_1(\tilde{u}_1, \tilde{u}_3)]\, \mathrm{d}s, \end{aligned} \tag{197}$$

$$\begin{aligned} \Psi_4(u) = {} & \int_l [\overline{\Phi}_1(\tilde{u}_3, \tilde{u}_3) + \overline{\Phi}_1(\tilde{u}_1^{(2)}, \tilde{u}_1^{(2)}) - 2\overline{\Phi}_1(\tilde{u}_3, \tilde{u}_1^{(2)}) + \\ & + \overline{\Phi}_1(\Delta\tilde{u}_1, \Delta\tilde{u}_1)]\, \mathrm{d}s + \int_l [2\overline{\Phi}_1(\tilde{u}_2, \tilde{u}_2^{(2)}) - \overline{\Phi}_1(\tilde{u}_2, \tilde{u}_4) - \\ & - \overline{\Phi}_1, (\tilde{u}^{(2)}\, \tilde{u}^{(2)}) - \overline{\Phi}_1\, \Delta(\tilde{u}, \Delta\tilde{u}_2)]\, \mathrm{d}s. \end{aligned} \tag{198}$$

The first integrals on the right-hand sides are the sums of integrals of the form:

$$\int \frac{\partial^k \tilde{u}}{\partial y_2^{2\alpha+1}\, \partial y_1^{k-2\alpha-1}} \cdot \frac{\partial^{k-1} \tilde{u}}{\partial y_2^{2\beta}\, \partial y_1^{k-1-2\beta}}\, \mathrm{d}s, \tag{199}$$

where

$$k = 2, 3, 4; \quad 0 \leqslant \alpha \leqslant \left[\frac{k-2}{2}\right]; \quad 0 \leqslant \beta \leqslant \left[\frac{k-1}{2}\right].$$

Here, $k - 2a - 1 \geqslant 1$, and we have differentiation with respect to y_1 in the first factor of the integrand of (199). By integrating the obvious identity:

$$\frac{\partial^k \tilde{u}}{\partial y_2^{2a+1} \partial y_1^{k-2a-1}} \cdot \frac{\partial^{k-1} \tilde{u}}{\partial y_2^{2\beta} \partial y_1^{k-1-2\beta}} =$$
$$= \frac{\partial}{\partial y_1} \left(\frac{\partial^{k-1} \tilde{u}}{\partial y_2^{2a+1} \partial y_1^{k-2a-2}} \cdot \frac{\partial^{k-1} \tilde{u}}{\partial y_2^{2\beta} \partial y_1^{k-1-2\beta}} \right) -$$
$$- \frac{\partial^{k-1} \tilde{u}}{\partial y_2^{2a+1} \partial y_1^{k-2a-2}} \cdot \frac{\partial^k \tilde{u}}{\partial y_2^{2\beta} \partial y_1^{k-2\beta}}$$

and applying (186) to the first term on the right-hand side, it will be seen that integrals (199) differ from integrals of the form

$$\int_l \frac{\partial^k \tilde{u}}{\partial y_2^{2\beta} \partial y_1^{k-2\beta}} \cdot \frac{\partial^{k-1} \tilde{u}}{\partial y_2^{2a+1} \partial y_1^{k-2a-2}} \, ds \tag{200}$$

by integrals of the products of derivatives of order $(k - 1)$ with $K(s)$. Furthermore, the second integrals on the right-hand sides of (196), (197) and (198) have the same structure as integrals (200). An important point for what follows is that the derivative

$$\left(\frac{\partial^m \tilde{u}}{\partial y_1^{m_1} \partial y_2^{m_2}} \right)_N$$

is a linear combination of derivatives of order m of (x_1, x_2), with coefficients which are products of powers of $\cos(n_1, x_1)$, $\cos(n_1, x_2)$. If these latter are replaced by $a_1(x_1, x_2)$, $a_2(x_1, x_2)$, which are defined by (178), we obtain a linear combination of derivatives of order m of $u(x_1, x_2)$ with coefficients smooth in $\overline{B}$, and this linear combination becomes $\partial \ \tilde{u}/\partial y_1^{m_1} \partial y_2^{m_2}$ at every point N of the boundary l.

It remains for us to show that integrals of form (200) are expressible in terms of integrals of products of $(k - 1)$ first derivatives of $u(x_1, x_2)$, if this function satisfies conditions (191).

If $\beta = 0$, this follows at once from Lemma 1. Let $\beta = 1$. We have:

$$\frac{\partial^2 \tilde{u}}{\partial y_2^2} = \Delta \tilde{u} - \frac{\partial^2 \tilde{u}}{\partial y_1^2}$$

so that

$$\frac{\partial^k u}{\partial y_2^2 \partial y_1^{k-2}} = \frac{\partial^{k-2} \Delta \tilde{u}}{\partial y_1^{k-2}} - \frac{\partial^k \tilde{u}}{\partial y_1^k},$$

and by again applying Lemma 1, which is permissible in view of (191), we see that the right-hand side of the last formula is expressed in terms of the derivatives up to order $(k - 3)$ of $\Delta \tilde{u}$ and the derivatives up to order $(k - 1)$ of $\tilde{u}$. Since 0 and 1 are naturally possible values of β, the theorem is proved.

260. Proof of the fundamental lemma. It follows from the last theorem that $\Psi_k(u)$ is the sum of integrals of the form

$$\int_l [\omega_N^{(s)}(0)]^r \frac{\partial^{k-1} \tilde{u}}{\partial y_2^a \partial y_1^{k-1-a}} \cdot \frac{\partial^m u}{\partial y_2^\beta \partial y_1^{m-\beta}} \, ds,$$

where $1 \leqslant m \leqslant k-1$, $2 \leqslant s \leqslant k$. But, as we have already indicated, the derivatives of $\tilde{u}$ with respect to (y_1, y_2) are expressible linearly in terms of the derivatives of $u(x_1, x_2)$ with respect to (x_1, x_2) with bounded coefficients. On also taking into account the boundedness of $\omega_N^{(s)}(0)$, we see that $\Psi_k(u)$ does not exceed a finite sum of terms of the form:

$$C \int_l \left| \frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right| \cdot \left| \frac{\partial^m u}{\partial x_1^\nu \, \partial x_2^{m-\nu}} \right| ds \leqslant$$

$$\leqslant \frac{C}{2} \int_l \left(\frac{\partial^{k-1} u}{\partial x^\mu \, \partial x_2^{k-1-\mu}} \right)^2 ds + \frac{C}{2} \int_l \left(\frac{\partial^m u}{\partial x_1^\nu \, \partial x_2^{m-\nu}} \right)^2 ds, \quad (201)$$

where C is constant. Let us find the bounds of the integrals on the right-hand side. For this, we return to the functions $a_1(x_1, x_2)$ and $a_2(x_1, x_2)$ defined by (178). They are equal to $\cos(n, x_1)$ and $\cos(n, x_2)$ at points of l, so that

$$a_1 \cos(n, x_1) + a_2 \cos(n, x_2) = 1 \qquad \text{on } l.$$

Application of Ostrogradskii's formula gives us

$$\int_l \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right)^2 ds =$$

$$= \int_l \left[\cos(n, x_1)\, a_1 \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right)^2 + \cos(n, x_2)\, a_2 \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right)^2 \right] ds =$$

$$= \iint_B \left\{ \frac{\partial}{\partial x_1} \left[a_1 \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right)^2 \right] + \frac{\partial}{\partial x_2} \left[a_2 \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right)^2 \right] \right\} dS =$$

$$= \iint_B \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \right)^2 \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \right) dS + 2 \iint_B \frac{\partial^{k-1} u}{\partial x_1^\mu \, \partial x_2^{k-1-\mu}} \times$$

$$\times \left(a_1 \frac{\partial^k u}{\partial x_1^{\mu+1} \, \partial x_2^{k-1-\mu}} + a_2 \frac{\partial^k u}{\partial x_1^\mu \, \partial x_2^{k-\mu}} \right) dS.$$

On introducing the notation:

$$M = \max_B \left\{ |a_1|, |a_2|, \left| \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \right| \right\}$$

and using the obvious inequality:

$$2\,|ab| \leqslant 2 \left| a \sqrt{\varepsilon_k} \cdot \frac{b}{\sqrt{\varepsilon_k}} \right| \leqslant (a \sqrt{\varepsilon_k})^2 + \left(\frac{b}{\sqrt{\varepsilon_k}} \right)^2 = \varepsilon_k a^2 + \frac{b^2}{\varepsilon_k},$$

where $\varepsilon_k > 0$, we obtain the inequalities:

$$\left| \iint\limits_B \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \partial x_2^{k-1-\mu}} \right)^2 \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \right) \mathrm{d}S \right| < M I_{k-1}(u, u),$$

$$\left| 2 \iint\limits_B \frac{\partial^{k-1} u}{\partial x_1^\mu \partial x_2^{k-1-\mu}} \left(a_1 \frac{\partial^k u}{\partial x_1^{\mu+1} \partial x_2^{k-1-\mu}} + a_2 \frac{\partial^k u}{\partial x_1^\mu \partial x_2^{k-\mu}} \right) \mathrm{d}S \right| \leqslant$$

$$\leqslant 2M \left[\varepsilon_k I_k(u, u) + \frac{1}{\varepsilon_k} I_{k-1}(u, u) \right],$$

whence

$$\int\limits_l \left(\frac{\partial^{k-1} u}{\partial x_1^\mu \partial x_2^{k-1-\mu}} \right)^2 \mathrm{d}s \leqslant M I_{k-1}(u, u) +$$

$$+ 2M \left[\varepsilon_k I_k(u_1 u) + \frac{1}{\varepsilon_k} I_{k-1}(u, u) \right]. \tag{202}$$

If $m < k - 1$, we choose $\varepsilon = 1$ and obtain the inequality:

$$\int\limits_l \left(\frac{\partial^m u}{\partial x_1^\nu \partial x_2^{m-\nu}} \right)^2 \mathrm{d}s \leqslant 2M I_{m+1}(u, u) + 3M I_m(u, u). \tag{203}$$

As we have seen, $\Psi_k(u)$ do not exceed products of the constant $C/2$ appearing in (201) with the sum of a finite number of integrals, for which we have obtained inequalities (202) and (203). Therefore:

$$|\Psi_k(u)| \leqslant A_1^{(k)} \varepsilon_k I_k(u, u) + \frac{A_2^{(k)}}{\varepsilon_k} I_{k-1}(u, u) + A_3^{(k)} \sum_{r=1}^{k-1} I_r(u, u), \tag{204}$$

where $A_s^{(k)}$ ($s = 1, 2, 3$) are constants depending on k, C and the number of terms. On now using (189), we get

$$I_2(u, u) \leqslant A_1^{(2)} \varepsilon_2 I_2(u, u) + \left(\frac{A_2^{(2)}}{\varepsilon_2} + A_3^{(2)} \right) I_1(u, u) + I_0(\Delta u, \Delta u),$$

and, after choosing ε_2 such that $1 - \varepsilon_2 A_1^{(2)} \geqslant 1/2$, we find that

$$I_2(u, u) \leqslant 2 \left(\frac{A_2^{(2)}}{\varepsilon_2} + A_3^{(2)} \right) I_1(u, u) + 2 I_0(\Delta u, \Delta u) =$$

$$= 2 I_0(\Delta u, \Delta u) + B_2 I_1(u, u). \tag{205}$$

Similarly, on choosing ε_3 and ε_4 such that $1 - \varepsilon_k A_1^{(k)} > 1/2$, we find by using (189) and (204) that

$$I_3(u, u) \leqslant 2 I_1(\Delta u, \Delta u) + B_3 [I_2(u, u) + I_1(u, u)] \tag{206}$$

$$I_4(u, u) \leqslant 2 I_0(\Delta^2 u, \Delta^2 u) + B_4 [I_3(u, u) + I_2(u, u) + I_1(u, u)], \tag{207}$$

where B_s are constants.

If we replace $I_2(u, u)$ in the right-hand side of (206) by the right-hand side of (205), we get:

$$I_3(u, u) \leqslant 2I_1(\Delta u, \Delta u) + 2B_3 I_0(\Delta u, \Delta u) + B_3(B_2 + 1) I_1(u, u). \tag{208}$$

If we replace $I_2(u, u)$ and $I_3(u, u)$ in the right-hand side of (207) by the right-hand sides of (205) and (208) respectively, we obtain

$$I_4(u, u) \leqslant 2I_0(\Delta^2 u, \Delta^2 u) + C_1 I_1(\Delta u, \Delta u) + C_2 I_0(\Delta u, \Delta u) + C_3 I_1(u, u), \tag{209}$$

where the C_s are constants.

On adding (205), (208) and (209), we in fact obtain the fundamental inequality:

$$H_4(u) \leqslant A\,[I_0(\Delta^2 u, \Delta^2 u) + I_1(\Delta u, \Delta u) + I_0(\Delta u, \Delta u) + I_1(u, u)],$$

where A is a constant.

The proof given above, that series (158) admits of term by term differentiation with respect to x_1, x_2 and t and hence gives the solution of problem (138), (139), (140), was essentially based on inequality (156). The whole of the material, starting from [256] and ending with the present section, represents the treatment, as applied to the wave equation, of Ladyzhenskaya's studies on the convergence of the Fourier series for linear hyperbolic equations (see Chap. II of the monograph *The Mixed Problem for the Hyperbolic Equation* (Smeshannaya zadacha dlya giperbolicheskogo uravneniya)).

261. Derivatives of eigenfunctions. When justifying Fourier's method, we made use of the fact that the derivatives of the eigenfunctions $v_m(P)$ of the equation

$$\Delta v_m + \lambda_m v_m = 0 \tag{210}$$

with boundary condition

$$v_m|_l = 0 \tag{211}$$

have derivatives up to the fifth order continuous as far as l. We stated the sufficient conditions for this in [226] in the three-dimensional case. We shall now use a conformal mapping of the circle on to the singly-connected domain B to prove this property of the eigenfunctions, given certain assumptions regarding the function that performs the conformal mapping. Let

$$z = f(\zeta) \qquad (z = x + yi;\quad \zeta = \xi + \eta i) \tag{212}$$

be the function accomplishing the conformal mapping of the circle $|\zeta| \leqslant 1$ on to the singly-connected domain B. We assume that $f(\zeta)$

is continuous together with its derivatives up to the fifth order in the closed circle $|\zeta| \leqslant 1$ and $f'(\zeta) \neq 0$. The sufficient conditions that must be imposed on the contour l for this to be the case may be found e.g. in V. I. Smirnov's *The Correspondence of Boundaries under Conformal Transformation* (O sootvetstvii granits pri konformnom preobrazovanii) *(Math. Annal.*, t. 107, 1932). On passing from variables (x, y) to variables (ξ, η), we obtain for the functions $\tilde{v}_m(\xi, \eta) = v_m(x, y)$, instead of (210), the equation

$$\Delta\tilde{v}_m = -\lambda_m |f'(\zeta)|^2 \tilde{v}_m. \tag{213}$$

Our future arguments are based on the following two lemmas, the proofs of which will be given in the next section.

LEMMA 1. *If the function $\psi(x, y)$ is continuous in the closed circle $\beta(x^2 + y^2 \leqslant 1)$ and has continuous first order derivatives inside β, the solution of the equation $\Delta u = \psi$ inside β, satisfying the condition $u = 0$ on the circumference $\lambda(x^2 + y^2 = 1)$, has continuous first order derivatives continuous as far as λ.*

LEMMA 2. *If $\psi(x, y)$ has derivatives up to order p, continuous as far as λ, $u(x, y)$ has derivatives up to order $(p + 1)$, continuous as far as λ.*

On applying Lemma 1, then Lemma 2 to equation (213), where the role of ψ is played by $-\lambda_m |f'(\zeta)|^2 \tilde{v}_m$, and observing that $|f'(\zeta)|^2$ has derivatives up to the fourth order, continuous as far as λ, the function $\tilde{v}_m(\xi, \eta)$ will be seen to have derivatives up to the fifth order, continuous as far as λ, so that $v_m(x, y)$ has derivatives up to the fifth order, continuous as far as l.

262. Proof of the auxiliary propositions. To prove Lemma 2, we need to prove a theorem on the logarithmic potential of a simple layer with differentiable density.

THEOREM. *If, in the potential of a simple layer:*

$$V(\mu) = \int_\lambda \mu(s) \log \frac{1}{r} \, ds \quad \left(r = \sqrt{(\xi - x)^2 + (\eta - y)^2}\right), \tag{214}$$

the density $\mu(s)$ has continuous derivatives up to some order k, the potential $V(\mu)$ will itself have derivatives up to order k inside β, continuous as far as λ.

The proof given below is suitable for any curve whose equation $\xi = \xi(s)$; $\eta = \eta(s)$ is such that the periodic functions $\xi(s)$ and $\eta(s)$ have continuous derivatives up to order $(k + 1)$.

Along with potential (214) we introduce the double layer potential:

$$W(\mu) = \int_\lambda \mu(s) \frac{\cos(n, r)}{r} \mathrm{d}s = \int_\lambda \mu(s) \frac{\eta'(\xi - x) - \xi'(\eta - y)}{r^2} \mathrm{d}s, \quad (215)$$

where ξ' and η' are the derivatives of $\xi(s)$ and $\eta(s)$ with respect to s. On taking into account the fact that $\xi'^2 + \eta'^2 = 1$, and differentiating with respect to x, we obtain:

$$\frac{\partial V(\mu)}{\partial x} = -\int_\lambda \mu(s) \frac{x - \xi}{r^2} \mathrm{d}s =$$

$$= \int_\lambda \mu(s)\, \xi' \,\frac{\xi'(\xi - x) + \eta'(\eta - y)}{r^2} \mathrm{d}s + \int_\lambda \mu(s)\, \eta' \,\frac{\eta'(\xi - x) - \xi'(\eta - y)}{r^2} \mathrm{d}s.$$

On using the formula:

$$\frac{\xi'(\xi - x) + \eta'(\eta - y)}{r^2} = -\frac{\mathrm{d}}{\mathrm{d}s} \log \frac{1}{r}$$

and integrating the first integral by parts, we obtain

$$\frac{\partial V(\mu)}{\partial x} = V[(\mu\xi')'] + W(\mu\eta'), \quad (216)$$

where the primes denote differentiation with respect to s. Similarly:

$$\frac{\partial V(\mu)}{\partial y} = V[(\mu\eta')'] - W(\mu\xi'). \quad (217)$$

We now differentiate the double layer potential:

$$\frac{\partial W(\mu)}{\partial x} = \int_\lambda \mu(s) \frac{-\eta' r^2 - 2(x - \xi)[\eta'(\xi - x) - \xi'(\eta - y)]}{r^4} \mathrm{d}s =$$

$$= \int_\lambda \mu(s) \frac{-\eta' r^2 + (y - \eta)[\xi'(x - \xi) + 2\eta'(y - \eta)]}{r^4} \mathrm{d}s =$$

$$= \int_\lambda \mu(s) \frac{\mathrm{d}}{\mathrm{d}s} \frac{y - \eta}{r^2} \mathrm{d}s = -\int_\lambda \mu'(s) \frac{y - \eta}{r^2} \mathrm{d}s = \frac{\partial V(\mu')}{\partial y}.$$

Therefore:

$$\frac{\partial W(\mu)}{\partial x} = \frac{\partial V(\mu')}{\partial y} \quad (218)$$

and similarly:

$$\frac{\partial W(\mu)}{\partial y} = -\frac{\partial V(\mu')}{\partial x}. \quad (219)$$

If $\mu'(s)$ is a continuous function, it follows from (216) and (217) that $V(\mu)$ has first order derivatives continuous as far as λ, i.e. the theorem is proved for $k = 1$. If $\mu'(s)$ and $\mu''(s)$ are continuous, in view of what has been said $V(\mu')$ will have first order derivatives continuous as far as λ, and it follows from (218) and (219) that $W(\mu)$ also has first order derivatives continuous as far as λ. But now $W(\mu\eta')$ and $W(\mu\xi')$, as also $V[(\mu\xi')']$ and $V[(\mu\eta')']$, have continuous first order derivatives as far as λ, and it follows from (216) and (217) that $V(\mu)$ has second order derivatives continuous as far as λ, i.e. the theorem is also proved for $k = 2$. Let us now prove the theorem for any $k \geqslant 3$, on the assumption that the theorem is valid for $(k - 1)$. Let $\mu(s)$ have continuous derivatives up to order k, i.e. $\mu'(s)$ has continuous derivatives up to order $(k - 1)$. In view of our assumption that the theorem holds for $(k - 1)$, $V(\mu')$ will have derivatives up to order $(k - 1)$, continuous as far as λ. It now follows from (218) and (219) that $W(\mu)$ has derivatives up to order $(k - 1)$, continuous as far as λ. By virtue of this, we can say that $W(\mu\xi')$ and $W(\mu\eta')$ have derivatives up to order $(k - 1)$, continuous as far as λ. Further, by the assumption stated above, $V[(\mu\xi')']$ and $V[(\mu\eta')']$ have derivatives up to order $(k - 1)$, continuous as far as λ, and it follows from (216) and (217) that $V(\mu)$ has derivatives up to the kth order continuous as far as λ. The theorem is therefore fully proved.

We now turn to the proof of the lemmas stated in the previous section. We bring in the point (x_1, y_1), conjugate to (x, y) with respect to the circumference λ:

$$x_1 = \frac{x}{x^2 + y^2}; \quad y_1 = \frac{y}{x^2 + y^2}, \tag{220}$$

and write:

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}; \qquad r_1 = \sqrt{(\xi - x_1)^2 + (\eta - y_1)^2}. \tag{221}$$

Green's function for the circular domain β with boundary condition $u = 0$ on λ has the form [222]:

$$G(\xi, \eta; x, y) = \frac{1}{2\pi} \log \frac{1}{r} - \frac{1}{2\pi} \log \frac{1}{r_1} - \frac{1}{2\pi} \log \frac{1}{\sqrt{x^2 + y^2}}, \tag{222}$$

and can be written, in view of the symmetry of Green's function, as

$$u(x, y) = -\iint_\beta G(\xi, \eta; x, y)\, \psi(\xi, \eta)\, d\xi\, d\eta. \tag{223}$$

This formula can be rewritten as

$$u(x, y) = - \iint\limits_{\xi^2+\eta^2 \leqslant \frac{1}{4}} G(\xi, \eta; x, y)\, \psi(\xi, \eta)\, d\xi\, d\eta +$$

$$+ \frac{1}{2\pi} \iint\limits_{\frac{1}{4} \leqslant \xi^2+\eta^2 \leqslant 1} \psi(\xi, \eta) \log \frac{1}{r_1}\, d\xi\, d\eta - \frac{1}{2\pi} \iint\limits_{\frac{1}{4} \leqslant \xi^2+\eta^2 \leqslant 1} \psi(\xi, \eta) \log \frac{1}{r}\, d\xi\, d\eta +$$

$$+ \frac{1}{4\pi} \log (x^2 + y^2) \iint\limits_{\frac{1}{4} \leqslant \xi^2+\eta^2 \leqslant 1} \psi(\xi, \eta)\, d\xi\, d\eta. \tag{224}$$

If the point (x, y) lies in some neighbourhood of the circumference λ, say $3/4 \leqslant \sqrt{x^2 + y^2} \leqslant 1$, the first and last terms on the right-hand side have continuous derivatives of all orders. This is obvious as regards the last term, whilst we have to make use of (222) in regard to the first. Thus we only need to consider the second and third terms on the right-hand side in the proof of both lemmas. These terms are logarithmic potentials with density $\psi(\xi, \eta)$, distributed over the domain $1/4 \leqslant \xi^2 + \eta^2 \leqslant 1$, the second term being the value of this potential at the point (x_1, y_1), and the third at (x, y). It follows at once from (220) that (x_1, y_1) have derivatives of all orders with respect to (x, y) in the neighbourhood of λ, where the points (x, y) and (x_1, y_1) approach λ simultaneously. It is therefore sufficient to show that the second term has corresponding derivatives with respect to (x_1, y_1), continuous as far as λ, when (x_1, y_1) tends to λ, and the same for the derivatives of the third term with respect to (x, y) when (x, y) tends to λ. We turn to the proof of Lemma 1. We know that $u(x, y)$ has continuous derivatives up to the second order inside β and is continuous as far as λ. We have to show that its first derivatives are also continuous as far as λ. But this follows immediately from the fact that the logarithmic potential with continuous density has continuous first order derivatives throughout the plane [II, 200, 201]. We turn to the proof of Lemma 2, where we shall investigate only the third term on the right-hand side of (224). The investigation of the second term is exactly similar. We have, on using Green's formula [II, 69]:

$$\frac{\partial}{\partial x} \iint\limits_{\frac{1}{4} \leqslant \xi^2+\eta^2 \leqslant 1} \psi(\xi, \eta) \log \frac{1}{r}\, d\xi\, d\eta = \iint\limits_{\frac{1}{4} \leqslant \xi^2+\eta^2 \leqslant 1} \frac{\partial \psi}{\partial \xi} \log \frac{1}{r}\, d\xi\, d\eta +$$

$$+ \int\limits_{\lambda} \psi \cos (n, \xi) \log \frac{1}{r}\, ds - \int\limits_{\xi^2+\eta^2 = \frac{1}{4}} \psi \cos (n, \xi) \log \frac{1}{r}\, ds. \tag{225}$$

The last integral has continuous derivatives of all orders everywhere except on the circumference $\xi^2 + \eta^2 = 1/4$, and it is sufficient to investigate the first and second terms on the right-hand side. If ψ has first order derivatives, continuous as far as λ, the second term has the same property by virtue of the theorem proved above, whilst the first term has first order derivatives, continuous throughout the plane, inasmuch as it is the logarithmic potential over the ring $1/4 \leqslant \xi^2 + \eta^2 \leqslant 1$ with continuous density $\partial\psi/\partial\xi$. Hence the left-hand side of (225) has first order derivatives, continuous as far as λ, and Lemma 2 is proved for $k = 1$.

We turn to the case $k = 2$. Let ψ have derivatives up to the second order, continuous as far as λ. We differentiate both sides of (225) say with respect to y and again use Green's formula:

$$\frac{\partial^2}{\partial x\,\partial y}\iint\limits_{\frac{1}{4}\leqslant\xi^2+\eta^2\leqslant 1}\psi(\xi,\eta)\log\frac{1}{r}\,d\xi\,d\eta = \iint\limits_{\frac{1}{4}\leqslant\xi^2+\eta^2\leqslant 1}\frac{\partial^2\psi}{\partial\xi\,\partial\eta}\log\frac{1}{r}\,d\xi\,d\eta +$$

$$+\int\limits_{\lambda}\frac{\partial\psi}{\partial\xi}\cos(n,\eta)\log\frac{1}{r}\,ds + \frac{\partial}{\partial y}\int\limits_{\lambda}\psi\cos(n,\xi)\log\frac{1}{r}\,ds + \ldots, \quad (226)$$

where the terms written contain integrals over the circumference $\xi^2 + \eta^2 = 1/4$ and have continuous derivatives of all orders everywhere except on this circumference. It may be seen by using the theorem proved above that the terms containing an integral over the circumference λ have first order derivatives, continuous as far as this circumference, and the double integral has first order derivatives, continuous throughout the plane. Hence the left-hand side of (226) has first order derivatives, continuous as far as λ, and Lemma 2 is proved for $k = 2$. The lemma is proved in precisely the same way for further values of k.

The proofs of the last two sections are due to Smolitskii.

263. The boundary value problem for a sphere. We shall now consider the boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad (227)$$

in the case of a sphere. We must prove the preliminary lemma: *if* $u = \varphi(x, y, z, t) = \varphi(M, t)$ *is the solution of equation* (227) *which is*

homogeneous and of zero degree in the variables (x, y, z, t), *and if it vanishes on the sphere* $r = t$, *where* $r = \sqrt{x^2 + y^2 + z^2}$, *the expression*

$$u = \int_0^{t-r} \omega(\tau)\, \varphi(M, t - \tau)\, d\tau, \tag{228}$$

where $\omega(\tau)$ *is any continuous function and the lower limit can be any given number, is also a solution of* (227).

Differentiation of solution (228) gives us:

$$\frac{\partial u}{\partial x} = \int_0^{t-r} \omega(\tau) \frac{\partial \varphi(M, t-\tau)}{\partial x}\, d\tau - \omega(t - r)\, \varphi(M, r) \frac{x}{r}.$$

But $\varphi(M, r) = 0$ by hypothesis, so that

$$\frac{\partial u}{\partial x} = \int_0^{t-r} \omega(\tau) \frac{\partial \varphi(M, t-\tau)}{\partial x}\, d\tau.$$

We differentiate again:

$$\frac{\partial^2 u}{\partial x^2} = \int_0^{t-r} \omega(\tau) \frac{\partial^2 \varphi(M, t-\tau)}{\partial x^2}\, d\tau - \omega(t-r) \left[\frac{\partial \varphi(M, t-\tau)}{\partial x}\right]_{\tau=t-r} \cdot \frac{x}{r}.$$

Analogous expressions may be obtained in the same way for the second derivatives with respect to y and z. We have for the second derivative with respect to t:

$$\frac{\partial^2 y}{\partial t^2} = \int_0^{t-r} \omega(\tau) \frac{\partial^2 \varphi(M, t-\tau)}{\partial t^2}\, d\tau + \omega(t-r) \left[\frac{\partial \varphi(M, t-\tau)}{\partial t}\right]_{\tau=t-r}.$$

On substituting in equation (227) and recalling that, by hypothesis, $\varphi(M, t - \tau)$ satisfies (227), we obtain as a result of the substitution:

$$\frac{\omega(t-r)}{r} \left[\frac{\partial \varphi(M, t-\tau)}{\partial t} r + \frac{\partial \varphi(M, t-\tau)}{\partial x} x + \frac{\partial \varphi(M, t-\tau)}{\partial y} y + \right.$$
$$\left. + \frac{\partial \varphi(M, t-\tau)}{\partial z} z\right]_{\tau=t-r} = 0. \tag{229}$$

But, by Euler's theorem on homogeneous functions, we have [I, 149]:

$$\frac{\partial \varphi(M, t-\tau)}{\partial t}(t-\tau) + \frac{\partial \varphi(M, t-\tau)}{\partial x} x + \frac{\partial \varphi(M, t-\tau)}{\partial y} y + \frac{\partial \varphi(M, t-\tau)}{\partial z} z = 0.$$

On substituting $\tau = t - r$ here, we see that equation (229) is satisfied, so that (228) in fact yields a solution of (227).

We shall now seek a special form of the solution of (227), namely

$$u = \psi\left(\frac{t}{r}\right) Y_n(\theta, \varphi), \tag{230}$$

where $Y_n(\theta, \varphi)$ is a spherical function of order n and $\psi(x)$ is the required function.

On transforming equation (227) to spherical coordinates, we obtain [II, 119]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right]. \tag{231}$$

On substituting expression (230) and taking into account the fact that $Y(\theta, \varphi)$ satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\varphi^2} + n(n+1)\,Y_n = 0,$$

we arrive at the following equation for $\psi(t/r)$:

$$\psi''\left(\frac{t}{r}\right) = \frac{t^2}{r^2}\psi''\left(\frac{t}{r}\right) - n(n+1)\,\psi\left(\frac{t}{r}\right),$$

or

$$(1 - x^2)\,\psi''(x) + n(n+1)\,\psi(x) = 0. \tag{232}$$

In order to find $\psi(x)$, we recall the equation that is satisfied by the Legendre polynomials [III, 172]:

$$[(1 - x^2)\,P_n'(x)]' + n(n+1)\,P_n(x) = 0.$$

We bring in the polynomial of degree $(n + 1)$:

$$Q_{n+1}(x) = \int_1^x P_n(x)\,dx. \tag{233}$$

Integration of both sides of the previous equation over the interval $(1, x)$ gives

$$(1 - x^2)\,P_n'(x) + n(n+1)\,Q_{n+1}(x) = 0,$$

or, by (233):

$$(1 - x^2)\,Q_{n+1}''(x) + n(n+1)\,Q_{n+1}(x) = 0,$$

and a comparison with (232) shows that the function

$$u = Q_{n+1}\left(\frac{t}{r}\right) Y_n(\theta, \varphi) \tag{234}$$

is a solution of equation (227). By (233), $Q_{n+1}(1) = 0$, i.e. solution (234) vanishes for $r = t$. Moreover solution (234) is clearly a homogeneous function of zero degree in the variables (x, y, z, t). On using the lemma, we can say that the function

$$u(M, t) = Y_n(\theta, \varphi) \int_0^{t-r} \omega(\tau)\, Q_{n+1}\left(\frac{t-\tau}{r}\right) d\tau \tag{235}$$

is also a solution of equation (227) with any choice of continuous function $\omega(\tau)$.

After these preliminary remarks, we turn to the solution of the boundary value problem for a special form of boundary condition. Let the solution of equation (227) be required outside the sphere $r = 1$, satisfying the homogeneous initial conditions:

$$u|_{t=0} = 0; \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \tag{236}$$

and a boundary condition of the form:

$$u|_{r=1} = f(t)\, Y_n(\theta, \varphi), \tag{237}$$

where $f(t)$ is a given function. We assume that this function has continuous derivatives up to the second order and that

$$f(0) = f'(0) = 0. \tag{238}$$

We return to (235). If we replace t by $(t + 1)$ on the right-hand side, we again obtain a solution of (227), since the coefficients of this equation do not contain t. We shall seek the solution of our present boundary value problem in the form:

$$u = \begin{cases} Y_n(\theta, \varphi) \displaystyle\int_0^{t+1-r} \omega(\tau)\, Q_{n+1}\left(\frac{t+1-\tau}{r}\right) d\tau & t \geqslant r - 1 \\ 0 & t \leqslant r - 1, \end{cases} \tag{239}$$

where $\omega(\tau)$ is a required function of τ for $\tau \geqslant 0$. The first of conditions (236) follows at once from (239). On differentiating (239) with respect to t with $r = 1$ and then putting $t = 0$, we obtain the second of conditions (236), in view of the fact that $Q_{n+1}(1) = 0$. Boundary condition (237) gives us the integral equation for $\omega(\tau)$:

$$\int_0^t \omega(\tau)\, Q_{n+1}(t + 1 - \tau)\, d\tau = f(t).$$

The equation written above is a Volterra equation of the first kind. Differentiation of it term by term gives the equation

$$\int_0^t \omega(\tau)\, P_n(t+1-\tau)\,\mathrm{d}\tau = f'(t),$$

this equation being equivalent to the previous one by virtue of (238). On differentiating once more, we obtain, by (238), an equivalent equation of the second kind:

$$\omega(t) + \int_0^t \omega(\tau)\, P_n'(t+1-\tau)\,\mathrm{d}\tau = f''(t).$$

The kernels of these equations depend only on the difference $(t-\tau)$, and use of the method of [46] gives us a solution of the form

$$\omega(t) = f''(t) - \int_0^t H(t-x)\, f''(x)\,\mathrm{d}x,$$

where $H(z)$ is the sum of residues of the function

$$\frac{-s^n}{s^n + s^{n-1}P_n'(1) + s^{n-2}P_n''(1) + \ldots + P_n^{(n)}(1)}\,\mathrm{e}^{sz}$$

with respect to the zeros of its denominator.

Boundary condition (237) starts to operate at the instant $t = 0$. We have rest prior to this instant. The front of the disturbance moves with unit velocity. Outside the sphere with centre at the origin and radius $(t+1)$ we have rest from the instant t, by (239). The second order derivatives can have discontinuities on the wave-front itself. It may be remarked that we can approximate in mean on the sphere to any continuous boundary condition with the aid of boundary conditions of the form (237). This follows from the closure of the spherical functions. The above method can also be used on a plane for the exterior of a circular domain (V. I. Smirnov, *Dokl. Akad. Nauk SSSR*, t. XIV, no. 1, 1937).

264. Vibrations of the interior part of a sphere. We shall now form a solution of equation (227) in the presence of conditions (236) and (237) for the interior of a sphere. If $n \geqslant 1$, $Q_{n+1}(x)$ may easily be shown to be an even function for even $(n+1)$ and an odd function for odd $(n+1)$, and solution (235) can be written in the form

$$u_1(M,t) = Y_n(\theta,\varphi)\int_0^{t-r} \omega_1(\tau)\, Q_{n+1}\left(\frac{\tau-t}{r}\right)\mathrm{d}\tau. \tag{240}$$

On replacing t by $t - 1$ on the right-hand side, we obtain a solution of the form

$$u_2(M, t) = Y_n(\theta, \varphi) \int_0^{t+r-1} \omega_2(\tau)\, Q_{n+1}\left(\frac{\tau + 1 - t}{r}\right) d\tau, \tag{241}$$

where $\omega_2(\tau) = 0$ for $\tau < 0$. This solution corresponds to a wave moving inwards from the surface of the sphere. It ceases to be finite for $t > 1$ at the centre of the sphere, i.e. with $r = 0$. When $t = 1$ the wave reaches the centre of the sphere, and it is natural to add solution (240) to this solution, with t replaced in it by $t = 1$ and a specially chosen $\omega_1(\tau)$. This leads us to a solution of the form

$$u_3(M, t) = Y_n(\theta, \varphi) \int_{t-1-r}^{t-1+r} \omega_3(\tau)\, Q_{n+1}\left(\frac{\tau + 1 - t}{r}\right) d\tau, \tag{242}$$

where $\omega(\tau) = 0$ for $\tau < 0$. We have in the limits of the integration: $-r \leqslant \tau + 1 - t \leqslant r$, and solution (242) remains finite even for $r = 0$. It vanishes here. In order to make fewer assumptions regarding the derivatives of $f(t)$, appearing in boundary condition (237), we take as fundamental the solution which is obtained from (242) by differentiation with respect to t. On using the fact that $Q_{n+1}(\pm 1) = 0$ for $n \geqslant 1$, we obtain the solution:

$$u(M, t) = Y_n(\theta, \varphi)\, \varphi_n(r, t), \tag{243}$$

where

$$\varphi_n(r, t) = \begin{cases} \dfrac{1}{r} \displaystyle\int_{t-1-r}^{t-1+r} \omega(\tau)\, P_n\left(\frac{\tau + 1 - t}{r}\right) d\tau & t > 1 - r \\ 0 & t \leqslant 1 - r \end{cases} \tag{244}$$

and $\omega(\tau) = 0$ for $\tau \leqslant 0$. It follows from this expression, as in [263], that conditions (236) are observed for any $\omega(\tau)$. It is easy to verify directly that formulae (243) and (244) yield a solution of equation (227) even with $n = 0$, if $\omega(\tau)$ has a continuous derivative. It may be remarked that (242) does not give solutions of (227) with $n = 0$.

Boundary condition (237) leads to the following equation:

$$\int_{t-2}^{t} \omega(\tau)\, P_n(\tau + 1 - t)\, d\tau = f(t). \tag{245}$$

Suppose that $f(t)$ has a continuous derivative and $f(0) = f'(0) = 0$. Differentiation of equation (245) with respect to t gives

$$\omega(t) + (-1)^{n+1}\omega(t-2) - \int_{t-2}^{t} \omega(\tau) P_n'(\tau + 1 - t)\, d\tau = f'(t) \quad (n \geqslant 1) \tag{246_1}$$

and with $n = 0$:

$$\omega(t) - \omega(t-2) = f'(t). \tag{246_2}$$

Equation (246_1) gives us the possibility of constructing $\omega(\tau)$ by successive stages. We first determine $\omega(\tau)$ in the interval $0 \leqslant t \leqslant 2$ from the Volterra equation:

$$\omega(t) - \int_0^t \omega(\tau) P_n'(\tau + 1 - t)\, d\tau = f'(t).$$

We then determine $\omega(\tau)$ in the interval $2 < t \leqslant 4$ from the equation

$$\omega(t) - \int_2^t \omega(t) P_n'(\tau + 1 - t)\, d\tau =$$

$$= f'(t) + (-1)^n \omega(t-2) + \int_0^2 \omega(\tau) P_n'(\tau + 1 - t)\, d\tau,$$

the right-hand side of which is known, and so on. The function $\omega(\tau)$ obtained is substituted in the right-hand side of (244).

A one-sided Laplace transformation can be used for the solution of equation (246_1). The broad outlines of this method will be mentioned. The final formula will be obtained below by a different method.

We write the integral in equation (246_1) as the sum of two integrals with lower limits zero, multiply both sides by e^{-st}, where $s = \sigma_1 + \sigma_2 i$, and σ_1 is a sufficiently large positive number, and integrate with respect to t over the interval $0 \leqslant t < \infty$. We introduce the notation:

$$\Omega(s) = \int_0^\infty e^{-st}\omega(t)\, dt, \quad F(s) = \int_0^\infty e^{-st} f(t)\, dt; \tag{247}$$

then we make use of the convolution theorem [45] and the formulae:

$$\left.\begin{aligned} &\int_0^\infty e^{-st} P_n(1-t)\, dt = \sqrt{\frac{\pi}{2s}}\, e^{i\frac{\pi}{4}(2n+1)} e^{-s} H^{(1)}_{n+\frac{1}{2}}(-is), \\ &\int_0^\infty e^{-st} P_n(-1-t)\, dt = (-1)^n \sqrt{\frac{\pi}{2s}}\, e^{i\frac{\pi}{4}(2n-1)} e^{s} H^{(2)}_{n+\frac{1}{2}}(-is). \end{aligned}\right\} \tag{248}$$

These formulae are readily obtained by direct integration of the left-hand sides. This method leads to the following equation for $\Omega(s)$:

$$\sqrt{\frac{2\pi}{s}}\, e^{i\frac{\pi}{4}(2n+1)} e^{-s} J_{n+\frac{1}{2}}(-is)\,\Omega(s) = F(s).$$

After making use of the inversion formula for the Laplace transformation, we obtain:

$$\omega(t) = \frac{e^{-i\frac{\pi}{4}(2n+1)}}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \sqrt{\frac{s}{2\pi}}\, \frac{e^{(t+1)s} F(s)}{J_{n+\frac{1}{2}}(-is)}\, ds. \tag{249}$$

The real number σ_1 is chosen big enough for all the singularities of the function $F(s)$ to lie to the left of the line of integration.

The possibility of applying the Laplace transformation and the inverse transformation is more readily justified in the present case due to the fact that the existence of $\omega(t)$ has already been established by the step by step method, and an upper bound can be fixed for this function for large t, provided certain conditions are imposed on $f'(t)$ for large t. On substituting expression (249) in formula (244), changing the order of integration and using the fairly obvious equation

$$\int\limits_{-1}^{1} e^{px} P_n(x)\, dx = \sqrt{2\pi}\, e^{i\frac{\pi}{4}(2n+1)} \frac{J_{n+\frac{1}{2}}(-ip)}{\sqrt{p}}, \tag{250}$$

we obtain

$$\varphi_n(r, t) = \frac{1}{\sqrt{r}\, 2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{J_{n+\frac{1}{2}}(-irs)}{J_{n+\frac{1}{2}}(-is)} F(s)\, e^{st}\, ds \quad (n = 0, 1, 2 \ldots). \tag{251}$$

A shorter way of obtaining this last formula may be mentioned. On substituting expression (243) in equation (227) and using the equation for $Y_n(\theta, \varphi)$, we get the following equation for $\varphi_n(r, t)$:

$$\frac{\partial^2 \varphi_n}{\partial t^2} = \frac{\partial^2 \varphi_n}{\partial r^2} + \frac{2}{r} \frac{\partial \varphi_n}{\partial r} - \frac{n(n+1)}{r^2} \varphi_n. \tag{252}$$

We must add to this equation the conditions:

$$\varphi_n \big|_{t=0} = \frac{\partial \varphi_n}{\partial t}\Big|_{t=0} = 0; \quad (253) \qquad \varphi_n\big|_{r=1} = f(t). \quad (254)$$

On multiplying both sides of equation (252) by e^{-st}, integrating with respect to t over the interval $0 \leqslant t < \infty$ and taking (253) into

account, we obtain for the function

$$X_n(r, s) = \int_0^\infty e^{-st} \varphi_n(r, t)\, dt \tag{255}$$

the equation

$$\frac{d^2X_n}{dr^2} + \frac{2}{r}\frac{dX_n}{dr} + \left(-s^2 - \frac{n(n+1)}{r^2}\right)X_n = 0. \tag{256}$$

Application of the Laplace transformation to (254) gives

$$X_n\big|_{r=1} = F(s). \tag{257}$$

Furthermore, the function $X_n(r, s)$ must be finite for $r = 0$. Equation (256) reduces to Bessel's equation, and we obtain on taking into account (257) and the fact that X_n is finite at $r = 0$:

$$X_n(r, s) = \frac{1}{\sqrt{r}} \frac{J_{n+\frac{1}{2}}(-irs)}{J_{n+\frac{1}{2}}(-is)} F(s).$$

After this, inversion of transformation (255) leads us to (251).

A discussion of the conditions which must be imposed on $f(t)$ for the employment of the Laplace transformation and (251) to be justified may be found in Petrashen's article *Dynamic problems of the theory of elasticity in the case of an isotropic sphere* (Dinamicheskie zadachi teorii uprugosti v sluchae izotropnoi sferi) (Uchenye zapiski LGU, seriya matematicheskikh nauk, vyp. 21, 1950). The material of the present section and the next has been taken from this work.

265. Investigation of the solution. Let us investigate the solution that we have obtained:

$$\varphi_n(r, t) = \frac{1}{\sqrt{r}\,2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{J_{n+\frac{1}{2}}(-irs)}{J_{n+\frac{1}{2}}(-is)} F(s)\, e^{st}\, ds. \tag{258}$$

We shall assume for definiteness that the function $f(t)$ differs from zero only in some finite interval $[0, T]$, has continuous first and second derivatives and satisfies

$$f(0) = f'(0) = f(T) = f'(T) = 0.$$

In this case, integration twice by parts gives us

$$F(s) = \int_0^T e^{-st} f(t)\, dt = \frac{1}{s^2}\int_0^T e^{-st} f''(t)\, dt. \tag{259}$$

We shall assume that function $F(s)$ has the form

$$F(s) = F_1(s) + F_2(s)\, e^{-sT}, \tag{260}$$

where $F_1(s)$ and $F_2(s)$ are rational fractions, in which the degree of the denominator is at least two more than the degree of the numerator. It is easily seen that $F(s)$ will have this property in the two cases:

$$f(t) = t^2(T-t)^2; \quad f(t) = \sin^2 \frac{\pi t}{T} \quad (0 \leqslant t \leqslant T) .$$

As is clear from (259), $F(s)$ is an entire function.

The function $J_{n+1/2}(-is)$ has pure imaginary zeros, which we write as $\pm k_s i$ $(s = 1, 2, 3, \ldots)$, where k_s are the roots of the equation $J_{n+1/2}(k) = 0$ [III$_2$, 145].

We use the formula:

$$J_{n+\frac{1}{2}}(z) = \frac{1}{2}\left[H^{(1)}_{n+\frac{1}{2}}(z) + H^{(2)}_{n+\frac{1}{2}}(z)\, i\right]$$

and the following expressions for the Hankel functions:

$$\left.\begin{aligned} H^{(1)}_{n+\frac{1}{2}}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i\left(z-\frac{\pi n}{2}-\frac{\pi}{2}\right)}\left[1+\varphi_1\left(\frac{1}{z}\right)\right], \\ H^{(2)}_{n+\frac{1}{2}}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\left(z-\frac{\pi n}{2}-\frac{\pi}{2}\right)}\left[1+\varphi_2\left(\frac{1}{z}\right)\right], \end{aligned}\right\} \tag{261}$$

where $\varphi_1(1/z)$ and $\varphi_2(1/z)$ are polynomials in $1/z$ with no constant term [III$_2$, 148]. On substituting $z = -is$, it may easily be seen that, for sufficiently large σ, the modulus of the ratio

$$\frac{H^{(2)}_{n+\frac{1}{2}}(-is)}{H^{(1)}_{n+\frac{1}{2}}(-is)}$$

does not exceed some number less than unity at all points of the contour of integration. In view of this, we can write:

$$\frac{J_{n+\frac{1}{2}}(-irs)}{J_{n+\frac{1}{2}}(-is)} = \frac{H^{(1)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)} \sum_{p=0}^{\infty}(-1)^p\left[\frac{H^{(2)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)}\right]^p +$$
$$+ \frac{H^{(2)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)} \sum_{p=0}^{\infty}(-1)^p\left[\frac{H^{(2)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)}\right]^p .$$

On substituting in (258) and integrating the series term by term, we get:

$$\varphi_n(r,t) = \frac{1}{\sqrt{r}} \sum_{p=0}^{\infty} \frac{(-1)^p}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{H^{(1)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)}\left[\frac{H^{(2)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)}\right]^p F(s)\, e^{st}\, ds +$$
$$+ \frac{1}{\sqrt{r}} \sum_{p=0}^{\infty} \frac{(-1)^p}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)}\left[\frac{H^{(1)}_{n+\frac{1}{2}}(-irs)}{H^{(1)}_{n+\frac{1}{2}}(-is)}\right]^p F(s)\, e^{st}\, ds . \tag{262}$$

Let us show that the right-hand side in fact contains only a finite number of terms, and that this number increases with increasing t. We take as an example the terms of the first sum. By making use of (261), we can write the integrals appearing in this sum in the form

$$\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\,[1+O(|z|^{-1})]\,e^{s[t-(2p+1)+r]}\,ds\,. \tag{263}$$

Suppose that the number p is so large that

$$t-(2p+1)+r<0\,. \tag{264}$$

We draw to the right of the contour of integration a semi-circle with centre σ and sufficiently large radius R. On taking into account formula (260) for $F(s)$ and the above-mentioned properties of $F_1(s)$ and $F_2(s)$, we can say that, given condition (264), the integral over this semi-circle of the integrand of (263) tends to zero on indefinite increase of R.

On the other hand, the integral over the closed contour formed by this semi-circle and the segment $-R \leqslant \sigma_1 \leqslant R$ of the contour of integration of integral (263) is equal to zero, since there are no singular points of the integrand inside this contour. Hence it follows, that, given condition (264), integral (263) vanishes. Similarly, the terms of the second sum of the right-hand side of (262) vanish if the condition is fulfilled:

$$t-(2p+1)-r<0\,. \tag{265}$$

The remaining terms describe the spherical waves which have been reflected a given number of times from the sphere $r=1$. By using the first of formulae (248), it is readily shown that the integrand of the integrals on the right-hand side of (262) has a finite number of singular points at a finite distance, which are given by the roots of the equation:

$$z^n - P_n'(1)\,z^{n-1} + \ldots + (-1)^n P_n^{(n)}(1) = 0\,,$$

and that the value of the integral is the sum of the residues at these poles, i.e. the integrals in question are expressible in terms of elementary functions.

The use of (258) as above described thus leads to a "d'Alembert method" for the solution of the problem of the vibrations of a sphere given boundary condition (237).

We shall now mention another transformation of (258), which leads to a "Fourier method" or, to be more precise, to an expansion of the solution of the problem in a series in eigenvibrations of the sphere. We substitute expression (260) for $F(s)$ in formula (258).

On substituting the term $F_2\,e^{-sT}$, the integrand will contain the factor $e^{s(t-T)}$, and it can be shown in precisely the same way as above that the corresponding integral vanishes for $t<T$, i.e. we have

$$\varphi_n(r,t) = \frac{1}{\sqrt{r}\;2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{J_{n+\frac{1}{2}}(-irs)}{J_{n+\frac{1}{2}}(-is)}\,F_1(s)\,e^{st}\,ds. \tag{266}$$

It can be shown that the value of this integral is equal to the sum of the residues of its integrand, and we obtain, on the assumption that the poles of $F_1(s)$ do not coincide with the zeros of $J_{n+1/2}(-is)$ (absence of resonance):

$$\varphi_n(r, t) = \psi_n(r, t) - 2\sum_{p=1}^{\infty} \frac{A_p \sin(k_p t + \omega_p)}{J_{n-\frac{1}{2}}(k_p)} \cdot \frac{J_{n+\frac{1}{2}}(k_p r)}{\sqrt{r}}, \tag{267}$$

where $\psi_n(r, t)$ corresponds to the sum of the residues at the poles of $F_1(s)$, and the notation has been used:

$$F_1(k_p i) = A_p e^{i\omega_p}.$$

It can be shown that, given our assumptions regarding $F_1(s)$, the series written is uniformly convergent with respect to t and r. It represents the superposition of the eigenvibrations of the system. It follows from this that the term $\psi_p(r, t)$, which can be written in the closed form, satisfies equation (227) and the boundary condition (237). If a pole of $F_1(s)$ coincides with a zero of $J_{n+1/2}(-is)$, a resonance term, containing t outside the sign of the trigonometric functions, appears in the right-hand side of (267).

If $t > T$, we merely get a series in the eigenvibrations, since $F(s)$ is an entire function, and we can apply Jordan's lemma [III_2, 60] with $t > T$ for some system of semi-circles with centre σ_1 for the integrand, in which all the function $F(s)$ appears. With $t < T$, we do not have suitable inequalities for the integrand on the above-mentioned semi-circles. The absence of a supplementary term in addition to the series in eigenvibrations is bound up with the exclusion of an external force from the boundary condition.

266. The boundary value problem for the telegraphist's equation. We have solved the boundary value problems for equations of the elliptic and parabolic types by making use of potential theory, the whole of the working being based on a singular solution of the corresponding differential equation. This method of potential theory cannot be used for equations of the hyperbolic type. It is only in the one-dimensional case for the telegraphist's equation that the fundamental idea of this method can be used to reduce a boundary value problem to a Volterra integral equation.

We take the equation [II, 185]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + c^2 u \tag{268}$$

in the interval $0 \leqslant x \leqslant l$ with the homogeneous initial conditions:

$$u|_{t=0} = u_t|_{t=0} = 0 \tag{269}$$

and the boundary conditions:

$$u|_{x=0} = \omega_1(t); \quad u|_{x=l} = \omega_2(t). \tag{270}$$

It may be remarked that the initial conditions can always be made homogeneous if use is made of the solution for an unbounded interval [II, 185], as has already been done in [243] for the heat conduction equation. On introducing the func-

tion $I(z) = J_0(iz)$ as in [II, 185], it is readily seen that the function $I(c\sqrt{t^2 - x^2})$ is a solution of equation (268). It will serve as our basic solution.

On locating the continuously operative sources corresponding to this solution at the ends of the interval [0, 1], we obtain, as may easily be verified directly, the solutions of equation (268):

$$\int_0^{t-x} \varphi(\tau) I\left(c\sqrt{(t-\tau)^2 - x^2}\right) d\tau$$

and

$$\int_0^{t-(l-x)} \psi(\tau) I\left(c\sqrt{(t-\tau)^2 - x^2}\right) d\tau,$$

where the functions $\varphi(\tau)$ and $\psi(\tau)$ are assumed differentiable. On differentiating these solutions with respect to x, we obtain further solutions, and we seek the solution of problem (268), (269), (270) in the form of the sum:

$$u = \frac{\partial}{\partial x}\int_0^{t-x} \varphi(\tau) I\left(c\sqrt{(t-\tau)^2 - x^2}\right) d\tau +$$
$$+ \frac{\partial}{\partial x}\int_0^{t-(l-x)} \psi(\tau) I\left(c\sqrt{(t-\tau)^2 - (t-x)^2}\right) d\tau, \tag{271}$$

on the assumption that $\varphi(\tau) = \psi(\tau) = 0$ for $\tau < 0$.

Formula (271) can be written as

$$u = -\varphi(t-x) - \int_0^{t-x} \varphi(\tau) \frac{cx\, I'\left(c\sqrt{(t-\tau)^2 - x^2}\right)}{\sqrt{(t-\tau)^2 - x^2}} d\tau +$$
$$+ \psi(t-l+x) + \int_0^{t-(l-x)} \psi(\tau) \frac{c(l-x) I'\left(c\sqrt{(t-\tau)^2 - (l-x)^2}\right)}{\sqrt{(t-\tau)^2 - (l-x)^2}} d\tau. \tag{272}$$

We recall the expansion:

$$I(z) = \sum_{s=0}^{\infty} \frac{1}{(s!)^2}\left(\frac{z}{2}\right)^{2s}.$$

Equation (268) and initial conditions (269) are satisfied for any choice of $\varphi(\tau)$ and $\psi(\tau)$. Boundary conditions (270) reduce to the following system of equations for $\varphi(\tau)$ and $\psi(\tau)$:

$$\left.\begin{aligned} &-\varphi(\tau) + \psi(t-l) + \int_0^{t-l} \psi(\tau) \frac{cl\, I'\left(c\sqrt{(t-\tau)^2 - l^2}\right)}{\sqrt{(t-\tau)^2 - l^2}} d\tau = \omega_1(t), \\ &-\varphi(t-l) + \psi(t) - \int_0^{t-l} \varphi(\tau) \frac{cl\, I'\left(c\sqrt{(t-\tau)^2 - l^2}\right)}{\sqrt{(t-\tau)^2 - l^2}} d\tau = \omega_2(t). \end{aligned}\right\} \tag{273}$$

We assume that $\omega_1(\tau)$ and $\omega_2(t)$ are continuously differentiable. We put

$$\psi(t) - \varphi(t) = \varphi_1(t);$$
$$\psi(t) + \varphi(t) = \psi_1(t).$$

On adding and subtracting equations (273) term by term, we obtain separate equations for $\varphi_1(t)$ and $\psi_1(t)$:

$$\left.\begin{aligned} \varphi_1(t) + \varphi_1(t-l) + cl \int_0^{t-l} \varphi_1(\tau) \frac{I'(c\sqrt{(t-\tau)^2 - l^2})}{\sqrt{(t-\tau)^2 - l^2}} \, d\tau &= \omega_1(t) + \omega_2(t), \\ -\psi_1(t) + \psi_1(t-l) + cl \int_0^{t-l} \psi_1(\tau) \frac{I'(c\sqrt{(t-\tau)^2 - l^2})}{\sqrt{(t-\tau)^2 - l^2}} \, d\tau &= \omega_1(t) - \omega_2(t), \end{aligned}\right\} \quad (274)$$

where $\varphi_1(\tau) = \psi_1(\tau) = 0$ for $\tau < 0$.

We can find $\varphi_1(t)$ and $\psi_1(t)$ from these equations by means of successive steps over the intervals $[0, l]$, $[l, 2l]$, etc. We have:

$$\varphi_1(t) = \omega_1(t) + \omega_2(t); \ \psi_1(t) = \omega_2(t) - \omega_1(t) \ \text{for} \ 0 \leqslant t \leqslant l;$$

$$\left.\begin{aligned} \varphi_1(t) &= \omega_1(t) + \omega_2(t) - \varphi_1(t-l) - cl \int_0^{t-l} \varphi_1(\tau) \frac{I'(c\sqrt{(t-\tau)^2 - l^2})}{\sqrt{(t-\tau)^2 - l^2}} \, d\tau, \\ \psi_1(t) &= \omega_2(t) - \omega_1(t) + \psi_1(t-l) + cl \int_0^{t-l} \psi_1(\tau) \frac{I'(c\sqrt{(t-\tau)^2 - l^2})}{\sqrt{(t-\tau)^2 - l^2}} \, d\tau \end{aligned}\right\} \quad (275)$$

$$\text{for } l \leqslant t \leqslant 2l$$

and so on.

A Laplace transformation can be used instead of the step by step method for the solution of the integral equations.

The material for the present section has been taken from an unpublished work of D. A. Dobrotin.

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